Vietnam Journal of Mathematics **36**:1(2008) 83–101

Vietnam Journal of MATHEMATICS

Integral Transforms Related to the Fourier Sine Convolution with a Weight Function

Nguyen Xuan Thao and Nguyen Thanh Hong

Hanoi Water Resources University, 175 Tay Son, Dong Da, Hanoi, Vietnam

Received May 10, 2007
Revised December 04, 2007

Abstract. Integral transforms of the form

$$
\begin{aligned}
g(x) = \Big(1 - \frac{d^2}{dx^2}\Big)\Big\{ \int_0^{+\infty} & f(y)[\ \text{sign}\ (x+y-1)k_1(|x+y-1|) \\
&+ \ \text{sign}\ (x-y+1)k_1(|x-y+1|) - k_1(x+y+1) \\
&- \ \text{sign}\ (x-y-1)k_1(|x-y-1|)]dy \\
&+ \int_0^{+\infty} f(y)[k_2(|x-y|) - k_2(x+y)]dy\Big\}
\end{aligned}
$$

from $L_p(\mathbb{R}_+)$ to $L_q(\mathbb{R}_+)$, $(1 \leqslant p \leqslant 2,\ p^{-1} + q^{-1} = 1)$ are studied. Watson's and Plancherel's Theorems are obtained. Applications to solving integral equation and systems of integral equations are considered.

2000 Mathematics Subject Classification: 44A05, 44A35.
Keywords: Integral Transform, convolution, Fourier Sine Transform, Plancherel Theorem, integral equation.

1. Introduction

The theory of convolution for integral transforms were studied in the 20^{th} century. At first, the convolution for the Fourier transformation has been studied.

Namely, the convolution of two functions f and g for the Fourier transform has the form

$$(f \underset{F}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y)g(x-y)dy, \quad x \in \mathbb{R}, \tag{1}$$

with the factorization property

$$F(f \underset{F}{*} g)(y) = (Ff)(y)(Fg)(y), \ \ \forall y \in \mathbb{R},$$

where F is the Fourier integral transform

$$(Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixy} f(y)dy.$$

Later on, convolutions for integral transforms Laplace, Mellin, Hilbert, Hankel, Kontorovich - Lebedev and Stieltjes have been introduced and studied. At the same time, integral transforms of the Fourier convolution type, of the Laplace convolution type, of the Mellin convolution type,... have also been constructed and investigated.

In 1941, Churchill introduced the convolution of two functions f and g for the Fourier cosine integral transform defined by the formula below [2]

$$(f \underset{F_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} f(y)[g(x+y) + g(|x-y|)]dy, \ x > 0, \tag{2}$$

for which the following factorization equality holds [8]

$$F_c(f \underset{F_c}{*} g)(y) = (F_c f)(y)(F_c g)(y), \ \forall y > 0. \tag{3}$$

Here, F_c is the Fourier cosine transform [1]

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} \cos(xy) f(y)dy.$$

The first convolution with a weight function was found by Vilenkin in 1958 for the transform Mehler - Fock. In 1967, Kakichev proposed a constructive method for defining the convolution with a weight function for an arbitrary integral transform (see [3]). The convolution of two functions f and g with the weight function $\gamma(y) = \sin y$ for the Fourier sine integral transformation has been studied in [3, 15]

$$\begin{aligned}(f \overset{\gamma}{\underset{F_s}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{0}^{+\infty} f(y)[&\operatorname{sign}\,(x+y-1)\, g(|x+y-1|) \\ &+ \operatorname{sign}\,(x-y+1)\, g(|x-y+1|) - g(x+y+1) \\ &- \operatorname{sign}\,(x-y-1)\, g(|x-y-1|)]dy, \ \ x > 0, \end{aligned} \tag{4}$$

for which the following factorization identity holds ([3, 15])

$$F_s(f \overset{\gamma}{\underset{F_s}{*}} g)(y) = \sin y(F_s f)(y)(F_s g)(y), \quad \forall y > 0, \tag{5}$$

where F_s is the Fourier sine transform ([1])

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} \sin(xy) f(y) dy.$$

In 1941, the first generalized convolution for two integral transforms was introduced by Churchill. Namely, he defined the generalized convolution of two functions f and g for the Fourier sine and cosine transforms [2]

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^{+\infty} f(y)[g(|x-y|) - g(x+y)]dy, \quad x > 0, \tag{6}$$

and proved the following factorization identity [7]

$$F_s(f \underset{1}{*} g)(y) = (F_s f)(y).(F_c g)(y), \quad \forall y > 0. \tag{7}$$

In the nineties of the last century, Yakubovich introduced several generalized convolutions with index for the Mellin transform, Kontorovich-Lebedev transform, G-transform and H-transform. In 1998, Kakichev and Nguyen Xuan Thao proposed a constructive method for defining the generalized convolution for three arbitrary integral transforms (see [4]). Up to now, based on this method, several new generalized convolutions for integral transforms were established and investigated. For instance, the generalized convolution for Stieltjes, Hilbert, Fourier cosine and sine integral transforms have been introduced in [11]; the generalized convolution for the I- transform has been studied in [17]; the generalized convolution with a weight function for the Fourier sine, Kontorovich-Lebedev and the Fourier cosine integral transforms and the generalized convolution for the Kontorovich-Lebedev, Fourier sine and cosine transforms have also been investigated in [18], [19], respectively; the generalized convolution with a weight function for the Fourier sine and cosine transforms were introduced in [14]; the generalized convolution with a weight function for the Fourier, Fourier cosine and sine transforms were found in [12], and so on.

The first generalized convolution which was constructed basing on that method was introduced in 1998 in [5]. Namely, the generalized convolution of two functions f and g for the Fourier cosine and sine transforms has the form

$$(f \underset{2}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^{+\infty} f(y)[\text{ sign } (y-x) g(|y-x|) + g(y+x)]dy, \ \ x > 0, \tag{8}$$

where the following factorization property has been established [5]

$$F_c(f \underset{2}{*} g)(y) = (F_s f)(y)(F_s g)(y), \ \ \forall y > 0. \tag{9}$$

Another generalized convolution with the weight function $\gamma(y) = \sin y$ for the Fourier cosine and Fourier sine transforms has been studied in [16]

$$(f \overset{\gamma}{\underset{2}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(u)[g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) - g(|x-u-1|)]du, \; x > 0. \tag{10}$$

It satisfies the factorization property [16]

$$F_c(f \overset{\gamma}{\underset{2}{*}} g)(y) = \sin y (F_s f)(y)(F_c g)(y), \; \forall y > 0. \tag{11}$$

Recently, in 2000, classes of integral transforms related to the generalized convolutions (6) and (8) was constructed and investigated by Vu Kim Tuan in [6, 7]. In this paper we will consider a new class of integral transforms which is related to the convolution with a weight function for the Fourier sine transform (4) and the generalized convolution for the Fourier sine and cosine transforms (6), namely, the transforms of the form

$$\begin{aligned} g(x) = \Big(1 - \frac{d^2}{dx^2}\Big)\Big\{ \int_0^{+\infty} & f(y)[\text{ sign } (x+y-1)k_1(|x+y-1|) \\ & + \text{ sign } (x-y+1)k_1(|x-y+1|) \\ & - k_1(x+y+1) - \text{ sign } (x-y-1)k_1(|x-y-1|)]dy \\ & + \int_0^{+\infty} f(y)[k_2(|x-y|) - k_2(x+y)]dy\Big\}. \end{aligned} \tag{12}$$

We will show that with certain conditions of k_1 and k_2, transform (12) defines a bounded operator from $L_p(\mathbb{R}_+)$ to $L_q(\mathbb{R}_+)$ $(1 \leqslant p \leqslant 2)$, $p^{-1} + q^{-1} = 1$. Moreover, we will show that with these conditions of k_1 and k_2, the transform (12) is a unitary operator in $L_2(\mathbb{R}_+)$. Watson and Plancherel type Theorem for transform (12) in $L_2(\mathbb{R}_+)$ are also obtained.

2. A Watson Type Theorem

Lemma 1. *Let f and g be $L_2(\mathbb{R}_+)$ functions. Then the following Parseval identity holds*

$$\begin{aligned} \int_0^{+\infty} & f(u)[\text{ sign } (x+u-1)g(|x+u-1|) + \text{ sign } (x-u+1)g(|x-u+1|) \\ & - g(x+u+1) - \text{ sign } (x-u-1)g(|x-u-1|)]du \\ & = 2\sqrt{2\pi} F_s\big(\sin u (F_s f)(u)(F_s g)(u)\big)(x), \quad \forall x > 0. \end{aligned} \tag{13}$$

Proof. Let f_1 and g_1 be the odd extension of f and g from $\mathbb{R}_+$ to $\mathbb{R}$, respectively. Then on $\mathbb{R}_+$ we have $Ff_1 = -iF_sf$ and $Fg_1 = -iF_sg$. Applying the Parseval identity for Fourier transform

$$\int_{-\infty}^{+\infty} f(y)g(x-y)dy = \int_{-\infty}^{+\infty} (Ff)(y)(Fg)(y)e^{ixy}dy,$$

we have

$$\begin{aligned}
&\int_0^{+\infty} f(u)[\text{ sign }(x+u-1)g(|x+u-1|) + \text{ sign }(x-u+1)g(|x-u+1|)\\
&- g(x+u+1) - \text{ sign }(x-u-1)g(|x-u-1|)]du\\
=&\int_0^{+\infty} f_1(u)[g_1(x+u-1) + g_1(x-u+1) - g_1(x+u+1) - g_1(x-u-1)]du\\
=&\int_{-\infty}^{+\infty} f_1(u)g_1(x-u+1)du - \int_{-\infty}^{+\infty} f_1(u)g_1(x-u-1)du\\
=&\int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)e^{i(x+1)u}du - \int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)e^{i(x-1)u}du\\
=&\int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)\{\cos((x+1)u) + i\sin\big((x+1)u\big)\}du\\
&-\int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)\{cos((x-1)u) + i\sin\big((x-1)u\big)\}du.
\end{aligned}$$

On the other hand, note that $(Ff_1)(u)(Fg_1)(u)\sin\big((x+1)u\big)$ and $(Ff_1)(u)(Fg_1)(u)\sin\big((x-1)u\big)$ are odd functions in u. Hence their integrals over $\mathbb{R}$ vanish, and therefore,

$$\begin{aligned}
&\int_0^{+\infty} f(u)[\text{ sign }(x+u-1)g(|x+u-1|) + \text{ sign }(x-u+1)g(|x-u+1|)\\
&- g(x+u+1) - \text{ sign }(x-u-1)g(|x-u-1|)]du\\
=&\int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)\cos\big((x+1)u\big)du - \int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)\cos\big((x-1)u\big)du\\
=&-2\int_{-\infty}^{+\infty} (Ff_1)(u)(Fg_1)(u)\sin u\sin(xu)du
\end{aligned}$$

$$= -4 \int_0^{+\infty} (Ff_1)(u)(Fg_1)(u) \sin u \sin(xu) du$$
$$= 2\sqrt{2\pi} F_s\big(\sin u (F_s f)(u)(F_s g)(u)\big)(x).$$

This completes the proof of the lemma. ■

Theorem 1. *Let $k_1, k_2 \in L_2(\mathbb{R}_+)$. Then*

$$|2\sin y (F_s k_1)(y) + (F_c k_2)(y)| = \frac{1}{\sqrt{2\pi}(1+y^2)} \tag{14}$$

is a necessary and sufficient condition to ensure that the integral transform $f \mapsto g$:

$$\begin{aligned} g(x) = \Big(1 - \frac{d^2}{dx^2}\Big)\Big\{ \int_0^{+\infty} & f(u)[\operatorname{sign}(x+u-1)k_1(|x+u-1|) \\ & + \operatorname{sign}(x-u+1)k_1(|x-u+1|) - k_1(x+u+1) \\ & - \operatorname{sign}(x-u-1)k_1(|x-u-1|)]du \\ & + \int_0^{+\infty} f(u)[k_2(|x-u|) - k_2(x+u)]du \Big\} \end{aligned} \tag{15}$$

is unitary on $L_2(\mathbb{R}_+)$ and the inverse transformation has the form

$$\begin{aligned} f(x) = \Big(1 - \frac{d^2}{dx^2}\Big)\Big\{ \int_0^{+\infty} & \overline{k}_1(u)[\operatorname{sign}(x+u-1)g(|x+u-1|) \\ & + \operatorname{sign}(x-u+1)g(|x-u+1|) - g(x+u+1) \\ & - \operatorname{sign}(x-u-1)g(|x-u-1|)]du \\ & + \int_0^{+\infty} g(u)[\overline{k_2}(|x-u|) - \overline{k_2}(x+u)]du \Big\}. \end{aligned} \tag{16}$$

Proof.

Necessity. Suppose that k_1 and k_2 satisfy condition (14). It is well-known that $h(y)$, $yh(y)$, $y^2h(y)$ belong to $L_2(\mathbb{R}_+)$ if and only if $(Fh)(x)$, $\frac{d}{dx}(Fh)(x)$ and $\frac{d^2}{dx^2}(Fh)(x)$ are also $L_2(\mathbb{R}_+)$ functions (Theorem 68, page 92, [10]). Moreover,

$$\frac{d^2}{dx^2}(Fh)(x) = \frac{1}{\sqrt{2\pi}} \frac{d^2}{dx^2} \int_{-\infty}^{+\infty} h(y) e^{-ixy} dy = F\big((-iy)^2 h(y)\big)(x).$$

In particular, if h is an even or odd function such that $(1+y^2)h(y) \in L_2(\mathbb{R}_+)$, then the following equalities hold

$$\begin{aligned}\Big(1-\frac{d^2}{dx^2}\Big)(F_c h)(x) &= F_c\big((1+y^2)h(y)\big)(x),\\ \Big(1-\frac{d^2}{dx^2}\Big)(F_s h)(x) &= F_s\big((1+y^2)h(y)\big)(x).\end{aligned} \tag{17}$$

Using the Lemma 1 and the factorization equalities for generalized convolutions (6), (8) we have

$$\begin{aligned}g(x) =&\Big(1-\frac{d^2}{dx^2}\Big)F_s\big(2\sqrt{2\pi}\sin y(F_s k_1)(y)(F_s f)(y) + \sqrt{2\pi}(F_s f)(y)(F_c k_2)(y)\big)(x)\\ =&F_s\big(\sqrt{2\pi}(1+y^2)\big(2\sin y(F_s k_1)(y) + (F_c k_2)(y)\big)(F_s f)(y)\big)(x).\end{aligned}$$

By virtue of Parseval equality for the Fourier sine transform $\|f\|_{L_2(\mathbb{R}_+)} = \|F_s f\|_{L_2(\mathbb{R}_+)}$ and note that k_1 and k_2 satisfy condition (14) we have

$$\begin{aligned}\|g\|_{L_2(\mathbb{R}_+)} &= \|2\sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_s f)(y)\|_{L_2(\mathbb{R}_+)}\\ &= \|F_s f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.\end{aligned}$$

It follows that the transformation (15) is unitary.

On the other hand, in view of condition (14), $\sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))$ is bounded, hence $\sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y)+(F_c k_2)(y))(F_s f)(y) \in L_2(\mathbb{R}_+)$. We have

$$(F_s g)(y) = \sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_s f)(y).$$

It follows that

$$(F_s f)(y) = \sqrt{2\pi}(1+y^2)(2\sin y(F_s \overline{k}_1)(y) + (F_c \overline{k}_2)(y))(F_s g)(y).$$

Again, condition (14) of k_1, k_2 yields

$$\sqrt{2\pi}(1+y^2)(2\sin y(F_s \overline{k}_1)(y) + (F_c \overline{k}_2)(y))(F_s g)(y) \in L_2(\mathbb{R}_+).$$

Using formulae (17) we obtain

$$\begin{aligned}f(x) &= F_s\big[\sqrt{2\pi}(1+y^2)(2\sin y(F_s \overline{k}_1)(y) + (F_c \overline{k}_2)(y))(F_s g)(y)\big](x)\\ &=\Big(1-\frac{d^2}{dx^2}\Big)F_s\big(2\sqrt{2\pi}\sin y(F_s \overline{k}_1)(y)(F_s g)(y) + \sqrt{2\pi}(F_s g)(y)(F_c \overline{k}_2)(y)\big)(x)\\ &=\Big(1-\frac{d^2}{dx^2}\Big)\Big\{\int_0^{+\infty} \overline{k}_1(y)[\mathrm{sign}(x+y-1)g(|x+y-1|)\\ &\quad + \mathrm{sign}(x-y+1)g(|x-y+1|) - g(x+y+1)\\ &\quad - \mathrm{sign}(x-y-1)g(|x-y-1|)]dy\\ &\quad + \int_0^{+\infty} g(y)[\overline{k_2}(|x-y|) - \overline{k_2}(x+y)]dy\Big\}.\end{aligned}$$

Therefore the transformation (15) is unitary on $L_2(\mathbb{R}_+)$ and the inverse transformation has the form (16).

Sufficiency. Suppose that transform (15) is unitary on $L_2(\mathbb{R}_+)$ with the inverse transformation defined by (16). Then Parseval identity for the Fourier sine transform yields

$$\begin{aligned}\|g\|_{L_2(\mathbb{R}_+)} &= \|\sqrt{2\pi}(1+y^2)(2\sin y(F_sk_1)(y) + (F_ck_2)(y))(F_sf)(y)\|_{L_2(\mathbb{R}_+)}\\ &= \|F_sf\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.\end{aligned}$$

However, the middle equality holds for all $f \in L_2(\mathbb{R}_+)$ if and only if

$$|\sqrt{2\pi}(1+y^2)(2\sin y(F_sk_1)(y) + (F_ck_2)(y))(F_sf)(y)| = |(F_sf)(y)|.$$

It shows that k_1 and k_2 satisfy condition (14). The proof of the theorem has been completed. ∎

Let $h_1, h_2 \in L_2(\mathbb{R}_+)$ satisfy the following condition

$$|(F_sh_1)(y)(F_sh_2)(y)| = \frac{1}{(1+y^2)(1+\sin^2 y)}\ . \tag{18}$$

For example, consider

$$h_1(x) = F_s\Big(\frac{e^{iu(y)}}{\sqrt{(1+y^2)(1+\sin^2 y)}}\Big)(x),$$

$$h_2(x) = F_s\Big(\frac{e^{iv(y)}}{\sqrt{(1+y^2)(1+\sin^2 y)}}\Big)(x),$$

where u, v are some functions defined on $\mathbb{R}_+$.

Let k_1, k_2 be defined by

$$k_1(x) = \frac{1}{2\sqrt{2\pi}}(h_1 \overset{\gamma}{\underset{F_s}{*}} h_2)(x), \quad k_2(x) = \frac{1}{\sqrt{2\pi}}(h_1 \underset{2}{*} h_2)(x).$$

Then $k_1, k_2 \in L_2(\mathbb{R}_+)$ and from (5) and (9) we have

$$\begin{aligned}&\Big|2\sin y(F_sk_1)(y) + (F_ck_2)(y)\Big|\\ &= \Big|\frac{1}{\sqrt{2\pi}}\sin^2 y(F_sh_1)(y)(F_sh_2)(y) + \frac{1}{\sqrt{2\pi}}(F_sh_1)(y)(F_sh_2)(y)\Big|\\ &= \Big|\frac{1}{\sqrt{2\pi}}(1+\sin^2 y)(F_sh_1)(y)(F_sh_2)(y)\Big|\\ &= \frac{1}{\sqrt{2\pi}(1+y^2)}.\end{aligned}$$

Thus k_1 and k_2 satisfy condition (14).

For another example, let h_1 and h_2 be $L_2(\mathbb{R}_+)$ functions and satisfy the condition

$$|(F_c h_1)(y)(F_c h_2)(y)| = \frac{1}{(1+y^2)(1+\sin^2 y)} \tag{19}$$

and let k_1, k_2 be defined by

$$k_1(x) = \frac{1}{2\sqrt{2\pi}}(h_1 \overset{\gamma}{\underset{3}{*}} h_2)(x), \quad k_2(x) = \frac{1}{\sqrt{2\pi}}(h_1 \underset{F_c}{*} h_2)(x).$$

Then $k_1, k_2 \in L_2(\mathbb{R}_+)$ and we have

$$\begin{aligned}
&\Big|2\sin y(F_s k_1)(y) + (F_c k_2)(y)\Big| \\
&= \Big|\frac{1}{\sqrt{2\pi}}\sin^2 y(F_c h_1)(y)(F_c h_2)(y) + \frac{1}{\sqrt{2\pi}}(F_c h_1)(y)(F_c h_2)(y)\Big| \\
&= \Big|\frac{1}{\sqrt{2\pi}}(1+\sin^2 y)(F_c h_1)(y)(F_c h_2)(y)\Big| \\
&= \frac{1}{\sqrt{2\pi}(1+y^2)}.
\end{aligned}$$

Thus k_1 and k_2 satisfy condition (14).

3. A Plancherel Type Theorem

Theorem 2. *Let k_1, k_2 be functions satisfying condition* (14) *and suppose that $K_1(x) = (1 - \frac{d^2}{dx^2})k_1(x)$ and $K_2(x) = (1 - \frac{d^2}{dx^2})k_2(x)$ are locally bounded. Let $f \in L_2(\mathbb{R}_+)$ and for each positive integer N, put*

$$\begin{aligned}
g_N(x) = \Big\{ \int_0^N & f(u)[\,\text{sign }(x+u-1)K_1(|x+u-1|) \\
&+ sign(x-u+1)K_1(|x-u+1|) - K_1(x+u+1) \\
&- \text{ sign }(x-u-1)K_1(|x-u-1|)]du \\
&+ \int_0^N f(u)[K_2(|x-u|) - K_2(x+u)]du\Big\}.
\end{aligned} \tag{20}$$

Then

1) *$g_N \in L_2(\mathbb{R}_+)$ and as $N \to +\infty, g_N$ converges in $L_2(\mathbb{R}_+)$-norm to a function g, furthermore $\|g\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$.*

2) *Put* $g^N = g.\chi_{(0,N)}$, *then*

$$\begin{aligned} f_N(x) = \Big\{ \int_0^{+\infty} \overline{K}_1(u)[\ \text{sign}\ (x+u-1)g^N(|x+u-1|) \\ + \ \text{sign}\ (x-u+1)g^N(|x-u+1|) - g^N(x+u+1) \\ - \ \text{sign}\ (x-u-1)g^N(|x-u-1|)]du \\ + \int_0^N g(u)[\overline{k_2}(|x-u|) - \overline{k_2}(x+u)]du \Big\} \end{aligned} \tag{21}$$

belongs to $L_2(\mathbb{R}_+)$ *and converges in* $L_2(\mathbb{R}_+)$*-norm to* f *as* $N \to +\infty$.

Remark 1. The integrals defining f_N and g_N are defined over finite intervals and therefore converge.

Proof. Put $f^N = f.\chi_{(0,N)}$ then

$$\begin{aligned} g_N(x) = \int_0^N f(u)[\ \text{sign}\ (x+u-1)K_1(|x+u-1|) \\ + \ \text{sign}\ (x-u+1)K_1(|x-u+1|) \\ - K_1(x+u+1) - \ \text{sign}\ (x-u-1)K_1(|x-u-1|)]du \\ + \int_0^N f(u)[K_2(|x-u|) - K_2(x+u)]du \end{aligned}$$

$$\begin{aligned} = \Big(1 - \frac{d^2}{dx^2}\Big) \int_0^{+\infty} f^N(u)[\ \text{sign}\ (x+u-1)k_1(|x+u-1|) \\ + \ \text{sign}\ (x-u+1)k_1(|x-u+1|) - k_1(x+u+1) \\ - \ \text{sign}\ (x-u-1)k_1(|x-u-1|)]du \\ + \int_0^{+\infty} f^N(u)[k_2(|x-u|) - k_2(x+u)]du. \end{aligned}$$

Interchanging the order of integration and differentiation here is legitimate since the integral was over a finite interval. Theorem 1 guarantees that $g_N \in L_2(\mathbb{R}_+)$. Moreover, if g is the image of f under the transform (15), we obtain that

$\|g\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$ and the reciprocal formula (16) holds. We have

$$(g - g_N)(x) = \Big(1 - \frac{d^2}{dx^2}\Big) \int\limits_0^{+\infty} (f - f^N)(u)[\operatorname{sign}(x+u-1)k_1(|x+u-1|)$$
$$+ \operatorname{sign}(x-u+1)k_1(|x-u+1|) - k_1(x+u+1)$$
$$- \operatorname{sign}(x-u-1)k_1(|x-u-1|)]du$$
$$+ \int\limits_0^{+\infty} (f - f^N)(u)[k_2(|x-u|) - k_2(x+u)]du,$$

so again by Theorem 1, $g - g_N \in L_2(\mathbb{R}_+)$ and

$$\|g - g_N\|_{L_2(\mathbb{R}_+)} = \|f - f^N\|_{L_2(\mathbb{R}_+)},$$

and since $\|f - f^N\|_{L_2(\mathbb{R}_+)} \to 0$ as $N \to +\infty$, it follows that g_N converges in $L_2(\mathbb{R}_+)$-norm to g as $N \to +\infty$. This completes the first part of the theorem.

Note that the convolution of two functions f, g with the weight function $\gamma(y) = \sin y$ for the Fourier sine transform is commutative [15], hence the following identity holds

$$\int\limits_0^{+\infty} \overline{K}_1(u)[\operatorname{sign}(x+u-1)g^N(|x+u-1|) + \operatorname{sign}(x-u+1)g^N(|x-u+1|)$$
$$- g^N(x+u+1) - \operatorname{sign}(x-u-1)g^N(|x-u-1|)]du$$
$$= \int\limits_0^{+\infty} g^N(u)[\operatorname{sign}(x+u-1)\overline{K}_1(|x+u-1|) + \operatorname{sign}(x-u+1)\overline{K}_1(|x-u+1|)$$
$$- \overline{K}_1(x+u+1) - \operatorname{sign}(x-u-1)\overline{K}_1(|x-u-1|)]du.$$

Therefore, in a similar way, one can obtain the second part of the theorem. ■

Remark 2. Theorem 1 shows that transformation (15) is unitary in $L_2(\mathbb{R}_+)$ and the inverse transformation has the form (16). Furthermore, Theorem 2 proved that these transformations (15) and (16) can be approximated in $L_2(\mathbb{R}_+)$-norm by the integral operators (20) and (21), respectively.

We assume additionally now that $K_1(x)$ and $K_2(x)$ are bounded on $\mathbb{R}_+$. Then transform (15) is a bounded operator from the space $L_1(\mathbb{R}_+)$ into the space $L_\infty(\mathbb{R}_+)$. Moreover, by Theorem 2 transform (15) is bounded on $L_2(\mathbb{R}_+)$. Hence, Riesz interpolation theorem [9] yields the following.

Theorem 3. *Let k_1, k_2 be functions satisfying condition* (14) *and suppose that $K_1(x)$ and $K_2(x)$ are bounded on $\mathbb{R}_+$. Let $1 \leqslant p \leqslant 2$ and q be its conjugate* *exponent* $\frac{1}{p}+\frac{1}{q}=1$*. Then the transformations*

$$\begin{aligned}
g(x) = \lim_{N\to+\infty} \Big\{ \int\limits_0^N & f(u)[\ \mathit{sign}\ (x+u-1)K_1(|x+u-1|) \\
&+ \mathit{sign}\ (x-u+1)K_1(|x-u+1|) \\
&- K_1(x+u+1) - \mathit{sign}\ (x-u-1)K_1(|x-u-1|)]du \\
&+ \int\limits_0^N f(u)[K_2(|x-u|) - K_2(x+u)]du \Big\}
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
g(x) = \lim_{N\to+\infty} \Big\{ \int\limits_0^{+\infty} & \overline{K}_1(u)[\ \mathit{sign}\ (x+u-1)f^N(|x+u-1|) \\
&+ \mathit{sign}\ (x-u+1)f^N(|x-u+1|) - f^N(x+u+1) \\
&- \mathit{sign}\ (x-u-1)f^N(|x-u-1|)]du \\
&+ \int\limits_0^N f(u)[\overline{K_2}(|x-u|) - \overline{K_2}(x+u)]du \Big\}
\end{aligned} \tag{23}$$

are bounded operators from $L_p(\mathbb{R}_+)$ *into* $L_q(\mathbb{R}_+)$*, here the limits are understood in* $L_q(\mathbb{R}_+)$*-norm.*

4. Applications to Integral Equation and Systems of Integral Equations

4.1. Consider the System of Intergral Equations

$$\begin{aligned}
&f(x) + \lambda_1 \Big\{ \int\limits_0^{+\infty} f(y)\theta(x,y)dy + \int\limits_0^{+\infty} \psi(y)[g(|x-y|) - g(x+y)]dy \Big\} = h(x), \\
&\lambda_2 \int\limits_0^{+\infty} f(y)[\ \mathrm{sign}\ (y-x)\xi(|y-x|) + \xi(y+x)]dy + g(x) = k(x).
\end{aligned} \tag{24}$$

Here and throughout this section we will denote by $\theta(x,y)$ the following function

$$\begin{aligned}
\theta(x,y) =& \ \mathrm{sign}\ (x+y-1)\varphi(|x+y-1|) + \ \mathrm{sign}\ (x-y+1)\varphi(|x-y+1|) \\
&- \varphi(x+y+1) - \ \mathrm{sign}\ (x-u-1)\varphi(|x-u-1|), \\
\varphi(x) =& (\varphi_1 \underset{1}{*} \varphi_2)(x).
\end{aligned}$$

λ_1 and λ_2 are complex constants and $\varphi_1, \varphi_2, \psi, \xi, h, k$ are functions from $L_1(\mathbb{R}_+)$, f and g are unknown functions.

Theorem 4. *With the condition*

$$1 + 2\sqrt{2\pi}\lambda_1 \sin y(F_s\varphi)(y) - 2\pi\lambda_1\lambda_2(F_s\psi)(y)(F_s\xi)(y) \neq 0, \quad \forall y > 0$$

there exists a unique solution in $L_1(\mathbb{R}_+)$ *of system* (24) *which has the form*

$$\begin{aligned} f(x) =& h(x) - \lambda_1\sqrt{2\pi}(\psi \underset{1}{*} k)(x) - (h \underset{1}{*} l)(x) + \lambda_1\sqrt{2\pi}\big((\psi \underset{1}{*} k) \underset{1}{*} l\big)(x), \\ g(x) =& k(x) + \lambda_1 2\sqrt{2\pi}(\varphi \overset{\gamma}{\underset{2}{*}} k)(x) - \lambda_2\sqrt{2\pi}(\xi \underset{2}{*} h)(x) - (k \underset{F_c}{*} l)(x) \\ & - \lambda_1 2\sqrt{2\pi}\big((\varphi \overset{\gamma}{\underset{2}{*}} k) \underset{F_c}{*} l\big)(x) + \lambda_2\sqrt{2\pi}\big((\xi \underset{2}{*} h) \underset{F_c}{*} l\big)(x), \end{aligned} \tag{25}$$

where $l \in L_1(\mathbb{R}_+)$ *is defined by*

$$(F_c l)(y) = \frac{\lambda_1 2\sqrt{2\pi}\sin y(F_s\varphi)(y) - \lambda_1\lambda_2\sqrt{2\pi}(F_s\xi)(y)(F_s\psi)(y)}{1 + \lambda_1 2\sqrt{2\pi}\sin y(F_s\varphi)(y) - \lambda_1\lambda_2\sqrt{2\pi}(F_s\xi)(y)(F_s\psi)(y)}.$$

Remark 3. By the condition that φ_1 and φ_2 be $L_1(\mathbb{R}_+)$ functions, it is clear that φ belongs to $L_1(\mathbb{R}_+)$.

Proof. In view of the factorization properties of convolutions (4), (6) and (8), one can rewrite the system (24) in the form

$$\begin{aligned} &(F_s f)(y) + \lambda_1.2\sqrt{2\pi}\sin y(F_s f)(y)(F_s\varphi)(y) + \lambda_1\sqrt{2\pi}(F_s\psi)(y)(F_c g)(y) = (F_s h)(y) \\ &\lambda_2\sqrt{2\pi}(F_s f)(y)(F_s\xi)(y) + (F_c g)(y) = (F_c k)(y). \end{aligned} \tag{26}$$

Accordingly, we have

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 + \lambda_1 2\sqrt{2\pi}\sin y(F_s\varphi)(y) & \lambda_1\sqrt{2\pi}(F_s\psi)(y) \\ \lambda_2\sqrt{2\pi}(F_s\xi)(y) & 1 \end{vmatrix} \\ &= 1 + 2\sqrt{2\pi}\lambda_1\sin y(F_s\varphi)(y) - 2\pi\lambda_1\lambda_2(F_s\psi)(y)(F_s\xi)(y) \neq 0. \end{aligned}$$

Under the hypothesis and by Wiener - Levy's Theorem we have

$$\begin{aligned} \frac{1}{\Delta} =& \frac{1}{1 + 2\sqrt{2\pi}\lambda_1\sin y(F_s\varphi)(y) - 2\pi\lambda_1\lambda_2(F_s\psi)(y)(F_s\xi)(y)} \\ =& 1 - \frac{2\sqrt{2\pi}\lambda_1\sin y(F_s\varphi)(y) - 2\pi\lambda_1\lambda_2(F_s\psi)(y)(F_s\xi)(y)}{1 + 2\sqrt{2\pi}\lambda_1\sin y(F_s\varphi)(y) - 2\pi\lambda_1\lambda_2(F_s\psi)(y)(F_s\xi)(y)} \\ =& 1 - \frac{2\sqrt{2\pi}\lambda_1\sin y(F_s\varphi_1)(y)(F_c\varphi_2)(y) - 2\pi\lambda_1\lambda_2(F_s\psi)(y)(F_s\xi)(y)}{1 + 2\sqrt{2\pi}\lambda_1\sin y(F_s\varphi_1)(y)(F_c\varphi_2)(y) - 2\pi\lambda_1\lambda_2(F_s\psi)(y)(F_s\xi)(y)} \\ =& 1 - \frac{2\sqrt{2\pi}\lambda_1 F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} \varphi_2)(y) - 2\pi\lambda_1\lambda_2 F_c(\psi \underset{2}{*} \xi)(y)}{1 + 2\sqrt{2\pi}\lambda_1 F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} \varphi_2)(y) - 2\pi\lambda_1\lambda_2 F_c(\psi \underset{2}{*} \xi)(y)} \\ =& 1 - (F_c l)(y), \end{aligned}$$

for some $l \in L_1(\mathbb{R}_+)$.

On the other hand

$$\Delta_1 = \begin{vmatrix} (F_s h)(y) & \sqrt{2\pi}\lambda_1 (F_s\psi)(y) \\ (F_c k)(y) & 1 \end{vmatrix} = (F_s h)(y) - \sqrt{2\pi}\lambda_1 (F_s\psi)(y)(F_c k)(y).$$

Therefore

$$\begin{aligned}(F_s f)(y) = \frac{\Delta_1}{\Delta} =& [(F_s h)(y) - \sqrt{2\pi}\lambda_1 (F_s\psi)(y)(F_c k)(y)][1 - (F_c l)(y)] \\ =& (F_s h)(y) - \sqrt{2\pi}\lambda_1 F_s(\psi \underset{1}{*} k)(y) \\ & - F_s(h \underset{1}{*} l)(y) + \sqrt{2\pi}\lambda_1 F_s((\psi \underset{1}{*} k) \underset{1}{*} l)(y).\end{aligned}$$

It follows that

$$f(x) = h(x) - \sqrt{2\pi}\lambda_1 (\psi \underset{1}{*} k)(x) - (h \underset{1}{*} l)(x) + \sqrt{2\pi}\lambda_1 ((\psi \underset{1}{*} k) \underset{1}{*} l)(x).$$

Similarly,

$$\begin{aligned}\Delta_2 =& \begin{vmatrix} 1 + 2\sqrt{2\pi}\lambda_1 \sin y (F_s\varphi)(y) & (F_s h)(y) \\ \sqrt{2\pi}\lambda_2 (F_s\xi)(y) & (F_c k)(y) \end{vmatrix} \\ =& \ (F_c k)(y) + 2\sqrt{2\pi}\lambda_1 \sin y (F_s\varphi)(y)(F_c k)(y) - \sqrt{2\pi}\lambda_2 (F_s\xi)(y)(F_s h)(y).\end{aligned}$$

Hence,

$$\begin{aligned}(F_c g)(y) = \frac{\Delta_1}{\Delta} =& \big[(F_c k)(y) + 2\sqrt{2\pi}\lambda_1 \sin y (F_s\varphi)(y)(F_c k)(y) \\ & - \sqrt{2\pi}\lambda_2 (F_s\xi)(y)(F_s h)(y)][1 - (F_c l)(y)\big] \\ =& (F_c k)(y) + 2\sqrt{2\pi}\lambda_1 F_c(\varphi \overset{\gamma}{\underset{2}{*}} k)(y) - \sqrt{2\pi}\lambda_2 F_c(\xi \underset{2}{*} h)(y) \\ & - F_c(k \underset{F_c}{*} l)(y) - 2\sqrt{2\pi}\lambda_1 F_c((\varphi \overset{\gamma}{\underset{2}{*}} k) \underset{F_c}{*} h)(y) \\ & + \sqrt{2\pi}\lambda_2 F_c((\xi \underset{2}{*} h) \underset{F_c}{*} l)(y),\end{aligned}$$

consequently,

$$\begin{aligned}g(x) =& k(x) + 2\sqrt{2\pi}\lambda_1 (\varphi \overset{\gamma}{\underset{2}{*}} k)(x) - \sqrt{2\pi}\lambda_2 (\xi \underset{2}{*} h)(x) - (k \underset{F_c}{*} l)(x) \\ & - 2\sqrt{2\pi}\lambda_1 ((\varphi \overset{\gamma}{\underset{2}{*}} k) \underset{F_c}{*} h)(x) + \sqrt{2\pi}\lambda_2 ((\xi \underset{2}{*} h) \underset{F_c}{*} l)(x).\end{aligned}$$

This completes the proof of the theorem. ■

4.2. Consider the System of Integral Equations

$$
\begin{aligned}
&f(x) + \lambda_1 \int\limits_0^{+\infty} g(y)\theta(x,y)dy = h(x) \\
&\lambda_2 \Big\{ \int\limits_0^{+\infty} f(y)\theta(x,y)dy + \int\limits_0^{+\infty} f(y)[\xi(|x-y| - \xi(x+y)]dy \Big\} + g(x) = k(x),
\end{aligned} \tag{27}
$$

in which $\varphi(x) = (\varphi_1 \underset{1}{*} \varphi_2)(x)$, λ_1 and λ_2 are complex constants and $\varphi_1, \varphi_2, \psi, \xi, h, k \in L_1(\mathbb{R}_+)$, f and g are unknown functions.

Theorem 5. *With the condition*

$$1 - \lambda_1\lambda_2 \sin^2 y (F_s\varphi)(y)(F_s\psi)(y) - \lambda_1\lambda_2 \sin y (F_s\varphi)(y)(F_c\xi)(y) \neq 0, \quad \forall y > 0,$$

the system (27) *has a unique solution in* $L_1(\mathbb{R}_+)$ *which has the form*

$$
\begin{aligned}
f(x) =& h(x) + (h \underset{1}{*} l)(x) - 2\sqrt{2\pi}\lambda_1(\varphi \overset{\gamma}{\underset{F_s}{*}} k)(x) - 2\sqrt{2\pi}\lambda_1((\varphi \overset{\gamma}{\underset{F_s}{*}} k) \underset{1}{*} l)(x), \\
g(y) =& k(x) - 2\sqrt{2\pi}\lambda_2(\psi \overset{\gamma}{\underset{F_s}{*}} h)(x) + \sqrt{2\pi}\lambda_2(h \underset{1}{*} \xi)(x) + (k \underset{1}{*} l)(x) \\
& - 2\sqrt{2\pi}\lambda_2((\psi \overset{\gamma}{\underset{F_s}{*}} h) \underset{1}{*} l)(x) + \sqrt{2\pi}\lambda_2((h \underset{1}{*} \xi) \underset{1}{*} l)(x),
\end{aligned} \tag{28}
$$

where $l \in L_1(\mathbb{R}_+)$ *is defined by*

$$(F_c l)(y) = \frac{8\pi\lambda_1\lambda_2 \sin^2 y (F_s\varphi)(y)(F_s\psi)(y) - 4\pi\lambda_1\lambda_2 \sin y (F_s\varphi)(y)(F_c\xi)(y)}{1 - 8\pi\lambda_1\lambda_2 \sin^2 y (F_s\varphi)(y)(F_s\psi)(y) - 4\pi\lambda_1\lambda_2 \sin y (F_s\varphi)(y)(F_c\xi)(y)}.$$

Proof. It is obvious that φ also is a function in the space $L_1(\mathbb{R}_+)$. Using the factorization properties of convolutions (4), (6) we can rewrite system (27) as follows

$$
\begin{aligned}
&(F_s f)(y) + 2\sqrt{2\pi}\lambda_1 \sin y (F_s g)(y)(F_s\varphi)(y) = (F_s h)(y), \\
&2\sqrt{2\pi}\lambda_2 \sin y (F_s f)(y)(F_s\psi)(y) + \sqrt{2\pi}\lambda_2 (F_s f)(y)(F_c\xi)(y) + (F_s g)(y) = (F_s k)(y).
\end{aligned} \tag{29}
$$

We have

$$
\begin{aligned}
\Delta &= \begin{vmatrix} 1 & 2\sqrt{2\pi}\lambda_1 \sin y (F_s\varphi)(y) \\ 2\sqrt{2\pi}\lambda_2 \sin y (F_s\psi)(y) + \sqrt{2\pi}\lambda_2 (F_c\xi)(y) & 1 \end{vmatrix} \\
&= 1 - 8\pi\lambda_1\lambda_2 \sin^2 y (F_s\varphi)(y)(F_s\psi)(y) - 4\pi\lambda_1\lambda_2 \sin y (F_s\varphi)(y)(F_c\xi)(y).
\end{aligned}
$$

From the hypothesis that $\varphi(x) = (\varphi_1 \underset{1}{*} \varphi_2)(x)$, and in view of Wiener-Levy's Theorem we obtain

$$\begin{aligned}
\frac{1}{\Delta} =& \frac{1}{1 - 8\pi\lambda_1\lambda_2 \sin^2 y(F_s\varphi)(y)(F_s\psi)(y) - 4\pi\lambda_1\lambda_2 \sin y(F_s\varphi)(y)(F_c\xi)(y)} \\
=& \frac{1}{1 - 8\pi\lambda_1\lambda_2 \sin^2 y(F_s\varphi_1)(y)(F_c\varphi_2)(y)(F_s\psi)(y) - 4\pi\lambda_1\lambda_2 \sin y(F_s\varphi)(y)(F_c\xi)(y)} \\
=& \frac{1}{1 - 8\pi\lambda_1\lambda_2 F_c\big((\varphi_1 \overset{\gamma}{\underset{F_s}{*}} \psi) \overset{\gamma}{\underset{2}{*}} \varphi_2\big)(y) - 4\pi\lambda_1\lambda_2 F_c\big(\varphi \overset{\gamma}{\underset{2}{*}} \xi\big)(y)} \\
=& 1 + \frac{8\pi\lambda_1\lambda_2 F_c\big((\varphi_1 \overset{\gamma}{\underset{F_s}{*}} \psi) \overset{\gamma}{\underset{2}{*}} \varphi_2\big)(y) - 4\pi\lambda_1\lambda_2 F_c\big(\varphi \overset{\gamma}{\underset{2}{*}} \xi\big)(y)}{1 - 8\pi\lambda_1\lambda_2 F_c\big((\varphi_1 \overset{\gamma}{\underset{F_s}{*}} \psi) \overset{\gamma}{\underset{2}{*}} \varphi_2\big)(y) - 4\pi\lambda_1\lambda_2 F_c\big(\varphi \overset{\gamma}{\underset{2}{*}} \xi\big)(y)} \\
=& 1 + (F_c l)(y),
\end{aligned}$$

for some $l \in L_1(\mathbb{R}_+)$.

On the other hand

$$\begin{aligned}
\Delta_1 &= \begin{vmatrix} (F_s h)(y) & 2\sqrt{2\pi}\lambda_1 \sin y(F_s\varphi)(y) \\ (F_s k)(y) & 1 \end{vmatrix} \\
&= (F_s h)(y) - 2\sqrt{2\pi}\lambda_1 \sin y(F_s\varphi)(y)(F_s k)(y)
\end{aligned}$$

$$\begin{aligned}
\Delta_2 &= \begin{vmatrix} 1 & (F_s h)(y) \\ 2\sqrt{2\pi}\lambda_2 \sin y(F_s\psi)(y) + \sqrt{2\pi}\lambda_2(F_c\xi)(y) & (F_s k)(y) \end{vmatrix} \\
&= (F_s k)(y) - 2\sqrt{2\pi}\lambda_2 \sin y(F_s\psi)(y)(F_s h)(y) + \sqrt{2\pi}\lambda_2(F_c\xi)(y)(F_s h)(y)
\end{aligned}$$

So the system (29) has a solution defined by

$$\begin{aligned}
(F_s f)(y) =& \frac{\Delta_1}{\Delta} = [(F_s h)(y) - 2\sqrt{2\pi}\lambda_1 \sin y(F_s\varphi)(y)(F_s k)(y)][1 + (F_c l)(y)] \\
=& (F_s h)(y) + F_s(h \underset{1}{*} l)(y) - 2\sqrt{2\pi}\lambda_1 F_s(\varphi \overset{\gamma}{\underset{F_s}{*}} k)(y) \\
& - 2\sqrt{2\pi}\lambda_1 F_s((\varphi \overset{\gamma}{\underset{F_s}{*}} k) \underset{1}{*} l)(y), \\
(F_s g)(y) =& \frac{\Delta_2}{\Delta} = [(F_s k)(y) - 2\sqrt{2\pi}\lambda_2 \sin y(F_s\psi)(y)(F_s h)(y) \\
& + \sqrt{2\pi}\lambda_2(F_c\xi)(y)(F_s h)(y)][1 + (F_c l)(y)] \\
=& (F_s k)(y) - 2\sqrt{2\pi}\lambda_2 F_s(\psi \overset{\gamma}{\underset{F_s}{*}} h)(y) + \sqrt{2\pi}\lambda_2 F_s(h \underset{1}{*} \xi)(y) + F_s(k \underset{1}{*} l)(y) \\
& - 2\sqrt{2\pi}\lambda_2 F_s((\psi \overset{\gamma}{\underset{F_s}{*}} h) \underset{1}{*} l)(y) + \sqrt{2\pi}\lambda_2 F_s((h \underset{1}{*} \xi) \underset{1}{*} l)(y).
\end{aligned}$$

It follows that

$$\begin{aligned}
f(x) =& h(x) + (h \underset{1}{*} l)(x) - 2\sqrt{2\pi}\lambda_1(\varphi \overset{\gamma}{\underset{F_s}{*}} k)(x) - 2\sqrt{2\pi}\lambda_1((\varphi \overset{\gamma}{\underset{F_s}{*}} k) \underset{1}{*} l)(x),\\
g(y) =& k(x) - 2\sqrt{2\pi}\lambda_2(\psi \overset{\gamma}{\underset{F_s}{*}} h)(x) + \sqrt{2\pi}\lambda_2(h \underset{1}{*} \xi)(x) + (k \underset{1}{*} l)(x)\\
& - 2\sqrt{2\pi}\lambda_2((\psi \overset{\gamma}{\underset{F_s}{*}} h) \underset{1}{*} l)(x) + \sqrt{2\pi}\lambda_2((h \underset{1}{*} \xi) \underset{1}{*} l)(x).
\end{aligned}$$

This completes the proof of the theorem. ■

4.3. Consider the Integral Equation of the Form

$$f(x) + \lambda\Big\{ \int\limits_0^{+\infty} f(y)\theta(x,y)dy + \int\limits_0^{+\infty} f(y)[\psi(|x-y| - \psi(x+y)]dy\Big\} = h(x), \quad (30)$$

where $\varphi(x) = (\varphi_1 \underset{1}{*} \varphi_2)(x)$, λ is a complex parameter, $\varphi_1, \varphi_2, \psi$ and h are $L_1(\mathbb{R}_+)$ functions and f is an unknown function.

Theorem 6. *Assume that the condition*

$$1 + \lambda\big(2\sqrt{2\pi}\sin y(F_s\varphi)(y) + \sqrt{2\pi}(F_c\psi)(x)\big) \neq 0, \quad \forall x > 0,$$

is satisfied. Then the unique solution of integral equation (30) *in* $L_1(\mathbb{R}_+)$ *is of the form*

$$f(x) = h(x) - \lambda(h \underset{1}{*} l)(x).$$

Here $l \in L_1(\mathbb{R}_+)$ *is defined by*

$$(F_c l)(y) = \frac{2\sqrt{2\pi}F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} \varphi_2)(y) + \sqrt{2\pi}(F_c\psi)(y)}{1 + \lambda\big(2\sqrt{2\pi}F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} \varphi_2)(y) + \sqrt{2\pi}(F_c\psi)(y)\big)}.$$

Proof. Applying Fourier sine transform to (30) and using the factorization identities of convolutions (4) and (6), one gets

$$(F_s f)(y) + \lambda\big(2\sqrt{2\pi}\sin y(F_s f)(y)(F_s\varphi)(y) + \sqrt{2\pi}(F_s f)(y)(F_c\psi)(y)\big) = (F_s h)(y).$$

It follows that

$$\begin{aligned}
(F_s f)(y) =& (F_s h)(y)\frac{1}{1 + \lambda\big(2\sqrt{2\pi}\sin y(F_s\varphi)(y) + \sqrt{2\pi}(F_c\psi)(y)\big)}\\
=& (F_s h)(y)\Big(1 - \lambda\frac{2\sqrt{2\pi}\sin y(F_s\varphi_1)(y)(F_c\varphi_2)(y) + \sqrt{2\pi}(F_c\psi)(y)}{1 + \lambda\big(2\sqrt{2\pi}\sin y(F_s\varphi_1)(y)(F_c\varphi_2)(y) + \sqrt{2\pi}(F_c\psi)(y)\big)}\Big)\\
=& (F_s h)(y)\Big(1 - \lambda\frac{2\sqrt{2\pi}F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} \varphi_2)(y) + \sqrt{2\pi}(F_c\psi)(y)}{1 + \lambda\big(2\sqrt{2\pi}F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} \varphi_2)(y) + \sqrt{2\pi}(F_c\psi)(y)\big)}\Big).
\end{aligned}$$

By virtue of Wiener - Levy's Theorem, there exist functions $l \in L_1(\mathbb{R}_+)$ such that

$$\frac{2\sqrt{2\pi}F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} \varphi_2)(y) + \sqrt{2\pi}(F_c\psi)(y)}{1 + \lambda\big(2\sqrt{2\pi}F_c(\varphi_1 \overset{\gamma}{\underset{2}{*}} \varphi_2)(y) + \sqrt{2\pi}(F_c\psi)(y)\big)} = (F_c l)(y).$$

Therefore,

$$(F_s f)(y) = (F_s h)(y)\big(1 - \lambda(F_c l)(y)\big).$$

Hence

$$f(x) = h(x) - \lambda(h \underset{1}{*} l)(x).$$

The theorem is proved. ∎

References

1. S. Bochner and K. Chandrasekharan, *Fourier Transforms,* Princeton Univ. Press, Princeton., 1949.
2. R. V. Churchill, *Fourier Series and Boundary Value Problems,* New York, 1941.
3. V. A. Kakichev, On the convolution for integral transforms, *Izv. Vyssh. Uchebn. Zaved. Mat.* **2** (1967) 53-62 (In Russian).
4. V. A. Kakichev and Nguyen Xuan Thao, On the design method for the generalized integral convolutions, *Izv. Vyssh. Uchebn. Zaved. Mat.* **1** (1998) 31-40 (In Russian).
5. V. A. Kakichev, Nguyen Xuan Thao and Vu Kim Tuan, On the generalized convolutions for Fourier cosine and sine transforms, *East- West Journal of Mathematics.* **1** (1998) 85-90.
6. F. Al-Musallam and Vu Kim Tuan, Integral transforms related to a generalized convolution, *Results Math.* **38** (2000) 3–4, 197–208.
7. F. Al-Musallam, Fadhel and Vu Kim Tuan, A class of convolution transforms. *Fract. Calc. Appl. Anal.* **3** (2000) 303–314.
8. I.N. Sneddon, *The Use of Integral Transforms.* McGray-Hill, New York, 1972.
9. Stein E.M. and Weiss G, *Introduction to Fourier analysis on Euclidean space.* Princeton Univ. Press, Princeton, 1971.
10. H. M. Titchmarsh, *Introduction to the Theory of Fourier Integrals,* 2nd Ed. Clarendon Press, Oxford, 1967.
11. Nguyen Xuan Thao, On the generalized convolution for the Stieltjes, Hilbert, Fourier Cosine and Sine transforms, *Ukrain.. Math. J.* **53** (2001) 560–567 (in Russian).
12. Nguyen Xuan Thao and Nguyen Minh Khoa, Generalized convolution for integral transforms, *Methods of complex and clifford Analysis. Proc. Inter. Conf. Appl. Math. SAS Inter. Public.* (2004) 161–180.
13. Nguyen Xuan Thao and Nguyen Minh Khoa (2005), On the generalized convolution with a weight-function for Fourier, Fourier Cosine and Sine transforms, *Vietnam J. Math.* **33** (2005) 421–436.

14. Nguyen Xuan Thao and Nguyen Minh Khoa, On the generalized convolution with a weight-function for the Fourier Sine and Cosine transforms", *Integral Transforms and Special Function.* **17** (2006) 673– 685.
15. Nguyen Xuan Thao and Nguyen Thanh Hai, Convolutions for Integral Transform and their Application, *Computer Centre of the Russian Academy.* Moscow, 44 pages (Russian), 1997.
16. Nguyen Xuan Thao, Vu Kim Tuan and Nguyen Minh Khoa , On the generalized convolution with a weight-function for the Fourier Cosine and Sine transforms, *Frac. Cal. and Appl. Anal.* **7** (2004) 323–337.
17. Nguyen Xuan Thao and Trinh Tuan, On the generalized convolution for $I-$ transform, *Acta Math. Vietnam.* **28** (2003) 159 – 174.
18. Nguyen Xuan Thao and Trinh Tuan, On the generalized convolution with a weight - function for the Fourier Sine, Kontorovich-Lebedev and Fourier Cosine transforms, *Method of Complex and Clifford Analysis. Proc. Inter. Conf. Appl. Math. SAS Inter. Public.* (2004) 181–198.
19. Nguyen Xuan Thao and Trinh Tuan, Generalized convolutions of the integral Kontorovich - Lebedev, Fourier Sine and Cosine transforms, *Eotvos Lorand Tud. Inf. Kar.* **25** (2005) 37–51.