Vietnam Journal of Mathematics **36**:1(2008) 39–46

Vietnam Journal of MATHEMATICS

On the Fekete-Szegö Problem for Certain Subclasses of Analytic Functions

T. N. Shanmugam[1], S. Kavitha[2], and S. Sivasubramanian[3]

[1] *Department of Mathematics, College of Engineering, Anna University, Chennai-600 025, Tamilnadu, India*

[2] *Department of Mathematics, Madras Christian College, Chennai-59, India.*

[3] *Department of Mathematics, Easwari Engineering College, Ramapuram, Chennai-600 089, Tamilnadu, India*

Received March 20, 2006

Abstract. In this present investigation, the authors obtain Fekete-Szegö's inequality for certain normalized analytic functions $f(z)$ defined on the open unit disk for which $(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta}+\alpha f'(z)\left(\frac{z}{f(z)}\right)^{\beta-1}$, $(\beta \geq 0,\ 0 \leq \alpha < 1)$ lies in a region starlike with respect to 1 and is symmetric with respect to the real axis. Also certain applications of the main result for a class of functions defined by convolution are given. As a special case of this result, Fekete-Szegö's inequality for a class of functions defined through fractional derivatives is obtained. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities obtained by Srivastava and Mishra.

2000 AMS Mathematics Subject Classification: 30C45.

Keywords: Analytic functions, Starlike functions, subordination, Coefficient problem, Fekete-Szegö inequality.

1. Introduction

Let $\mathcal{A}$ denote the class of all *analytic* functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta := \{z \in \mathbb{C} \mid |z| < 1\}) \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \Delta),$$

and $C(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \quad (z \in \Delta),$$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [3]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. Recently, Shanmugam and Sivasubramanian [7] obtained Fekete- Szegö inequalities for the class of functions

$$\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} \prec \phi(z) \quad (\alpha \geq 0).$$

For a brief history of Fekete-Szegö problem for the class of starlike, convex and close-to-convex functions, see the recent paper by Srivastava *et al.* [9].

In the present paper, we obtain the Fekete-Szegö inequality for functions in a more general class $M_{\alpha,\beta}(\phi)$ of functions which we define below. Also we give applications of our results to certain functions defined through convolution (or Hadamard product) and in particular we consider a class $M^{\lambda}_{\alpha,\beta}(\phi)$ of functions defined by fractional derivatives. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities of Srivastava and Mishra [8].

Definition 1.1. *Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk Δ onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in \mathcal{A}$ is in the class $M_{\alpha,\beta}(\phi)$ if*

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha f'(z)\left(\frac{z}{f(z)}\right)^{\beta-1} \prec \phi(z), \quad (\beta \geq 0,\ 0 \leq \alpha < 1).$$

For fixed $g \in \mathcal{A}$, we define the class $M^{g}_{\alpha,\beta}(\phi)$ to be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in M_{\alpha,\beta}(\phi)$.

To prove our main result, we need the following.

Lemma 1.2. [3] *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic function with positive real part in Δ, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v+2 & \text{if } v \leq 0 \\ 2 & \text{if } 0 \leq v \leq 1 \\ 4v-2 & \text{if } v \geq 1 \end{cases}.$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then the equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Also the above upper bound is sharp, and it can be improved as follows when $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \quad (0 < v \leq 1/2)$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2 \quad (1/2 < v \leq 1).$$

2. Fekete-Szegö Problem

Our main result is the following.

Theorem 2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by* (1.1) *belongs to $M_{\alpha,\beta}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{B_2}{(\beta+2\alpha)} - \dfrac{\mu}{(\beta+\alpha)^2}B_1^2 + \dfrac{1-\beta}{2(\beta+\alpha)^2}B_1^2, & \text{if } \mu \leq \sigma_1 \\ \dfrac{B_1}{(\beta+2\alpha)}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ -\dfrac{B_2}{(\beta+2\alpha)} + \dfrac{\mu}{(\beta+\alpha)^2}B_1^2 - \dfrac{1-\beta}{2(\beta+\alpha)^2}B_1^2, & \text{if } \mu \geq \sigma_2 \end{cases}$$

where

$$\sigma_1 := \frac{2(\beta+\alpha)^2(B_2 - B_1) + (1-\beta)(\beta+2\alpha)B_1^2}{2(\beta+2\alpha)B_1^2},$$

$$\sigma_2 := \frac{2(\beta+\alpha)^2(B_2 + B_1) + (\beta+2\alpha)(1-\lambda)B_1^2}{2(\beta+2\alpha)B_1^2}.$$

The result is sharp.

Proof. For $f(z) \in M_{\alpha,\beta}(\phi)$, let

$$p(z) := (1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha f'(z)\left(\frac{z}{f(z)}\right)^{\beta-1} = 1 + b_1 z + b_2 z^2 + \cdots . \quad (2.1)$$

From (2.1), we obtain

$$(\beta+\alpha)a_2 = b_1 \quad \text{and} \quad (\beta+2\alpha)a_3 = b_2 - \frac{(\beta-1)(\beta+2\alpha)}{2}a_2^2.$$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic and has positive real part in Δ. Also we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \quad (2.2)$$

and from this equation (2.2), we obtain

$$b_1 = \frac{1}{2}B_1 c_1$$

and

$$b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2 c_1^2.$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{B_1}{2(\beta+2\alpha)}(c_2 - v c_1^2) \quad (2.3)$$

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{(\beta-1+2\mu)(2\alpha+\beta)}{2(\beta+\alpha)^2}B_1\right].$$

Our result now follows by an application of Lemma 1.2. To show that the bounds are sharp, we define the functions K^{ϕ_n} $(n = 2, 3, \dots)$ by

$$(1-\alpha)\left(\frac{K^{\phi_n}}{z}\right)^{\beta} + \alpha[K^{\phi_n}]'(z)\left(\frac{z}{K^{\phi_n}}\right)^{\beta-1} = \phi(z^{n-1}),$$

$$K^{\phi_n}(0) = 0 = [K^{\phi_n}]'(0) - 1$$

and the function F^{λ} and G^{λ} $(0 \le \lambda \le 1)$ by

$$(1-\alpha)\left(\frac{F^{\lambda}(z)}{z}\right)^{\beta} + \alpha[F^{\lambda}]'(z)\left(\frac{z}{F^{\lambda}(z)}\right)^{\beta-1} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right),$$

$$F^{\lambda}(0) = 0 = (F^{\lambda})'(0) - 1$$

and

$$(1-\alpha)\left(\frac{G^{\lambda}(z)}{z}\right)^{\beta} + \alpha[G^{\lambda}]'(z)\left(\frac{z}{G^{\lambda}(z)}\right)^{\beta-1} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right),$$

$$G^{\lambda}(0) = 0 = (G^{\lambda})'(0).$$

Clearly the functions $K_{\alpha}^{\phi n}, F_{\alpha}^{\lambda}, G_{\alpha}^{\lambda} \in M_{\alpha}(\phi)$. Also we write $K_{\alpha}^{\phi} := K_{\alpha}^{\phi_2}$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_{α}^{ϕ} or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is $K_{\alpha}^{\phi_3}$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_{α}^{λ} or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_{α}^{λ} or one of its rotations. ■

Remark 2.2. If $\sigma_1 \leq \mu \leq \sigma_2$, then, in view of Lemma 1.2, Theorem 2.1 can be improved. Let σ_3 be given by

$$\sigma_3 := \frac{2(\beta+\alpha)^2 B_2 + (\beta+2\alpha)(\beta-1)B_1^2}{2(\beta+2\alpha)B_1^2}$$

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{(\beta+\alpha)^2}{(\beta+2\alpha)B_1^2}\left[B_1 - B_2 + \frac{(\beta-1+2\mu)(2\alpha+\beta)}{2(\beta+\alpha)^2}B_1^2\right]|a_2|^2 \leq \frac{B_1}{(\beta+2\alpha)}.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{(\beta+\alpha)^2}{(\beta+2\alpha)B_1^2}\left[B_1 + B_2 + \frac{(\beta-1+2\mu)(2\alpha+\beta)}{2(\beta+\alpha)^2}B_1^2\right]|a_2|^2 \leq \frac{B_1}{(\beta+2\alpha)}.$$

3. Applications to Functions Defined by Fractional Derivatives

In order to introduce the class $M_{\alpha,\beta}^{\lambda}(\phi)$, we need the following.

Definition 3.1. (see [4, 5]; see also [10, 11]). *Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order λ is defined by*

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \leq \lambda < 1)$$

where the multiplicity of $(z-\zeta)^\lambda$ *is removed by requiring that* $\log(z-\zeta)$ *is real for* $z-\zeta>0$.

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [4] introduced the operator $\Omega^\lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$(\Omega^\lambda f)(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z), \quad (\lambda \neq 2,3,4,\ldots).$$

The class $M^\lambda_{\alpha,\beta}(\phi)$ consists of functions $f \in \mathcal{A}$ for which $\Omega^\lambda f \in M_{\alpha,\beta}(\phi)$. Note that $M^0_{1,0}(\phi) \equiv S^*(\phi)$ and $M^\lambda_{\alpha,\beta}(\phi)$ is the special case of the class $M^g_{\alpha,\beta}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n. \tag{3.1}$$

Let $g(z) = z + \sum_{n=2}^\infty g_n z^n$ $(g_n > 0)$. Since $f(z) = z + \sum_{n=2}^\infty a_n z^n \in M^g_{\alpha,\beta}(\phi)$ if and only if $(f*g) = z + \sum_{n=2}^\infty g_n a_n z^n \in M_{\alpha,\beta}(\phi)$, we obtain the coefficient estimate for functions in the class $M^g_{\alpha,\beta}(\phi)$, from the corresponding estimate for functions in the class $M_{\alpha,\beta}(\phi)$. Applying Theorem 2.1 for the function $(f*g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \cdots$, we get the following Theorem 3.2 after an obvious change of the parameter μ.

Theorem 3.2. *Let the function* $\phi(z)$ *be given by* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *If* $f(z)$ *given by* (1.1) *belongs to* $M^g_{\alpha,\beta}(\phi)$, *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{1}{g_3}\left[\dfrac{B_2}{\beta+2\alpha} - \dfrac{\mu g_3}{g_2^2(\beta+\alpha)^2} B_1^2 + \dfrac{1-\beta}{2(\beta+\alpha)^2} B_1^2\right], & \text{if } \mu \leq \sigma_1 \\ \dfrac{1}{g_3}\,\dfrac{B_1}{\beta+2\alpha}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ \dfrac{1}{g_3}\left[-\dfrac{B_2}{\beta+2\alpha} + \dfrac{\mu g_3}{g_2^2(\beta+\alpha)^2} B_1^2 - \dfrac{1-\beta}{2(\beta+\alpha)^2} B_1^2\right], & \text{if } \mu \geq \sigma_2 \end{cases}$$

where

$$\sigma_1 := \frac{g_2^2}{g_3} \frac{2(\beta+\alpha)^2(B_2 - B_1) + (1-\beta)(\beta+2\alpha)B_1^2}{2(\beta+2\alpha)B_1^2}$$

$$\sigma_2 := \frac{g_2^2}{g_3} \frac{2(\beta+\alpha)^2(B_2 + B_1) + (1-\beta)(\beta+2\alpha)B_1^2}{2(\beta+2\alpha)B_1^2}.$$

The result is sharp.

Since

$$(\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n,$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \tag{3.2}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{3.3}$$

For g_2 and g_3 given by (3.2) and (3.3), Theorem 3.2 reduces to the following:

Theorem 3.3. *Let the function* $\phi(z)$ *be given by* $\phi(z) = 1+B_1z+B_2z^2+B_3z^3+\cdots$. *If* $f(z)$ *given by* (1.1) *belongs to* $M^{\lambda}_{\alpha,\beta}(\phi)$, *then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{(2-\lambda)(3-\lambda)}{6}\gamma & \text{if } \mu \le \sigma_1 \\ \dfrac{(2-\lambda)(3-\lambda)}{6}\dfrac{B_1}{2(1+2\alpha)} & \text{if } \sigma_1 \le \mu \le \sigma_2 \\ -\dfrac{(2-\lambda)(3-\lambda)}{6}\gamma & \text{if } \mu \ge \sigma_2 \end{cases}$$

where

$$\gamma := \frac{B_2}{\beta+2\alpha} - \frac{3(2-\lambda)}{2(3-\lambda)}\frac{\mu}{(\beta+\alpha)^2}B_1^2 + \frac{1-\beta}{2(\beta+\alpha)^2}B_1^2$$

$$\sigma_1 := \frac{2(3-\lambda)}{3(2-\lambda)}.\frac{2(\beta+\alpha)^2(B_2-B_1)+(1-\beta)(\beta+2\alpha)B_1^2}{2(\beta+2\alpha)B_1^2}$$

$$\sigma_2 := \frac{2(3-\lambda)}{3(2-\lambda)}.\frac{2(\beta+\alpha)^2(B_2+B_1)+(1-\beta)(\beta+2\alpha)B_1^2}{2(\beta+2\alpha)B_1^2}$$

The result is sharp.

Remark 3.4. When $\alpha = 1$, $\beta = 0$, $B_1 = 8/\pi^2$ and $B_2 = 16/(3\pi^2)$, the above Theorem 3.2 reduces to a recent result of Srivastava and Mishra [8, Theorem 8, p. 64] for a class of functions for which $\Omega^{\lambda} f(z)$ is a parabolic starlike function [2, 6].

References

1. B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.* **15** (1984) 737–745.
2. A. W. Goodman, Uniformly convex functions, *Ann. Polon. Math.* **56** (1991) 87–92.
3. W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in: *Proceedings of the Conference on Complex Analysis*, Z. Li, F. Ren, L. Yang, and S. Zhang(Eds.), Int. Press (1994) 157–169.
4. S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.* **39** (1987) 1057–1077.

5. S. Owa, On the distortion theorems I, *Kyungpook Math. J.* **18** (1978) 53 – 58.
6. F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.* **118** (1993) 189–196.
7. T.N.Shanmugam and S.Sivasubramanian, On the Fekete-Szegö Problem for Some Subclasses of Analytic Functions, Preprint.
8. H. M. Srivastava and A. K. Mishra, Applications of fractional calculus to parabolic starlike and uniformly convex functions, *Computer Math. Appl.* **39** (2000) 57–69.
9. H. M. Srivastava, A. K. Mishra and M. K. Das, The Fekete-Szegö problem for a subclass of close-to-convex functions, *Complex Variables, Theory Appl.* **44** (2001) 145–163.
10. H. M. Srivastava and S. Owa, An application of the fractional derivative, *Math. Japon.* **29** (1984) 383–389.
11. H. M. Srivastava and S. Owa, *Univalent functions, Fractional Calculus, and their Applications*, Halsted Press/John Wiley and Songs, Chichester/New York, (1989).