On Some $p$-subgroups of Automorphism Group of a Finite $p$-group

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Abstract. Let $G$ be a group and let $\text{Aut}_{Z(G)}^G(G)$ denote the group of all automorphisms of $G$ fixing both $G/G'$ and $Z(G)$ elementwise. In this paper, using the notion of Frattinian groups, we give some necessary and sufficient conditions on a finite non-abelian $p$-group $G$ for the groups $\text{Aut}_{Z(G)}^G(G)$ and $\text{Inn}(G)$ coincide.

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1. Introduction

Let $G$ be a group and let $N$ be a normal subgroup of $G$. Let $\sigma$ be an automorphism of $G$. If $N^\sigma = N$ (or $Ng^\sigma = Ng$ for all $g$ in $G$), we shall say $\sigma$ normalizes $N$ (centralizes $G/N$ respectively). Now let $M$ and $N$ be normal subgroups of a group $G$. We let $\text{Aut}^N_G(G)$ denote the group of all automorphisms of $G$ normalizing $N$ and centralizing $G/N$, and $\text{Aut}_M(G)$ the group of all automorphisms of $G$ centralizing $M$. Moreover, $\text{Aut}_M^N(G) = \text{Aut}^N_G(G) \cap \text{Aut}_M(G)$.

Various authors have studied the groups $\text{Aut}^N_G(G)$ and $\text{Aut}_M^N(G)$ for some particular characteristic subgroups $M$ and $N$ of a finite $p$-group $G$. It is well known that if $G$ is a finite $p$-group, then so is the group $\text{Aut}^\Phi(G)$, where $\Phi$ denotes the Frattini subgroup of $G$, the intersection of all the maximal subgroups of $G$. Liebeck in [6] gave an upper bound for the nilpotency class of $\text{Aut}^\Phi(G)$. In [1], Adney and Yen proved that if $G$ is a finite $p$-group having no nontrivial abelian direct factor, then there is a one-to-one correspondence between $\text{Aut}^G(G)$ and the group $\text{Hom}(G/G', Z)$ of all homomorphisms of $G$ into $Z = Z(G)$, where $G'$ denotes the derived subgroup of $G$. For some special values of $M$ and $N$ the group of all inner automorphisms $\text{Inn}(G)$ of $G$ is contained in $\text{Aut}_M^N(G)$. 
Several papers have been devoted to study the group $\text{Aut}_N^G(G)/\text{Inn}(G)$ when $G$ is a finite nonabelian $p$-group. Müller in [7] proved, using techniques from cohomology, that if $G$ is a finite nonabelian $p$-group, then $\text{Aut}_N^G(G) = \text{Inn}(G)$ if and only if $\Phi \leq Z$ and $\Phi$ is cyclic. This turns out that $\text{Aut}_N^\Phi(G)/\text{Inn}(G)$ is nontrivial if and only if $G$ is neither elementary abelian nor extraspecial. Cheng [3] proved, among others, the following result. Let $G$ be a finite $p$-group such that $G' = \langle a \rangle$ is cyclic. Assume that either $p > 2$, or $p = 2$ and $[a, G] \leq \langle a^4 \rangle$. Then $\text{Aut}_N^G(G) = \text{Inn}(G)$. Curran and McCaughan in [5] proved that if $G$ is a finite $p$-group, then $\text{Aut}_N^Z(G) = \text{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. Finally Curran [4] showed that for any nonabelian group $G$, $\text{Aut}_N^Z(G) \cong \text{Hom}(G/G'Z, Z)$ obtaining some results concerning the group $\text{Aut}_N^Z(G)$, where $G$ is a finite nonabelian $p$-group. In particular, he showed that $\text{Aut}_N^Z(G) = Z(\text{Inn}(G))$ if and only if $\text{Hom}(G/G', Z) \cong Z(G/Z)$.

In this paper we study closely the groups $\text{Aut}_N^G(G)$ and $\text{Aut}_N^G(G)$ for a finite nonabelian $p$-group $G$. We also give an alternative short proof for the main result of Müller mentioned earlier using an elegant theorem of Schmid [8].

In Sec. 2 we give some preliminary results that are needed for the main results of the paper. In Sec. 3 we prove the main results of the paper. Finally in Sec. 4 we give a new short proof for the Müller’s result which was mentioned earlier. This proof, based on an elegant result of Schmid [8], simplifies greatly the Müller’s proof. We use standard notation in group theory: we use the notation $\text{Hom}(G, A)$ to denote the group of homomorphisms of $G$ into an abelian group $A$. $\Omega_i(G)$ the subgroup of $G$ generated by its elements of order dividing $p^i$. Recall that a group $G$ is called a central product of its subgroups $A$ and $B$ if $A$ and $B$ commute elementwise and together generate $G$. In this situation, we write $G = A * B$.

2. Some Basic Results

In this section we give some known results which will be used in the rest of the paper.

Let $G$ be a finite $p$-group. Following Schmid, we call $G$ Frattinian provided $Z(G) \neq Z(M)$ for all maximal subgroups $M$ of $G$. In [8], Schmid proved the following structural theorem for the Frattinian groups.

**Theorem 2.1** [8]. Suppose $G$ is a nonabelian Frattinian $p$-group. Then one of the following holds:

(i) $G$ is the central product of nonabelian $p$-groups of order $p^2|Z(G)|$, amalgamating their centres.

(ii) $G = E * F$ is the central product of Frattinian subgroups $E$ and $F$ with $C_F(Z(\Phi(F))) = \Phi(F)$, $E = C_G(F)$ and $\Phi(E) \leq Z(G)$.

It is worth noting that in case (i) of the above theorem the factors of the central product are minimal nonabelian $p$-groups. Accordingly, in this case we have
$Z(G) = \Phi(G)$. The following simple lemmas will be used in the rest of the paper.

**Lemma 2.2.** Let $G$ be a group and let $M$, $N$ be normal subgroups of $G$ with $N \leq M$ and $C_N(M) \leq Z(G)$. Then $\text{Aut}_M^N(G) \cong \text{Hom}(G/M, C_N(M))$.

**Proof.** It is easy to verify that the map $f_\sigma : Mx \mapsto x^{-1}x^\sigma$ defines a homomorphism from $G/M$ into $C_N(M)$ for every $\sigma \in \text{Aut}_M^N(G)$. On the other hand, the map $\sigma \mapsto x \mapsto xf(x)$ defines an automorphism of $G$ for every $f \in \text{Hom}(G/M, C_N(M))$. This automorphism lies in $\text{Aut}_M^N(G)$ and the map $\sigma \mapsto f_\sigma$ is an isomorphism from $\text{Aut}_M^N(G)$ to $\text{Hom}(G/M, C_N(M))$. ■

**Lemma 2.3.** Let $G = E \ast F$ be a central product of subgroups $E$ and $F$. Assume that $\psi(G)$ is $\Phi(G)$, $G'$ or $Z(G)$. If $\alpha \in \text{Aut}_{Z(G)}^\psi(E)$ then the map $\hat{\alpha} : xy \mapsto x^\alpha y$, where $x \in E$ and $y \in F$, defines an automorphism of $G$ lying in $\text{Aut}_{Z(G)}^\psi(G)$.

**Proof.** Straightforward.

Throughout the paper we write $Z$ and $\Phi$ for $Z(G)$ and $\Phi(G)$, respectively.

### 3. The Groups $\text{Aut}_Z^G(G)$ and $\text{Aut}_Z^{G'}(G)$

In this section we study the groups $\text{Aut}_Z^G(G)$ and $\text{Aut}_Z^{G'}(G)$ for a finite non-abelian $p$-group $G$.

We begin by an elementary lemma which is a consequence of Lemma 2.2.

**Lemma 3.1.** If $G$ is a group of class 2, then
(i) $\text{Aut}_Z^G(G) \cong \text{Hom}(G/G', G')$,
(ii) $\text{Aut}_Z^{G'}(G) \cong \text{Hom}(G/Z(G), G')$.

**Proposition 3.2.** Let $G$ be a finite $p$-group of class 2. Then $\text{Aut}_Z^G(G) = \text{Inn}(G)$ if and only if $G'$ is cyclic.

**Proof.** Assume that $G'$ is cyclic. Since $\exp(G/Z) = \exp(G')$, $\text{Aut}_Z^G(G) \cong \text{Hom}(G/Z, G') \cong G/Z$, as required. The converse of the result is evident from the fact that $\text{Hom}(G/Z, G') \cong G/Z$. ■

**Theorem 3.3.** Let $G$ be a finite nonabelian $p$-group of class 2. Then $\text{Aut}_Z^{G'}(G) = \text{Inn}(G)$ if and only if $G'$ is cyclic and $Z(G) = G'Gp^n$ where $|G'| = p^n$.

**Proof.** Assume that $G'$ is cyclic and $Z(G) = G'Gp^n$, where $|G'| = p^n$. By Proposition 3.2, $\text{Aut}_Z^G(G) = \text{Inn}(G)$. Let $\alpha \in \text{Aut}_Z^{G'}(G)$ and $a \in G$. We may write $\alpha(a) = ad$ with $d \in G'$. Now we observe that $\alpha(a^p) = (\alpha(a))^p = a^p d^p = a^p$, which shows that $\alpha$ fixes any element of $Z(G)$. Consequently $\text{Aut}_Z^{G'}(G) \leq \text{Aut}_Z^G(G)$, and the proof is complete.

Conversely suppose that $\text{Aut}_Z^{G'}(G) = \text{Inn}(G)$. We deduce that $G'$ is cyclic,
because \( \text{Aut}_Z^G(G) = \text{Inn}(G) \). Since \( G \) is of class 2, \( G' \leq G'G^{p^n} \leq Z(G) \). It follows that

\[
\text{Inn}(G) \cong \text{Hom}(G/Z(G), G') \hookrightarrow \text{Hom}(G/G^{p^n}, G') \hookrightarrow \text{Hom}(G/G', G')
\]

\[
\cong \text{Aut}^G(G) = \text{Inn}(G).
\]

So that \( \text{Hom}(G/G^{p^n}, G') \cong \text{Hom}(G/Z(G), G') \).

However \( \exp(G') = \exp(G/Z(G)) = |G'| \), which gives \( |G/Z(G)| = |G/G^{p^n}| \), as required.

**Remark.** In [2], Berkovich shows that if \( G \) is a finite \( p \)-group with \( \text{rank}(G/G') = r \) and \( |G'| \leq |G/Z| \), then \( \text{Aut}^G(G) = \text{Inn}(G) \).

As an application of Theorem 3.3, we get another proof of the main result of [5].

**Corollary 3.4.** [5]. If \( G \) is a finite \( p \)-group then \( \text{Aut}^Z(G) = \text{Inn}(G) \) if and only if \( G' = Z(G) \) and \( Z(G) \) is cyclic.

**Proof.** If \( G' = Z(G) \) and \( Z(G) \) is cyclic then \( G' \) is cyclic and obviously \( Z(G) = G'G^{p^n} \), and hence \( \text{Aut}^G(G) = \text{Aut}^Z(G) = \text{Inn}(G) \), by Theorem 3.3. Conversely, suppose that \( \text{Aut}^Z(G) = \text{Inn}(G) \). So \( G \) is of class 2 and we have \( \text{Aut}^G(G) \leq \text{Aut}^Z(G) \). It follows that \( \text{Aut}^G(G) = \text{Inn}(G) \). Therefore \( G' \) is cyclic and \( Z(G) = G'G^{p^n} \), from which we conclude that \( G \) has no nontrivial abelian direct factor. So, by [1], we have

\[
|\text{Hom}(G/G', Z(G))| = |\text{Aut}^Z(G)| = |\text{Aut}^G(G)| = |\text{Hom}(G/G', G')|.
\]

Using [5, Lemma 1],

\[
|\text{Aut}^Z(G)| = |\text{Hom}(G/G', Z(G))| \geq |\text{Hom}(G/Z(G), G')||Z(G) : G'|
\]

\[
= |\text{Aut}^G(G)||Z(G) : G'|.
\]

Thus \( Z(G) = G' \) as required.

**Corollary 3.5.** If \( G \) is a finite nonabelian \( p \)-group, then \( \text{Aut}^Z(G) = \text{Inn}(G) \) if and only if \( G \) is of class 2 and \( Z(G) \) is cyclic.

**Proof.** Let \( \text{Aut}^Z(G) = \text{Inn}(G) \). Obviously \( G \) is of class 2. By Lemma 2.2, \( \text{Aut}^Z(G) \cong \text{Hom}(G/Z, Z) \). Now since \( \exp(G/Z) = \exp(G') \leq \exp(Z) \), we conclude that \( Z \) is cyclic. The converse of the result is immediate.

**Theorem 3.6.** Let \( G \) be a finite nonabelian \( p \)-group such that \( Z(\Phi(G)) \leq Z(G) \). Then \( \text{Aut}^G(G) = \text{Inn}(G) \) if and only if \( G \) is of class 2 and \( G' \) is cyclic.

**Proof.** Assume that \( \text{Aut}^G(G) = \text{Inn}(G) \). We distinguish two cases:
Case I. $Z(G) \not\subseteq \Phi(G)$.
We may write $G = MZ(G)$ for some maximal subgroup $M$ of $G$. It is evident that $Z(\Phi(M)) \subseteq Z(\Phi(G))$, whence $Z(\Phi(M)) \subseteq Z(G) \cap M = Z(M)$. Let $\alpha \in \text{Aut}_{Z(M)}^M(M)$. Then the map $\tilde{\alpha} : xz \mapsto x^\alpha z$, where $x \in M$ and $z \in Z(G)$, defines an automorphism of $G$ which lies in $\text{Aut}_{Z}^G(G) = \text{Inn}(G)$. Since $G = MZ(G)$, it implies that $\alpha \in \text{Inn}(M)$. Therefore $\text{Aut}_{Z(M)}^M(M) = \text{Inn}(M)$. Using induction, we conclude that $M'$ is cyclic and $M$ is of class 2. It follows that $G'$ is cyclic and $G$ is of class 2.

Case II. $Z(G) \subseteq \Phi(G)$.
In this case we show that $G$ is Frattinian. Let $M$ be an arbitrary maximal subgroup of $G$, and $z \in G \setminus M$. We write $G = M \langle z \rangle$ and choose an element $u$ in $\Omega_1(G' \cap Z(G))$. Clearly the map $\alpha : hz^i \mapsto h(zu)^i$, where $h \in M$ and $0 \leq i < p$, defines an automorphism of $G$ which is in $\text{Aut}_{Z}^G(G) = \text{Inn}(G)$. Assume that $\alpha$ is the inner automorphism of $G$ induced by $x$. It turns out that $x \in C_G(M) = Z(M)$ which shows that $Z(G) \neq Z(M)$. So $G$ is Frattinian and one of the statements (i),(ii) of Theorem 2.1 holds. If (i) is fulfilled, then $\Phi(G) = Z(G)$ and $G$ is of class 2. So the result follows at once from Proposition 3.2. However, the second statement of Theorem 2.1 cannot occur, because in this case, by $\Phi(G) = \Phi(E) \Phi(F) \leq Z(G) \Phi(F)$, we have $Z(\Phi(F)) \leq Z(\Phi(G)) \leq Z(G)$, which gives the contradiction $F = C_F(Z(\Phi(F))) = \Phi(F)$. The converse follows at once from Proposition 3.2.

**Theorem 3.7.** Let $G$ be a finite nonabelian $p$-group such that $Z(\Phi(G)) \leq Z(G)$. Then $\text{Aut}_{Z}^G(G) = \text{Inn}(G)$ if and only if $Z(G) = \Phi(G)$ and $G'$ is of order $p$.

**Proof.** We claim that $Z(G) \subseteq \Phi(G)$. Assume that this is false, then $G = M \langle z \rangle$ for some maximal subgroup $M$ of $G$ and for some $z$ in $Z(G) \setminus M$. We choose an element $u$ in $\Omega_1(G' \cap Z(G))$. The map $\alpha : hz^i \mapsto h(zu)^i$, where $h \in M$ and $0 \leq i < p$, is in $\text{Aut}_{Z}^G(G) = \text{Inn}(G)$ from which we conclude that $u = 1$, a contradiction. So $Z(G) \subseteq \Phi(G)$. By a similar argument given for the proof of Theorem 3.6, $G$ is Frattinian. Thus one of the statements of Theorem 2.1 holds. However, the second statement of Theorem 2.1 cannot occur by a similar argument given for the proof of Theorem 3.6. If the first statement occurs, then $\Phi(G) = Z(G)$. Hence by Proposition 3.2, $G$ is of class 2 and $G'$ is cyclic. Now since $\exp(G') = \exp(G/Z(G)) = \exp(G/\Phi(G))$, we conclude that $|G'| = p$.

The converse is immediate.

**4. The Groups $\text{Aut}_{Z}^\Phi(G)$ and $\text{Aut}^\Phi(G)$**

In this section we give an alternative proof for the Müller’s result on the groups $\text{Aut}_{Z}^\Phi(G)/\text{Inn}(G)$ and $\text{Aut}^\Phi(G)/\text{Inn}(G)$ using Theorem 2.1 and the following Proposition due to Schmid [8] which is readily proved by cohomological methods.

**Proposition 4.1** [8, Proposition 3]. Let $G$ be a finite Frattinian $p$-group. If $\text{Aut}_{Z}^\Phi(G) = \text{Inn}(G)$ then $C_G(Z(\Phi(G))) \neq \Phi(G)$.
Theorem 4.2 [7, Proposition 3.1]. Let $G$ be a finite nonabelian $p$-group. Then $\text{Aut}_Z^G(G) = \text{Inn}(G)$ if and only if $\Phi(G) \leq Z(G)$ and $\Phi(G)$ is cyclic.

Proof. Assume first that $\Phi(G) \leq Z(G)$ and $\Phi(G)$ is cyclic. By Lemma 2.2, $\text{Aut}_Z^G(G) \cong \text{Hom}(G/Z, \Phi)$. Now since $G$ is of class 2, $\exp(G/Z) = \exp(G') \leq \exp(\Phi(G))$, whence $\text{Aut}_Z^G(G) \cong G/Z$.

Conversely let $\text{Aut}_Z^G(G) = \text{Inn}(G)$. Assume either $\Phi(G) \not\leq Z(G)$ or $\Phi(G)$ is noncyclic. We consider two cases:

Case I. $Z(G) \not\leq \Phi(G)$.

We choose a maximal subgroup $M$ of $G$ such that $Z(G) \not\leq M$. So $G = MZ(G)$ and $Z(M) = Z(G) \cap M$. Now if $\Phi(M) \leq Z(M)$ then $\Phi(G) \leq Z(G)$ and hence by Lemma 2.2, $\text{Aut}_Z^G(Z(M)) \cong \text{Hom}(G/Z, \Phi)$. Since $\Phi(G)$ is noncyclic, it follows that $|\text{Aut}_Z^G(Z(M))| > |G/Z|$ which is impossible. So we suppose that $\Phi(M) \not\leq Z(M)$. In this situation we may use induction to deduce that $\text{Aut}_Z^{\Phi(M)}(M) \neq \text{Inn}(M)$. Let $\beta \in \text{Aut}_Z^{\Phi(M)}(M) \setminus \text{Inn}(M)$. We write $G = M(z)$ where $z \in Z(G)/M$, and extend $\beta$ to an automorphism $\hat{\beta} \in \text{Aut}_Z^G(G)$ by setting $(hz^i)^{\hat{\beta}} = h^\beta z^i$, where $h \in M$ and $0 \leq i < p$. We therefore have $\hat{\beta} \in \text{Inn}(G)$. It follows that $\beta \in \text{Inn}(M)$, a contradiction.

Case II. $Z(G) \leq \Phi(G)$.

In this case we claim that $G$ is Frattinian. To see this, let $M$ be an arbitrary maximal subgroup of $G$. Choose an element $z$ in $G \setminus M$ and let $u \in \Omega_1(Z(G))$. The map $\alpha : hz^i \mapsto h(zu)^i$, where $h \in M$ and $0 \leq i < p$, defines an automorphism of $G$ which is in $\text{Aut}_Z^G(G)$. So $\alpha$ is an inner automorphism of $G$ induced by an element $t$ in $G$. It follows that $t \in C_G(M) = Z(M)$. Now since $t \notin Z(G)$, we see that $G$ is Frattinian. By Theorem 2.1, one of the statements (i),(ii) of the theorem holds. If the statement (i) holds then $Z(G) = \Phi(G)$ and hence $\text{Inn}(G) = \text{Aut}_Z^G(G) = \text{Aut}_Z^G(G)$. Consequently $Z(G)$ is cyclic by Corollary 3.5, a contradiction. We therefore suppose that the second statement of Theorem 2.1 is fulfilled. If $E$ is abelian then $E \leq Z(G) \leq \Phi(G)$ and we have $G = F$ whence $C_G(Z(\Phi(G))) = \Phi(G)$, a contradiction to Proposition 4.1. So we may suppose that $E$ is nonabelian. Now let $\alpha \in \text{Aut}_Z^{Z(E)}(E)$ and extend $\alpha$ to an automorphism $\hat{\alpha} \in \text{Aut}_Z^G(G)$ according to Lemma 2.3. It follows that $\hat{\alpha}$ is an inner automorphism of $G$ induced by some element in $E$. Therefore, $\alpha \in \text{Inn}(E)$, and hence $\text{Aut}_Z^{Z(E)}(E) = \text{Inn}(E)$. By Corollary 3.5, $Z(E)$ is cyclic. Since $E = C_G(F)$, we deduce that $Z(G) = Z(E)$ is cyclic. Now if $\Phi(F) \leq Z(G)$, then $\Phi(G) = \Phi(E)\Phi(F) \leq Z(G)$ and hence $Z(G) = \Phi(G)$, a contradiction. Thus $\Phi(F) \not\leq Z(G)$ from which we deduce that $\Phi(F) \not\leq Z(F)$. Again by induction hypothesis $\text{Aut}_Z^{\Phi(F)}(F) \neq \text{Inn}(F)$, which is impossible, by a similar argument given in Case I.

Corollary 4.3 [7]. If $G$ is a finite nonabelian $p$-group then $\text{Aut}_Z^G(G) = \text{Inn}(G)$ if and only if $G$ is extraspecial.

Proof. If $\text{Aut}_Z^G(G) = \text{Inn}(G)$, then $\Phi(G) \leq Z(G)$ and $\Phi(G)$ is cyclic by Theorem 4.2. So $G$ is of class 2 and hence $Z(G) \leq \Phi(G)$ by Theorem 3.3. It follows that
$Z(G) = \Phi(G)$. Now according to Corollary 3.4, $G' = Z(G)$ and $Z(G)$ is cyclic. Finally $\exp(G') = \exp(G/\Phi) = p$ which completes the proof of the first part.

The converse is straightforward.

References