Integral Transforms Related to the Fourier Sine Convolution with a Weight Function

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Abstract. Integral transforms of the form

\[ g(x) = \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_0^\infty f(y) \left[ \text{sign} \ (x + y - 1)k_1(|x + y - 1|) \right. \right. \]
\[ + \ \left. \left. \text{sign} \ (x - y + 1)k_1(|x - y + 1|) - k_1(x + y + 1) \right. \right. \]
\[ - \ \left. \left. \text{sign} \ (x - y - 1)k_1(|x - y - 1|) \right] \right. \]
\[ + \int_0^\infty f(y)[k_2(|x - y|) - k_2(x + y)]dy \}\]

from \(L_p(\mathbb{R}^+)\) to \(L_q(\mathbb{R}^+)\), \((1 \leq p \leq 2, \ p^{-1} + q^{-1} = 1)\) are studied. Watson’s and Plancherel’s Theorems are obtained. Applications to solving integral equation and systems of integral equations are considered.

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1. Introduction

The theory of convolution for integral transforms were studied in the 20th century. At first, the convolution for the Fourier transformation has been studied.
Namely, the convolution of two functions $f$ and $g$ for the Fourier transform has the form

$$ (f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)g(x - y)dy, \quad x \in \mathbb{R}, $$

with the factorization property

$$ F(f * g)(y) = (Ff)(y)(Fg)(y), \quad \forall y \in \mathbb{R}, $$

where $F$ is the Fourier integral transform

$$ (Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixy} f(y)dy. $$

Later on, convolutions for integral transforms Laplace, Mellin, Hilbert, Hankel, Kontorovich - Lebedev and Stieltjes have been introduced and studied. At the same time, integral transforms of the Fourier convolution type, of the Laplace convolution type, of the Mellin convolution type, ... have also been constructed and investigated.

In 1941, Churchill introduced the convolution of two functions $f$ and $g$ for the Fourier cosine integral transform defined by the formula below [2]

$$ (f *_{F_c} g)(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} f(y)[g(x + y) + g(|x - y|)]dy, \quad x > 0, $$

for which the following factorization equality holds [8]

$$ F_c(f *_{F_c} g)(y) = (F_c f)(y)(F_c g)(y), \quad \forall y > 0. $$

Here, $F_c$ is the Fourier cosine transform [1]

$$ (F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} \cos(xy)f(y)dy. $$

The first convolution with a weight function was found by Vilenkin in 1958 for the transform Mehler - Fock. In 1967, Kakichev proposed a constructive method for defining the convolution with a weight function for an arbitrary integral transform (see [3]). The convolution of two functions $f$ and $g$ with the weight function $\gamma(y) = \sin y$ for the Fourier sine integral transformation has been studied in [3, 15]

$$ (f *_{F_s} \gamma g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{+\infty} f(y)[\text{sign}(x + y - 1) g(|x + y - 1|) $$

$$ + \text{sign}(x - y + 1) g(|x - y + 1|) - g(x + y + 1) $$

$$ - \text{sign}(x - y - 1) g(|x - y - 1|)]dy, \quad x > 0, $$

(4)
for which the following factorization identity holds ([3, 15])

$$F_s(f \star_2 g)(y) = \sin y (F_s f)(y)(F_s g)(y), \quad \forall y > 0,$$

(5)

where $F_s$ is the Fourier sine transform ([1])

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \sin(xy) f(y) dy.$$

In 1941, the first generalized convolution for two integral transforms was introduced by Churchill. Namely, he defined the generalized convolution of two functions $f$ and $g$ for the Fourier sine and cosine transforms [2]

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y)(|x - y|) - g(x + y) dy, \quad x > 0,$$

(6)

and proved the following factorization identity [7]

$$F_s(f * g)(y) = (F_s f)(y)(F_s g)(y), \quad \forall y > 0.$$

(7)

In the nineties of the last century, Yakubovich introduced several generalized convolutions with index for the Mellin transform, Kontorovich-Lebedev transform, $G$-transform and $H$-transform. In 1998, Kachichev and Nguyen Xuan Thao proposed a constructive method for defining the generalized convolution for three arbitrary integral transforms (see [4]). Up to now, based on this method, several new generalized convolutions for integral transforms were established and investigated. For instance, the generalized convolution for Stieltjes, Hilbert, Fourier cosine and sine integral transforms have been introduced in [11]; the generalized convolution for the $I$- transform has been studied in [17]; the generalized convolution with a weight function for the Fourier sine, Kontorovich-Lebedev and the Fourier cosine integral transforms and the generalized convolution for the Kontorovich-Lebedev, Fourier sine and cosine transforms have also been investigated in [18], [19], respectively; the generalized convolution with a weight function for the Fourier sine and cosine transforms were introduced in [14]; the generalized convolution with a weight function for the Fourier, Fourier cosine and sine transforms were found in [12], and so on.

The first generalized convolution which was constructed basing on that method was introduced in 1998 in [5]. Namely, the generalized convolution of two functions $f$ and $g$ for the Fourier cosine and sine transforms has the form

$$(f \star_1 g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y)[ \text{sign} (y - x)(|y - x|) + g(y + x)] dy, \quad x > 0,$$

(8)

where the following factorization property has been established [5]

$$F_c(f \star_1 g)(y) = (F_s f)(y)(F_s g)(y), \quad \forall y > 0.$$  

(9)
Another generalized convolution with the weight function $\gamma(y) = \sin y$ for the Fourier cosine and Fourier sine transforms has been studied in [16]

$$
(f \ast g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(u)[g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u-1|)] du, \quad x > 0.
$$

It satisfies the factorization property [16]

$$
F_c(f \ast g)(y) = \sin y (F_s f)(y)(F_c g)(y), \quad \forall y > 0.
$$

Recently, in 2000, classes of integral transforms related to the generalized convolutions (6) and (8) was constructed and investigated by Vu Kim Tuan in [6, 7]. In this paper we will consider a new class of integral transforms which is related to the convolution with a weight function for the Fourier sine transform (4) and the generalized convolution for the Fourier sine and cosine transforms (6), namely, the transforms of the form

$$
g(x) = \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_0^{+\infty} f(y)[ \text{sign} (x+y-1)k_1(|x+y-1|) \\
+ \text{sign} (x-y+1)k_1(|x-y+1|) \\
- k_1(x+y+1) - \text{sign} (x-y-1)k_1(|x-y-1|)] dy \\
+ \int_0^{+\infty} f(y)[k_2(|x-y|) - k_2(x+y)] dy \right\}.
$$

We will show that with certain conditions of $k_1$ and $k_2$, transform (12) defines a bounded operator from $L_p(\mathbb{R}^+)$ to $L_q(\mathbb{R}^+)$ ($1 \leq p \leq 2$), $p^{-1} + q^{-1} = 1$. Moreover, we will show that with these conditions of $k_1$ and $k_2$, the transform (12) is a unitary operator in $L_2(\mathbb{R}^+)$. Watson and Plancherel type Theorem for transform (12) in $L_2(\mathbb{R}^+)$ are also obtained.

2. A Watson Type Theorem

**Lemma 1.** Let $f$ and $g$ be $L_2(\mathbb{R}^+)$ functions. Then the following Parseval identity holds

$$
\int_0^{+\infty} f(u)[ \text{sign} (x+u-1)g(|x+u-1|) + \text{sign} (x-u+1)g(|x-u+1|) \\
- g(x+u+1) - \text{sign} (x-u-1)g(|x-u-1|)] du \\
= 2\sqrt{2\pi} F_s \left( \sin u(F_s f)(u)(F_s g)(u) \right)(x), \quad \forall x > 0.
$$


Proof. Let \( f_1 \) and \( g_1 \) be the odd extension of \( f \) and \( g \) from \( \mathbb{R}_+ \) to \( \mathbb{R} \), respectively. Then on \( \mathbb{R}_+ \) we have \( Ff_1 = -i F_x f \) and \( Fg_1 = -i F_x g \). Applying the Parseval identity for Fourier transform

\[
\int_{-\infty}^{+\infty} f(y)g(x-y)dy = \int_{-\infty}^{+\infty} (Ff)(y)(Fg)(y)e^{ixy}dy,
\]

we have

\[
\begin{align*}
&\int_{0}^{+\infty} f(u)[\text{sign } (x+u-1)g(|x+u-1|) + \text{sign } (x-u+1)g(|x-u+1|) \\
&- g(x+u+1) - \text{sign } (x-u-1)g(|x-u-1|)]du \\
&= \int_{0}^{+\infty} f_1(u)[g_1(x-u+1) + g_1(x-u-1) - g_1(x+u+1) - g_1(x+u-1)]du \\
&= \int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)e^{i(x+1)u}du - \int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)e^{i(x-1)u}du \\
&= \int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)\{\cos((x+1)u) + i\sin((x+1)u)\}du \\
&\quad - \int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)\{\cos((x-1)u) + i\sin((x-1)u)\}du.
\end{align*}
\]

On the other hand, note that \((Ff_1)(u)(Fg_1)(u)\sin((x+1)u)\) and \((Ff_1)(u)(Fg_1)(u)\sin((x-1)u)\) are odd functions in \( u \). Hence their integrals over \( \mathbb{R} \) vanish, and therefore,

\[
\begin{align*}
&\int_{0}^{+\infty} f(u)[\text{sign } (x+u-1)g(|x+u-1|) + \text{sign } (x-u+1)g(|x-u+1|) \\
&- g(x+u+1) - \text{sign } (x-u-1)g(|x-u-1|)]du \\
&= \int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)\cos((x+1)u)du - \int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)\cos((x-1)u)du \\
&= -2 \int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)\sin u \sin(xu)du.
\end{align*}
\]
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\[\begin{align*}
&= -4 \int_{0}^{+\infty} (Ff_1)(u) (Fg_1)(u) \sin u \sin(xu) du \\
&= 2\sqrt{2\pi} F_s (\sin u(F_s f)(u)(F_s g)(u))(x).
\end{align*}\]

This completes the proof of the lemma.

\[\square\]

Theorem 1. Let \(k_1, k_2 \in L_2(\mathbb{R}^+)\). Then

\[|2 \sin y (F_s k_1)(y) + (F_s k_2)(y)| = \frac{1}{\sqrt{2\pi(1 + y^2)}}\]

is a necessary and sufficient condition to ensure that the integral transform \(f \mapsto g:\)

\[g(x) = \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_{0}^{+\infty} f(u)[\text{sign}(x + u - 1)k_1(|x + u - 1|) \\
+ \text{sign}(x - u + 1)k_1(|x - u + 1|) - k_1(x + u + 1) \\
- \text{sign}(x - u - 1)k_1(|x - u - 1|)]du \\
+ \int_{0}^{+\infty} f(u)[k_2(|x - u|) - k_2(x + u)]du \right\}\]

(15)

is unitary on \(L_2(\mathbb{R}^+)\) and the inverse transformation has the form

\[f(x) = \left(1 - \frac{d^2}{dx^2}\right) \left\{ \int_{0}^{+\infty} k_1(u)[\text{sign}(x + u - 1)g(|x + u - 1|) \\
+ \text{sign}(x - u + 1)g(|x - u + 1|) - g(x + u + 1) \\
- \text{sign}(x - u - 1)g(|x - u - 1|)]du \\
+ \int_{0}^{+\infty} g(u)[k_2(|x - u|) - k_2(x + u)]du \right\}\]

(16)

Proof.

Necessity. Suppose that \(k_1\) and \(k_2\) satisfy condition (14). It is well-known that \(h(y), gh(y), g^2h(y)\) belong to \(L_2(\mathbb{R}^+)\) if and only if \((Fh)(x), \frac{d^2}{dx^2}(Fh)(x)\) and \(\frac{d^2}{dx^2}(Fh)(x)\) are also \(L_2(\mathbb{R}^+)\) functions (Theorem 68, page 92, [10]). Moreover,

\[\frac{d^2}{dx^2}(Fh)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(y)e^{-i\pi y} dy = F((iy)^2h(y))(x).\]
In particular, if $h$ is an even or odd function such that $(1 + y^2)h(y) \in L_2(\mathbb{R}_+)$, then the following equalities hold

$$
(1 - \frac{d^2}{dx^2})(F_s h)(x) = F_s ((1 + y^2)h(y))(x),
$$

and

$$
(1 - \frac{d^2}{dx^2})(F_s h)(x) = F_s ((1 + y^2)h(y))(x).
$$

Using the Lemma 1 and the factorization equalities for generalized convolutions (6), (8) we have

$$
g(x) = (1 - \frac{d^2}{dx^2})F_s(2\sqrt{2\pi} \sin y(F_s k_1)(y)(F_s f)(y) + \sqrt{2\pi}(F_s f)(y)(F_s k_2)(y))(x)
$$

$$
= F_s(\sqrt{2\pi}(1 + y^2)(2\sin y(F_s k_1)(y) + (F_s k_2)(y))(F_s f)(y))(x).
$$

By virtue of Parseval equality for the Fourier sine transform $\|f\|_{L_2(\mathbb{R}_+)} = \|F_s f\|_{L_2(\mathbb{R}_+)}$ and note that $k_1$ and $k_2$ satisfy condition (14) we have

$$
\|g\|_{L_2(\mathbb{R}_+)} = \|2\sqrt{2\pi}(1 + y^2)(2\sin y(F_s k_1)(y) + (F_s k_2)(y))(F_s f)(y)\|_{L_2(\mathbb{R}_+)}
$$

$$
= \|F_s f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.
$$

It follows that the transformation (15) is unitary.

On the other hand, in view of condition (14), $\sqrt{2\pi}(1 + y^2)(2\sin y(F_s k_1)(y) + (F_s k_2)(y))$ is bounded, hence $\sqrt{2\pi}(1 + y^2)(2\sin y(F_s k_1)(y) + (F_s k_2)(y))(F_s f)(y) \in L_2(\mathbb{R}_+)$. We have

$$
(F_s g)(y) = \sqrt{2\pi}(1 + y^2)(2\sin y(F_s k_1)(y) + (F_s k_2)(y))(F_s f)(y).
$$

It follows that

$$
(F_s f)(y) = \sqrt{2\pi}(1 + y^2)(2\sin y(F_s \overline{k_1})(y) + (F_s \overline{k_2})(y))(F_s g)(y).
$$

Again, condition (14) of $k_1, k_2$ yields

$$
\sqrt{2\pi}(1 + y^2)(2\sin y(F_s \overline{k_1})(y) + (F_s \overline{k_2})(y))(F_s g)(y) \in L_2(\mathbb{R}_+).
$$

Using formulae (17) we obtain

$$
f(x) = F_s[\sqrt{2\pi}(1 + y^2)(2\sin y(F_s \overline{k_1})(y) + (F_s \overline{k_2})(y))(F_s g)(y)](x)
$$

$$
= (1 - \frac{d^2}{dx^2})F_s(2\sqrt{2\pi} \sin y(F_s \overline{k_1})(y)(F_s g)(y) + \sqrt{2\pi}(F_s g)(y)(F_s \overline{k_2})(y))(x)
$$

$$
= (1 - \frac{d^2}{dx^2})\left\{ \int_0^{+\infty} \overline{k_1}(y)[\text{sign}(x + y - 1)g(|x + y - 1|)
$$

$$
+ \text{sign}(x - y + 1)g(|x - y + 1|) - g(x + y + 1)
$$

$$
- \text{sign}(x - y - 1)g(|x - y - 1|)]dy
$$

$$
+ \int_0^{+\infty} g(y)[\overline{k_2}(|x - y|) - \overline{k_2}(x + y)]dy \right\}.$$


Therefore the transformation (15) is unitary on $L_2(\mathbb{R}_+)$ and the inverse transformation has the form (16).

**Sufficiency.** Suppose that transform (15) is unitary on $L_2(\mathbb{R}_+)$ with the inverse transformation defined by (16). Then Parseval identity for the Fourier sine transform yields

\[
\|g\|_{L_2(\mathbb{R}_+)} = \|\sqrt{2\pi}(1 + y^2)(2 \sin y(F_s k_1)(y) + (F_c k_2)(y))(F_s f)(y)\|_{L_2(\mathbb{R}_+)} = \|F_s f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.
\]

However, the middle equality holds for all $f \in L_2(\mathbb{R}_+)$ if and only if

\[
|\sqrt{2\pi}(1 + y^2)(2 \sin y(F_s k_1)(y) + (F_c k_2)(y))(F_s f)(y)| = |(F_s f)(y)|.
\]

It shows that $k_1$ and $k_2$ satisfy condition (14). The proof of the theorem has been completed. ■

Let $h_1, h_2 \in L_2(\mathbb{R}_+)$ satisfy the following condition

\[
|\langle F_s h_1, y \rangle \langle F_s h_2, y \rangle| = \frac{1}{(1 + y^2)(1 + \sin^2 y)}.
\]

(18)

For example, consider

\[
h_1(x) = F_s \left( \frac{e^{iu(y)}}{(1 + y^2)^{1/2}(1 + \sin^2 y)^{1/2}} \right)(x),
\]

\[
h_2(x) = F_s \left( \frac{e^{iv(y)}}{(1 + y^2)^{1/2}(1 + \sin^2 y)^{1/2}} \right)(x),
\]

where $u, v$ are some functions defined on $\mathbb{R}_+$.

Let $k_1, k_2$ be defined by

\[
k_1(x) = \frac{1}{2\sqrt{2\pi}}(h_1 \ast F_s h_2)(x), \quad k_2(x) = \frac{1}{\sqrt{2\pi}}(h_1 \ast 2h_2)(x).
\]

Then $k_1, k_2 \in L_2(\mathbb{R}_+)$ and from (5) and (9) we have

\[
\left| 2 \sin y(F_s k_1)(y) + (F_c k_2)(y) \right| = \left| \frac{1}{\sqrt{2\pi}} \sin^2 y(F_s h_1)(y)(F_s h_2)(y) + \frac{1}{\sqrt{2\pi}} (F_s h_1)(y)(F_s h_2)(y) \right|
\]

\[
= \left| \frac{1}{\sqrt{2\pi}} (1 + \sin^2 y)(F_s h_1)(y)(F_s h_2)(y) \right|
\]

\[
= \frac{1}{\sqrt{2\pi}(1 + y^2)}.
\]
Thus $k_1$ and $k_2$ satisfy condition (14).

For another example, let $h_1$ and $h_2$ be $L_2(\mathbb{R}_+)$ functions and satisfy the condition

$$|(F_c h_1)(y)(F_c h_2)(y)| = \frac{1}{(1 + y^2)(1 + \sin^2 y)}$$

(19)

and let $k_1, k_2$ be defined by

$$k_1(x) = \frac{1}{2\sqrt{2\pi}} (h_1 \ast h_2)(x), \quad k_2(x) = \frac{1}{\sqrt{2\pi}} (h_1 \ast F_c h_2)(x).$$

Then $k_1, k_2 \in L_2(\mathbb{R}_+)$ and we have

$$\left| 2 \sin(y F_c k_1)(y) + (F_c k_2)(y) \right| = \frac{1}{\sqrt{2\pi}} \sin^2(y (F_c h_1)(y)(F_c h_2)(y)) + \frac{1}{\sqrt{2\pi}} (F_c h_1)(y)(F_c h_2)(y)$$

$$= \frac{1}{\sqrt{2\pi}} (1 + \sin^2 y)(F_c h_1)(y)(F_c h_2)(y)$$

$$= \frac{1}{\sqrt{2\pi}} (1 + y^2).$$

Thus $k_1$ and $k_2$ satisfy condition (14).

3. A Plancherel Type Theorem

**Theorem 2.** Let $k_1, k_2$ be functions satisfying condition (14) and suppose that $K_1(x) = (1 - \frac{d^2}{dx^2}) k_1(x)$ and $K_2(x) = (1 - \frac{d^2}{dx^2}) k_2(x)$ are locally bounded. Let $f \in L_2(\mathbb{R}_+)$ and for each positive integer $N$, put

$$g_N(x) = \left\{ \int_0^N f(u) \left[ \text{sign} (x + u - 1) K_1(|x + u - 1|) ight. \\
+ \text{sign}(x - u + 1) K_1(|x - u + 1|) - K_1(x + u + 1) \\
\left. - \text{sign} (x - u - 1) K_1(|x - u - 1|) \right] du \\
+ \int_0^N f(u) [K_2(|x - u|) - K_2(x + u)] du \right\}.$$  

(20)

Then

1) $g_N \in L_2(\mathbb{R}_+)$ and as $N \to +\infty$, $g_N$ converges in $L_2(\mathbb{R}_+)$-norm to a function $g$, furthermore $\|g\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$. 

2) Put \( g^N = g \cdot \chi_{(0,N)} \), then

\[
 f_N(x) = \left\{ \int_0^{+\infty} K_1(u) \left[ \text{sign} \,(x + u - 1) g^N(|x + u - 1|) \right] \right. \\
+ \left. \text{sign} \,(x - u + 1) g^N(|x - u + 1|) - g^N(x + u + 1) \right) \\
- \left. \text{sign} \,(x - u - 1) g^N(|x - u - 1|) \right\} \\
+ \int_0^{N} g(u) [K_2(|x - u|) - K_2(x + u)] du \tag{21}
\]

belongs to \( L_2(\mathbb{R}+) \) and converges in \( L_2(\mathbb{R}^+)-\text{norm} \) to \( f \) as \( N \to +\infty \).

**Remark 1.** The integrals defining \( f_N \) and \( g_N \) are defined over finite intervals and therefore converge.

**Proof.** Put \( f_N = f \cdot \chi_{(0,N)} \) then

\[
 g_N(x) = \int_0^{N} f(u) \left[ \text{sign} \,(x + u - 1) K_1(|x + u - 1|) \right] \\
+ \left. \text{sign} \,(x - u + 1) K_1(|x - u + 1|) \right) \\
- \left. K_1(x + u + 1) - \text{sign} \,(x - u - 1) K_1(|x - u - 1|) \right] du \\
+ \int_0^{N} f(u) [K_2(|x - u|) - K_2(x + u)] du \\
= \left( 1 - \frac{d^2}{dx^2} \right) \int_0^{+\infty} f^N(u) \left[ \text{sign} \,(x + u - 1) k_1(|x + u - 1|) \right] \\
+ \left. \text{sign} \,(x - u + 1) k_1(|x - u + 1|) - k_1(x + u + 1) \right) \\
- \left. \text{sign} \,(x - u - 1) k_1(|x - u - 1|) \right] du \\
+ \int_0^{+\infty} f^N(u) [k_2(|x - u|) - k_2(x + u)] du.
\]

Interchanging the order of integration and differentiation here is legitimate since the integral was over a finite interval. Theorem 1 guarantees that \( g_N \in L_2(\mathbb{R}^+) \). Moreover, if \( g \) is the image of \( f \) under the transform (15), we obtain that
Remark 2. Theorem 1 shows that transformation (15) is unitary in $\int_0^\infty g(y) dy$ and the reciprocal formula (16) holds. We have

$$\|g\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$$

so again by Theorem 1, $g - g_N \in L_2(\mathbb{R}_+)$ and

$$\|g - g_N\|_{L_2(\mathbb{R}_+)} = \|f - f^N\|_{L_2(\mathbb{R}_+)}$$

and since $\|f - f^N\|_{L_2(\mathbb{R}_+)} \to 0$ as $N \to +\infty$, it follows that $g_N$ converges in $L_2(\mathbb{R}_+)$-norm to $g$ as $N \to +\infty$. This completes the first part of the theorem.

Note that the convolution of two functions $f, g$ with the weight function $\gamma(y) = \sin y$ for the Fourier sine transform is commutative [15], hence the following identity holds

$$\int_0^\infty K_1(u) \sin (x - u) g^N(\sin (x + u)) + \sin (x - u) g^N(\sin (x - u)) + \sin (x + u) g^N(\sin (x - u)) + \sin (x - u) g^N(\sin (x + u)) du$$

Therefore, in a similar way, one can obtain the second part of the theorem. □

Remark 2. Theorem 1 shows that transformation (15) is unitary in $L_2(\mathbb{R}_+)$ and the inverse transformation has the form (16). Furthermore, Theorem 2 proved that these transformations (15) and (16) can be approximated in $L_2(\mathbb{R}_+)$-norm by the integral operators (20) and (21), respectively.

We assume additionally now that $K_1(x)$ and $K_2(x)$ are bounded on $\mathbb{R}_+$. Then transform (15) is a bounded operator from the space $L_1(\mathbb{R}_+)$ into the space $L_\infty(\mathbb{R}_+)$. Moreover, by Theorem 2 transform (15) is bounded on $L_2(\mathbb{R}_+)$. Hence, Riesz interpolation theorem [9] yields the following.

Theorem 3. Let $k_1, k_2$ be functions satisfying condition (14) and suppose that $K_1(x)$ and $K_2(x)$ are bounded on $\mathbb{R}_+$. Let $1 \leq p \leq 2$ and $q$ be its conjugate
exponent \( \frac{1}{p} + \frac{1}{q} = 1 \). Then the transformations

\[
g(x) = \lim_{N \to +\infty} \left\{ \int_0^N f(u) \left[ \text{sign} (x + u - 1) K_1(|x + u - 1|) ight. \right.
\]
\[
+ \text{sign} (x - u + 1) K_1(|x - u + 1|)
\]
\[
- K_1(x + u + 1) - \text{sign} (x - u - 1) K_1(|x - u - 1|) \left. \right] du
\]
\[
+ \int_0^N f(u) [K_2(|x - u|) - K_2(x + u)] du \right\} \quad (22)
\]

and

\[
g(x) = \lim_{N \to +\infty} \left\{ \int_0^{+\infty} K_1(u) \left[ \text{sign} (x + u - 1) f^N(|x + u - 1|) \right. \right.
\]
\[
+ \text{sign} (x - u + 1) f^N(|x - u + 1|) - f^N(x + u + 1)
\]
\[
- \text{sign} (x - u - 1) f^N(|x - u - 1|) \left. \right] du
\]
\[
+ \int_0^N f(u) [K_2(|x - u|) - K_2(x + u)] du \right\} \quad (23)
\]

are bounded operators from \( L_p(\mathbb{R}_+) \) into \( L_q(\mathbb{R}_+) \), here the limits are understood in \( L_q(\mathbb{R}_+) \)-norm.

4. Applications to Integral Equation and Systems of Integral Equations

4.1. Consider the System of Integral Equations

\[
f(x) + \lambda_1 \left\{ \int_0^{+\infty} f(y) \theta(x, y) dy + \int_0^{+\infty} \psi(y) [g(|x - y|) - g(x + y)] dy \right\} = h(x),
\]
\[
\lambda_2 \int_0^{+\infty} f(y) [\text{sign} (y - x) \xi(|y - x|) + \xi(y + x)] dy + g(x) = k(x). \quad (24)
\]

Here and throughout this section we will denote by \( \theta(x, y) \) the following function

\[
\theta(x, y) = \text{sign} (x + y - 1) \varphi(|x + y - 1|) + \text{sign} (x - y + 1) \varphi(|x - y + 1|)
\]
\[
- \varphi(x + y + 1) - \text{sign} (x - u - 1) \varphi(|x - u - 1|),
\]
\[
\varphi(x) = (\varphi_1 + \varphi_2)(x).
\]
\( \lambda_1 \) and \( \lambda_2 \) are complex constants and \( \varphi_1, \varphi_2, \psi, \xi, h, k \) are functions from \( L_1(\mathbb{R}_+) \), \( f \) and \( g \) are unknown functions.

**Theorem 4.** With the condition
\[
1 + 2\sqrt{2\pi}\lambda_1 \sin y(F_s \varphi)(y) - 2\pi \lambda_1 \lambda_2 (F_s \psi)(y)(F_s \xi)(y) \neq 0, \quad \forall y > 0
\]
there exists a unique solution in \( L_1(\mathbb{R}_+) \) of system (24) which has the form
\[
f(x) = h(x) - \lambda_1 \sqrt{2\pi}(\psi * k)(x) + (h * l)(x) + \lambda_1 \sqrt{2\pi}((\psi * k) * l)(x),
\]
\[
g(x) = k(x) + \lambda_1 \sqrt{2\pi}(\varphi_2 * k)(x) - \lambda_2 \sqrt{2\pi}(\xi * h)(x) - (k * l)(x)
- \lambda_1 \sqrt{2\pi}(\varphi_2 * k)(x) + \lambda_2 \sqrt{2\pi}((\xi * h) * l)(x),
\]
where \( l \in L_1(\mathbb{R}_+) \) is defined by
\[
(F.l)(y) = \frac{\lambda_1 \sqrt{2\pi} \sin y(F_s \varphi)(y) - \lambda_1 \lambda_2 \sqrt{2\pi} (F_s \xi)(y)(F_s \psi)(y)}{1 + \lambda_1 \sqrt{2\pi} \sin y(F_s \varphi)(y) - \lambda_1 \lambda_2 \sqrt{2\pi} (F_s \xi)(y)(F_s \psi)(y)}.
\]

**Remark 3.** By the condition that \( \varphi_1 \) and \( \varphi_2 \) be \( L_1(\mathbb{R}_+) \) functions, it is clear that \( \varphi \) belongs to \( L_1(\mathbb{R}_+) \).

**Proof.** In view of the factorization properties of convolutions (4), (6) and (8), one can rewrite the system (24) in the form
\[
(F_s f)(y) + \lambda_1 \sqrt{2\pi} \sin y(F_s \varphi)(y)(F_s \psi)(y) + \lambda_1 \sqrt{2\pi} (F_s \psi)(y)(F_s g)(y) = (F_s h)(y)
\]
\[
\lambda_1 \lambda_2 \sqrt{2\pi} (F_s f)(y)(F_s \xi)(y) + (F_s g)(y) = (F_s k)(y).
\]
Accordingly, we have
\[
\Delta = \begin{vmatrix}
1 + \lambda_1 \sqrt{2\pi} \sin y(F_s \varphi)(y) & \lambda_1 \sqrt{2\pi} (F_s \psi)(y) \\
\lambda_1 \lambda_2 \sqrt{2\pi} (F_s \xi)(y) & 1
\end{vmatrix}
\]
\[
= 1 + 2\sqrt{2\pi}\lambda_1 \sin y(F_s \varphi)(y) - 2\pi \lambda_1 \lambda_2 (F_s \psi)(y)(F_s \xi)(y) \neq 0.
\]
Under the hypothesis and by Wiener - Levy’s Theorem we have
\[
\frac{1}{\Delta} = \frac{1}{1 + 2\sqrt{2\pi}\lambda_1 \sin y(F_s \varphi)(y) - 2\pi \lambda_1 \lambda_2 (F_s \psi)(y)(F_s \xi)(y)}
\]
for some \( l \in L_1(\mathbb{R}_+) \).

On the other hand

\[
\Delta_1 = \left| \begin{array}{c}
(F_s h)(y) \\
(F_c k)(y)
\end{array} \right| = \sqrt{2\pi} \lambda_1(F_s \psi)(y) - \sqrt{2\pi} \lambda_1(F_s \psi)(y)(F_s k)(y).
\]

Therefore

\[
(F_s f)(y) = \frac{\Delta_1}{\Delta} = [(F_s h)(y) - \sqrt{2\pi} \lambda_1(F_s \psi)(y)(F_s k)(y)][1 - (F_s l)(y)]
\]

\[
= (F_s h)(y) - \sqrt{2\pi} \lambda_1(F_s \psi)(y) + (\psi \ast k)(y) - F_s (h \ast l)(y) + \sqrt{2\pi} \lambda_1 F_s ((\psi \ast k) \ast l)(y).
\]

It follows that

\[
f(x) = h(x) - \sqrt{2\pi} \lambda_1(\psi \ast k)(x) - (h \ast l)(x) + \sqrt{2\pi} \lambda_1((\psi \ast k) \ast l)(x).
\]

Similarly,

\[
\Delta_2 = \left| \begin{array}{c}
1 + 2\sqrt{2\pi} \lambda_1 \sin y(F_s \varphi)(y) \\
\sqrt{2\pi} \lambda_2(F_s \xi)(y)
\end{array} \right| = (F_s k)(y) + 2\sqrt{2\pi} \lambda_1 \sin y(F_s \varphi)(y)(F_c k)(y) - \sqrt{2\pi} \lambda_2(F_s \xi)(y)(F_s h)(y).
\]

Hence,

\[
(F_c g)(y) = \frac{\Delta_1}{\Delta} = [(F_c k)(y) + 2\sqrt{2\pi} \lambda_1 \sin y(F_s \varphi)(y)(F_c k)(y)
\]

\[
- \sqrt{2\pi} \lambda_2(F_s \xi)(y)(F_s h)(y)][1 - (F_s l)(y)]
\]

\[
= (F_c k)(y) + 2\sqrt{2\pi} \lambda_1 F_c ((\varphi \ast k) \ast h)(y) - \sqrt{2\pi} \lambda_2 F_c ((\xi \ast h) \ast l)(y)
\]

\[
= F_c (k \ast l)(y) - 2\sqrt{2\pi} \lambda_1 F_c ((\varphi \ast k) \ast h)(y)
\]

\[
+ \sqrt{2\pi} \lambda_2 F_c ((\xi \ast h) \ast l)(y),
\]

consequently,

\[
g(x) = k(x) + 2\sqrt{2\pi} \lambda_1 ((\varphi \ast k)(x) - \sqrt{2\pi} \lambda_2 (\xi \ast h)(x) - (k \ast l)(x)
\]

\[
- 2\sqrt{2\pi} \lambda_1 ((\varphi \ast k) \ast h)(x) + \sqrt{2\pi} \lambda_2 ((\xi \ast h) \ast l)(x).
\]

This completes the proof of the theorem.
4.2. Consider the System of Integral Equations

\[ f(x) + \lambda_1 \int_0^{+\infty} g(y) \theta(x, y) dy = h(x) \]  
\[ \lambda_2 \left\{ \int_0^{+\infty} f(y) \theta(x, y) dy + \int_0^{+\infty} f(y) [\xi(|x-y|) - \xi(x+y)] dy \right\} + g(x) = k(x), \]

in which \( \varphi(x) = (\varphi_1 * \varphi_2)(x) \), \( \lambda_1 \) and \( \lambda_2 \) are complex constants and \( \varphi_1, \varphi_2, \psi, \xi, h, k \in L_1(\mathbb{R}_+) \), \( f \) and \( g \) are unknown functions.

**Theorem 5.** With the condition

\[ 1 - \lambda_1 \lambda_2 \sin^2 y(F_s \varphi)(y)(F_s \psi)(y) - \lambda_1 \lambda_2 \sin y(F_s \varphi)(F_s \xi)(y) \neq 0, \ \forall y > 0, \]

the system (27) has a unique solution in \( L_1(\mathbb{R}_+) \) which has the form

\[ f(x) = h(x) + (h \ast l)(x) - 2\sqrt{2\pi} \lambda_1 \phi \frac{\pi \lambda}{F_s}(k)(x) - 2\sqrt{2\pi} \lambda_1 \phi \frac{\pi \lambda}{F_s}(k \ast l)(x), \]
\[ g(y) = k(x) - 2\sqrt{2\pi} \lambda_2 \phi \frac{\pi \lambda}{F_s}(h \ast l)(x) + \sqrt{2\pi} \lambda_2 ((h \ast l)(x) + (k \ast l)(x)), \]

where \( l \in L_1(\mathbb{R}_+) \) is defined by

\[ (F_s l)(y) = \frac{8\pi \lambda_1 \lambda_2 \sin^2 y(F_s \varphi)(y)(F_s \psi)(y) - 4\pi \lambda_1 \lambda_2 \sin y(F_s \varphi)(y)(F_s \xi)(y)}{1 - 8\pi \lambda_1 \lambda_2 \sin^2 y(F_s \varphi)(y)(F_s \psi)(y) - 4\pi \lambda_1 \lambda_2 \sin y(F_s \varphi)(y)(F_s \xi)(y)}. \]

**Proof.** It is obvious that \( \varphi \) also is a function in the space \( L_1(\mathbb{R}_+) \). Using the factorization properties of convolutions (4), (6) we can rewrite system (27) as follows

\[ (F_s f)(y) + 2\sqrt{2\pi} \lambda_1 \sin y(F_s g)(y)(F_s \varphi)(y) = (F_s h)(y), \]
\[ 2\sqrt{2\pi} \lambda_2 \sin y(F_s f)(y)(F_s \psi)(y) + \sqrt{2\pi} \lambda_2 (F_s f)(y)(F_s \xi)(y) + (F_s g)(y) = (F_s k)(y). \]

We have

\[ \Delta = \begin{vmatrix} 1 & 2\sqrt{2\pi} \lambda_1 \sin y(F_s \varphi)(y) \\ 2\sqrt{2\pi} \lambda_2 \sin y(F_s \psi)(y) + \sqrt{2\pi} \lambda_2 (F_s \xi)(y) & 1 \end{vmatrix} = 1 - 8\pi \lambda_1 \lambda_2 \sin^2 y(F_s \varphi)(y)(F_s \psi)(y) - 4\pi \lambda_1 \lambda_2 \sin y(F_s \varphi)(y)(F_s \xi)(y). \]
From the hypothesis that \( \varphi(x) = (\varphi_1 + \varphi_2)(x) \), and in view of Wiener-Levy’s Theorem we obtain

\[
\frac{1}{\Delta} = \frac{1}{1 - 8\pi \lambda_1 \lambda_2 \sin^2 g(F_s \varphi)(y)(F_s \psi)(y) - 4\pi \lambda_1 \lambda_2 \sin y(F_s \varphi)(y)(F_c \xi)(y)}
\]

\[
= \frac{1}{1 - 8\pi \lambda_1 \lambda_2 \sin^2 g(F_s \varphi_1)(y)(F_c \varphi_2)(y)(F_s \psi)(y) - 4\pi \lambda_1 \lambda_2 \sin y(F_s \varphi)(y)(F_c \xi)(y)}
\]

\[
= \frac{1}{1 - 8\pi \lambda_1 \lambda_2 F_c((\varphi_1 \hat{\varphi} \psi) \hat{\varphi} \varphi)(y) - 4\pi \lambda_1 \lambda_2 F_c(\varphi \xi)(y)}
\]

\[
= 1 + \frac{8\pi \lambda_1 \lambda_2 F_c((\varphi_1 \hat{\varphi} \psi) \hat{\varphi} \varphi)(y) - 4\pi \lambda_1 \lambda_2 F_c(\varphi \xi)(y)}{1 - 8\pi \lambda_1 \lambda_2 F_c((\varphi_1 \hat{\varphi} \psi) \hat{\varphi} \varphi)(y) - 4\pi \lambda_1 \lambda_2 F_c(\varphi \xi)(y)}
\]

\[
= 1 + (F_s, l)(y),
\]

for some \( l \in L_1(\mathbb{R}_+) \).

On the other hand

\[
\Delta_1 = \begin{vmatrix}
(F_s h)(y) & 2\sqrt{2\pi} \lambda_1 \sin y(F_s \varphi)(y) \\
(F_s k)(y) & 1
\end{vmatrix}
\]

\[
= (F_s, h)(y) - 2\sqrt{2\pi} \lambda_1 \sin y(F_s \varphi)(y)(F_s, k)(y)
\]

\[
\Delta_2 = \begin{vmatrix}
(F_s, h)(y) & 1 \\
2\sqrt{2\pi} \lambda_2 \sin y(F_s \psi)(y) + \sqrt{2\pi} \lambda_2 (F_c, \xi)(y) & (F_s, k)(y)
\end{vmatrix}
\]

\[
= (F_s, k)(y) - 2\sqrt{2\pi} \lambda_2 \sin y(F_s \psi)(y)(F_s, h)(y) + \sqrt{2\pi} \lambda_2 (F_c, \xi)(y)(F_s, h)(y)
\]

So the system (29) has a solution defined by

\[
(F_s f)(y) = \Delta_1 = [(F_s, h)(y) - 2\sqrt{2\pi} \lambda_1 \sin y(F_s \varphi)(y)(F_s, k)(y)][1 + (F_s, l)(y)]
\]

\[
= (F_s, h)(y) + F_s((h \hat{\varphi} \psi) \hat{\varphi} \varphi_F)(y) - 2\sqrt{2\pi} \lambda_1 F_s((\varphi \hat{\varphi} \psi) \hat{\varphi} \varphi_F)(y)
\]

\[
- 2\sqrt{2\pi} \lambda_1 F_s((\varphi \hat{\varphi} \psi) \hat{\varphi} \varphi_F) \hat{\varphi} \varphi_F(1 + (F_s, l)(y)).
\]

\[
(F_s g)(y) = \Delta_2 = [(F_s, k)(y) - 2\sqrt{2\pi} \lambda_2 \sin y(F_s \psi)(y)(F_s, h)(y)
\]

\[
+ \sqrt{2\pi} \lambda_2 (F_c, \xi)(y)(F_s, h)(y)][1 + (F_s, l)(y)]
\]

\[
= (F_s, k)(y) - 2\sqrt{2\pi} \lambda_2 F_s((\varphi \hat{\varphi} \psi) \hat{\varphi} \varphi_F)(y) + \sqrt{2\pi} \lambda_2 (h \hat{\varphi} \psi) \hat{\varphi} \varphi_F(1 + (F_s, l)(y)
\]

\[
- 2\sqrt{2\pi} \lambda_2 (\varphi \hat{\varphi} \psi) \hat{\varphi} \varphi_F (h \hat{\varphi} \psi) \hat{\varphi} \varphi_F(1 + (h \hat{\varphi} \psi) \hat{\varphi} \varphi_F(1 + (F_s, l)(y)).
\]
It follows that
\[
\begin{align*}
f(x) &= h(x) + (h \ast l)(x) - 2\sqrt{2\pi}\lambda_1(\varphi_{\frac{\gamma}{2}} k_1)(x) - 2\sqrt{2\pi}\lambda_1(\varphi_{\frac{\gamma}{2}} k_1 \ast l)(x), \\
g(y) &= k(x) - 2\sqrt{2\pi}\lambda_2(\psi_{\frac{\gamma}{2}} h_1)(x) + \sqrt{2\pi}\lambda_2(h_1 \ast \xi)(x) + (k \ast l)(x) \\
&\quad - 2\sqrt{2\pi}\lambda_2((\psi_{\frac{\gamma}{2}} h_1 \ast l)(x) + \sqrt{2\pi}\lambda_2((h_1 \ast \xi) \ast l)(x).
\end{align*}
\]

This completes the proof of the theorem. \[\blacksquare\]

4.3. Consider the Integral Equation of the Form
\[
f(x) + \lambda \left\{ \int_0^{+\infty} f(y)\theta(x,y)\,dy + \int_0^{+\infty} f(y)[\psi(|x-y|) - \psi(x+y)]\,dy \right\} = h(x), \quad (30)
\]
where \(\varphi(x) = (\varphi_1 \ast \varphi_2)(x)\), \(\lambda\) is a complex parameter, \(\varphi_1, \varphi_2, \psi\) and \(h\) are \(L_1(\mathbb{R}_+)\) functions and \(f\) is an unknown function.

**Theorem 6.** Assume that the condition
\[
1 + \lambda(2\sqrt{2\pi}\sin y(F_x \varphi)(y) + \sqrt{2\pi}(F_c \psi)(x)) \neq 0, \quad \forall x > 0,
\]
is satisfied. Then the unique solution of integral equation (30) in \(L_1(\mathbb{R}_+)\) is of the form
\[
f(x) = h(x) - \lambda(h \ast l)(x).
\]

Here \(l \in L_1(\mathbb{R}_+)\) is defined by
\[
(F_c l)(y) = \frac{2\sqrt{2\pi}F_c(\varphi_1 \ast \varphi_2)(y) + \sqrt{2\pi}(F_c \psi)(y)}{1 + \lambda(2\sqrt{2\pi}F_c(\varphi_1 \ast \varphi_2)(y) + \sqrt{2\pi}(F_c \psi)(y)).}
\]

**Proof.** Applying Fourier sine transform to (30) and using the factorization identities of convolutions (4) and (6), one gets
\[
(F_s f)(y) + \lambda(2\sqrt{2\pi}\sin y(F_s \varphi)(y)(F_s \varphi)(y) + \sqrt{2\pi}(F_s f)(y)(F_c \psi)(y)) = (F_s h)(y).
\]
It follows that
\[
(F_s f)(y) = (F_s h)(y) \frac{1}{1 + \lambda(2\sqrt{2\pi}\sin y(F_s \varphi)(y) + \sqrt{2\pi}(F_c \psi)(y))}
\]
\[
= (F_s h)(y) \left( 1 - \frac{2\sqrt{2\pi}\sin y(F_s \varphi)(y)(F_c \varphi_2)(y) + \sqrt{2\pi}(F_c \psi)(y))}{1 + \lambda(2\sqrt{2\pi}\sin y(F_s \varphi)(y)(F_c \varphi_2)(y) + \sqrt{2\pi}(F_c \psi)(y))} \right)
\]
\[
= (F_s h)(y) \left( 1 - \frac{2\sqrt{2\pi}F_c(\varphi_1 \ast \varphi_2)(y) + \sqrt{2\pi}(F_c \psi)(y))}{1 + \lambda(2\sqrt{2\pi}F_c(\varphi_1 \ast \varphi_2)(y) + \sqrt{2\pi}(F_c \psi)(y))} \right).
\]
By virtue of Wiener - Levy’s Theorem, there exist functions $l \in L_1(\mathbb{R}_+)$ such that

$$\frac{2\sqrt{2\pi}F_c(\varphi_1 \frac{\gamma}{2} \varphi_2)(y) + \sqrt{2\pi}(F_c\psi)(y)}{1 + \lambda\left(2\sqrt{2\pi}F_c(\varphi_1 \frac{\gamma}{2} \varphi_2)(y) + \sqrt{2\pi}(F_c\psi)(y)\right)} = (F_c l)(y).$$

Therefore,

$$(F_s f)(y) = (F_s h)(y) \left(1 - \lambda(F_c l)(y)\right).$$

Hence

$$f(x) = h(x) - \lambda(h * l)(x).$$

The theorem is proved.

References