Vietnam Journal of Mathematics **36**:2(2008) 125–136

Vietnam Journal
of
MATHEMATICS

$T1$ Theorems for Inhomogeneous Besov and Triebel-Lizorkin Spaces over Space of Homogeneous Type*

Yanchang Han

Department of Mathematics,
South China Normal University, Guangzhou, 510631, P.R. China

Received November 17, 2006

Abstract. The author establishes $T1$ theorems for inhomogeneous Besov and Triebel-Lizorkin spaces by discrete Calderón type reproducing formula and the Plancherel-Pôlya characterization for inhomogeneous Besov and Triebel-Lizorkin spaces. These results are new even for $\mathbb{R}^d$.

2000 Mathematics Subject Classification: 42B25, 42B35, 46E35.
Keywords: $T1$ theorem, inhomogeneous Besov and Triebel-Lizorkin spaces, spaces of homogeneous type.

1. Introduction

In the past years, there has been significant progress on the problem of proving the boundedness of generalized Calderón-Zygmund operators on various function spaces. A remarkable result is the famous $T1$ theorem of David and Journé in [3]. $T1$ theorem has been extended for Besov and Triebel-Lizorkin spaces. For a broader view of this active area of research, see e.g. [5, 10, 12–14, 16, 17] and references therein.

The main purpose of this paper is to establish $T1$ theorems for the inhomogeneous spaces $B_p^{\alpha,q}(X)$ when $\frac{d}{d+\alpha} < p \leq \infty, 0 < q \leq \infty, 0 < \alpha < \epsilon$ and for $F_p^{\alpha,q}(X)$ when $\frac{d}{d+\alpha} < p < \infty, \frac{d}{d+\alpha} < q \leq \infty, 0 < \alpha < \epsilon$, and for $B_p^{\alpha,q}(X)$ when $\frac{d}{d+\alpha+\epsilon} < p \leq \infty, 0 < q \leq \infty, -\epsilon < \alpha < 0$ and for $F_p^{\alpha,q}(X)$ when $\frac{d}{d+\alpha+\epsilon} < p < \infty, \frac{d}{d+\alpha+\epsilon} < q \leq \infty, -\epsilon < \alpha < 0$ for some $\epsilon > 0$ by discrete

*Project supported by the NNSF (No. 10726071) of China.

Calderón type reproducing formula and Plancherel-Pôlya characterization for the inhomogeneous Besov and Triebel-Lizorkin spaces. Roughly speaking, T is bounded on $B_p^{\alpha,q}(X)$ and $F_p^{\alpha,q}(X)$ for the range of α, p, and q indicated above, respectively, if its kernel satisfies only half smoothness and moment conditions. An application of these results is given in [4].

To state main results of this paper, we begin by recalling the definitions necessary for inhomogeneous Besov and Triebel-Lizorkin spaces on spaces of homogeneous type and some basic facts about the Calderón-Zygmund operator theory. A *quasi-metric* ρ on a set X is a function $\rho : X \times X \to [0, \infty)$ satisfying:

(i) $\rho(x, y) = 0$ if and only if $x = y$;

(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(iii) There exists a constant $A \in [1, \infty)$ such that for all $x, y, z \in X$,

$$\rho(x, y) \leq A[\rho(x, z) + \rho(z, y)].$$

Any quasi-metric defines a topology, for which the balls $B(x, r) = \{y \in X : \rho(y, x) < r\}$ for all $x \in X$ and all $r > 0$ form a basis.

The following spaces of homogeneous type are variants of those introduced by Coifman and Weiss in [2].

Definition 1.1. [13] *Let $d > 0$ and $0 < \theta \leq 1$. A space of homogeneous type $(X, \rho, \mu)_{d,\theta}$ is a set X together with a quasi-metric ρ and a nonnegative Borel measure μ on X with* $\text{supp}\mu = X$*, and there exists a constant $0 < C < \infty$ such that for all $0 < r < \text{diam}X$ and all $x, x', y \in X$,*

$$\mu(B(x, r)) \sim r^d, \tag{1.1}$$

$$|\rho(x, y) - \rho(x', y)| \leq C\rho(x, x')^\theta[\rho(x, y) + \rho(x', y)]^{1-\theta}. \tag{1.2}$$

In [14], Macías and Segovia have proved that one can replace a quasi-metric ρ of spaces of homogeneous type in the sense of Coifman and Weiss by another quasi-metric ϱ which yields the same topology on X as ρ such that (X, ϱ, μ) is the space defined by Definition 1.1 with $d = 1$.

Suppose that T is a continuous linear mapping from $C_0^\eta(X)$ to $(C_0^\eta(X))'$, associated to a kernel $K(x, y)$ in the following sense that

$$\langle Tf, g\rangle = \int\int g(x)K(x, y)f(y)d\mu(x)d\mu(y)$$

for all test functions f and g in C_0^η with disjoint supports.

Assume that $K(x, y)$ satisfies the pointwise conditions:

$$|K(x, y)| \leq C\rho(x, y)^{-d} \text{ for } \rho(x, y) \neq 0, \tag{1.3}$$

$$|K(x, y)| \leq C\rho(x, y)^{-d-\sigma} \text{ for } \rho(x, y) \geq 1, \tag{1.4}$$

$$|K(x, y) - K(x', y)| \leq C\rho(x, x')^\epsilon\rho(x, y)^{-d-\epsilon} \text{ for } \rho(x, x') \leq \frac{\rho(x, y)}{(2A)}, \tag{1.5}$$

$$|K(x, y) - K(x, y')| \leq C\rho(y, y')^\epsilon\rho(x, y)^{-d-\epsilon} \text{ for } \rho(y, y') \leq \frac{\rho(x, y)}{(2A)}, \tag{1.6}$$

where $\epsilon \in (0,\theta), \sigma > 0$.

The conditions (1.3)-(1.6) are natural when one considers the boundedness of Calderón-Zygmund operators on inhomogeneous function spaces, which were pointed out by Meyer in [16].

Assume also that T satisfies the Weak Boundedness Property, denote this by $T \in \mathrm{WBP}$,

$$|\langle Tf, g\rangle| \le Cr^{d+2\eta}\|f\|_{C_0^\eta(X)}\|g\|_{C_0^\eta(X)}$$

for all f and g in $C_0^\eta(X)$ with diameters of supports not greater than r.

To state the definition of the inhomogeneous Besov and Triebel-Lizorkin spaces, we need the following definitions. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Definition 1.2. [9] *A sequence $\{S_k\}_{k\in\mathbb{Z}_+}$ of operators is said to be an approximation to the identity if $S_k(x,y)$, the kernel of S_k, are functions from $X \times X$ into $\mathbb{C}$ such that for all $k \in \mathbb{Z}_+$ and all x, x', y and y' in X, and some $0 < \epsilon \le \theta$ and $C > 0$,*

$$|S_k(x,y)| \le C\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}}; \tag{1.7}$$

$$|S_k(x,y) - S_k(x',y)| \le C\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}} \tag{1.8}$$

for $\rho(x,x') \le \frac{1}{2A}(2^{-k}+\rho(x,y))$;

$$|S_k(x,y) - S_k(x,y')| \le C\left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon}\frac{2^{-k\epsilon}}{(2^{-k}+\rho(x,y))^{d+\epsilon}} \tag{1.9}$$

for $\rho(y,y') \le \frac{1}{2A}(2^{-k}+\rho(x,y))$;

$$\begin{aligned}&\big|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]\big|\\ &\le C\left(\frac{\rho(x,x')}{2^{-k}+\rho(x,y)}\right)^{\epsilon'}\left(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\right)^{\epsilon'}\frac{2^{-k\sigma}}{(2^{-k}+\rho(x,y))^{d+\sigma}}\end{aligned}$$

for $0 < \epsilon' < \epsilon, \sigma = \epsilon - \epsilon' > 0, \rho(x,x') \le \frac{1}{2A}(2^{-k}+\rho(x,y))$ and $\rho(y,y') \le \frac{1}{2A}(2^{-k}+\rho(x,y))$;

$$\int S_k(x,y)d\mu(x) = 1 \tag{1.10}$$

for all $k \in \mathbb{Z}_+$;

$$\int S_k(x,y)d\mu(y) = 1 \tag{1.11}$$

for all $k \in \mathbb{Z}_+$.

Definition 1.3. [12] *Fix two exponents $0 < \beta \le \theta$ and $\gamma > 0$. A function f defined on X is said to be a test function of type (β,γ) centered at $x_0 \in X$ with width $d > 0$ if f satisfies the following conditions:*

$$|f(x)| \le C\frac{d^\gamma}{(d+\rho(x,x_0))^{d+\gamma}}; \tag{1.12}$$

$$|f(x)-f(x')| \le C\left(\frac{\rho(x,x')}{d+\rho(x,x_0)}\right)^{\beta}\frac{d^{\gamma}}{(d+\rho(x,x_0))^{d+\gamma}} \tag{1.13}$$

for $\rho(x,x') \le \frac{1}{2A}(d+\rho(x,x_0))$.

If f is a test function of type (β,γ) centered at x_0 with width $d>0$, we write $f \in \mathcal{M}(x_0,d,\beta,\gamma)$, and the norm of f in $\mathcal{M}(x_0,d,\beta,\gamma)$ is defined by

$$\|f\|_{\mathcal{M}(x_0,d,\beta,\gamma)} = \inf\{C \ge 0 : (1.12) \text{ and } (1.13) \text{ hold}\}.$$

We denote by $\mathcal{M}(\beta,\gamma)$ the class of all $f \in \mathcal{M}(x_0,1,\beta,\gamma)$. It is easy to see that $\mathcal{M}(x_1,d,\beta,\gamma)=\mathcal{M}(\beta,\gamma)$ with equivalent norms for all $x_1 \in X$ and $d>0$. Furthermore, it is also easy to check that $\mathcal{M}(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{M}(\beta,\gamma)$. We denote by $(\mathcal{M}(\beta,\gamma))'$ the dual space of $\mathcal{M}(\beta,\gamma)$ consisting of all linear functionals $\mathcal{L}$ from $\mathcal{M}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists a constant C such that for all $f \in \mathcal{M}(\beta,\gamma)$,

$$|\mathcal{L}(f)| \le C\|f\|_{\mathcal{M}(\beta,\gamma)}.$$

We denote by $\langle h,f\rangle$ the natural pairing of elements $h \in (\mathcal{M}(\beta,\gamma))'$ and $f \in \mathcal{M}(\beta,\gamma)$. Since $\mathcal{M}(x_1,d,\beta,\gamma)=\mathcal{M}(\beta,\gamma)$ with the equivalent norms for all $x_1 \in X$ and $d>0$, thus, for all $h \in (\mathcal{M}(\beta,\gamma))', \langle h,f\rangle$ is well defined for all $f \in \mathcal{M}(x_0,d,\beta,\gamma)$ with $x_0 \in X$ and $d>0$. In what follows, we let $\widetilde{\mathcal{M}}(\beta,\gamma)$ be the completion of $\mathcal{M}(\theta,\theta)$ in $\mathcal{M}(\beta,\gamma)$ when $0<\beta,\gamma<\theta$.

We also need the following construction of Christ in [1], which provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type.

Lemma 1.4. *Let X be a space of homogeneous type. Then there exist a collection $\{Q_\alpha^k \subset X : k \in \mathbb{Z}_+, \alpha \in I_k\}$ of open subsets, where I_k is some (possible finite) index set, and constants $\delta \in (0,1)$ and $C_1, C_2 > 0$ such that*
(i) $\mu(X\setminus \cup_\alpha Q_\alpha^k)=0$ *for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ if $\alpha \ne \beta$;*
(ii) *for any α,β,k,l with $l \ge k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$;*
(iii) *for each (k,α) and each $l<k$ there is a unique β such that $Q_\alpha^k \subset Q_\beta^l$;*
(iv) $\operatorname{diam}(Q_\alpha^k) \le C_1\delta^k$*;*
(v) *each Q_α^k contains some ball $B(z_\alpha^k, C_2\delta^k)$, where $z_\alpha^k \in X$.*

In fact, we can think of Q_α^k as being a dyadic cube with diameter roughly δ^k and centered at z_α^k. In what follows, we always suppose $\delta = 1/2$. See [12] for how to remove this restriction. Also, in the following, for $k \in \mathbb{Z}_+, \tau \in I_k$, we will denote by $Q_\tau^{k,\nu}, \nu = 1,\ldots,N(k,\tau,M)$, the set of all cubes $Q_{\tau'}^{k+M} \subset Q_\tau^k$, where M is a fixed large positive integer.

Now, we can introduce the inhomogeneous Besov spaces $B_p^{\alpha,q}(X)$ and Triebel-Lizorkin spaces $F_p^{\alpha,q}(X)$ via approximations to the identity.

Definition 1.5. *Suppose that $-\theta<\alpha<\theta$, and β and γ satisfying*

$$\max(0,-\alpha+\max(0,d(1/p-1))) < \beta < \theta, 0<\gamma<\theta. \tag{1.14}$$

Suppose $\{S_k\}_{k\in\mathbb{Z}_+}$ *is an approximation to identity and let* $D_0 = S_0$, *and* $D_k = S_k - S_{k-1}$ *for* $k \in \mathbb{N}$. *Let* M *be a fixed large positive integer,* $Q_\tau^{0,\nu}$ *be as above. Inhomogeneous Besov space* $B_p^{\alpha,q}(X)$ *for* $\max\left(\frac{d}{d+\theta}, \frac{d}{d+\theta+\alpha}\right) < p \le \infty$, $0 < q \le \infty$ *is the collection of all* $f \in (\widetilde{\mathcal{M}}(\beta,\gamma))'$ *such that*

$$\|f\|_{B_p^{\alpha,q}(X)} = \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p\right\}^{\frac{1}{p}} + \left\{\sum_{k=1}^{\infty}\left[2^{k\alpha}\|D_k(f)\|_{L^p(X)}\right]^q\right\}^{\frac{1}{q}} < \infty.$$

Inhomogeneous Triebel-Lizorkin space $F_p^{\alpha,q}(X)$ *for* $\max\left(\frac{d}{d+\theta}, \frac{d}{d+\theta+\alpha}\right) < p < \infty$ *and* $\max\left(\frac{d}{d+\theta}, \frac{d}{d+\theta+\alpha}\right) < q \le \infty$ *is the collection of* $f \in (\widetilde{\mathcal{M}}(\beta,\gamma))'$ *such that*

$$\|f\|_{F_p^{\alpha,q}(X)} = \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p\right\}^{\frac{1}{p}} + \left\|\left\{\sum_{k=1}^{\infty}\left[2^{k\alpha}|D_k(f)|\right]^q\right\}^{\frac{1}{q}}\right\|_{L^p(X)} < \infty,$$

where $m_{Q_\tau^{0,\nu}}(D_0(f))$ *are averages of* $D_0(f)$ *over* $Q_\tau^{0,\nu}$.

$T1$ theorems for inhomogeneous Besov and Triebel-Lizorkin spaces were proved in [10]. Roughly speaking, T is bounded on $B_p^{\alpha,q}, 1 \le p,q \le \infty$ and $0 < \alpha < \epsilon$, and on $F_p^{\alpha,q}, 1 < p,q < \infty$ and $0 < \alpha < \epsilon$, if T has the weak boundedness property, $T1 = 0$ and the conditions (1.3)–(1.5) hold in [10]. In this paper, we will prove the following results.

Theorem A. *Let* $0 < \epsilon \le \theta, 0 < \alpha < \epsilon$. *Suppose that* $T(1) = 0$, $T \in$ *WBP, and* $K(x,y)$, *the kernel of* T, *satisfies* $(1.3)-(1.5)$ *with* $\sigma > \max(0, d(\frac{1}{p}-1))$. *Then* T *is bounded on* $B_p^{\alpha,q}(X)$, *for* $\frac{d}{d+\alpha} < p \le \infty, 0 < q \le \infty$, *and on* $F_p^{\alpha,q}(X)$, *for* $\frac{d}{d+\alpha} < p < \infty, \frac{d}{d+\alpha} < q \le \infty$.

Theorem B. *Let* $0 < \epsilon \le \theta, -\epsilon < \alpha < 0$. *Suppose that* $T^*(1) = 0, T \in$ *WBP, and* $K(x,y)$, *the kernel of* T, *satisfies* (1.3), (1.4) *and* (1.6) *with* $\sigma > \max(0, d(\frac{1}{p}-1))$. *Then* T *is bounded on* $B_p^{\alpha,q}(X)$, *for* $\frac{d}{d+\alpha+\epsilon} < p \le \infty, 0 < q \le \infty$, *and on* $F_p^{\alpha,q}(X)$, *for* $\frac{d}{d+\alpha+\epsilon} < p < \infty, \frac{d}{d+\alpha+\epsilon} < q \le \infty$.

Theorems A and B are to give a uniform treatment in [10]. To be precise, to deal with the case where $0 < \alpha < \epsilon, p, q > 1$, the main tools used were the continuous Calderón reproducing formula. The proof of the case where

$-\epsilon < \alpha < 0$, and $p,\ q > 1$ then follows from the duality argument. However, the continuous Calderón reproducing formula and duality argument do not work for the cases where either p or q, or both p and q are less than or equal to 1. The key point of the present paper is to use discrete Calderón reproducing formula and Plancherel-Pôlya characterization of the Besov and Triebel-Lizorkin spaces developed in [6, 11]. $T1$ theorems for inhomogeneous Triebel-Lizorkin space $F_p^{\alpha,q}(X)$ with $-\epsilon < \alpha < \epsilon$, $\max\{\frac{d}{d+\epsilon}, \frac{d}{d+\alpha+\epsilon}\} < p < \infty$ and $\max\{\frac{d}{d+\epsilon}, \frac{d}{d+\alpha+\epsilon}\} < q \leq \infty$ in [17] are also stated, if T has the weak boundedness property, $T(1) = 0$, $T^*(1) = 0$ and the conditions (1.3)–(1.6) hold. Furthermore, by use of the real interpolation theorems the author obtained the $T1$ theorem for inhomogeneous Besov space $B_p^{\alpha,q}(X)$ with $-\epsilon < \alpha < \epsilon$, $\max\{\frac{d}{d+\epsilon}, \frac{d}{d+\alpha+\epsilon}\} < p \leq \infty$ and $0 < q \leq \infty$ under the same conditions. The range of index p and q has the change from $\frac{d}{d+\epsilon}$ to $\frac{d}{d+\alpha}$ with $0 < \alpha < \epsilon$, the main reason is that the smoothness and moment conditions of Theorems A and B decrease a half compared with the corresponding results of [17].

2. Proofs of Theorems A and B

The basic tool to show main results is the discrete Calderón reproducing formulae in [6]. It can be stated as follows.

Lemma 2.1. *Suppose that $\{S_k\}_{k\in\mathbb{Z}_+}$ is an approximation to the identity as in Definition 1.2. Set $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. Then there exist functions $\widetilde{D}_{Q_\tau^{0,\nu}}, \tau \in I_0$ and $\nu \in \{1, \dots, N(0,\tau,M)\}$ and $\{\widetilde{D}_k(x,y)\}_{k\in\mathbb{N}}$ such that for any fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$, $k \in \mathbb{N}, \tau \in I_k$ and $\nu \in \{1, \dots, N(k,\tau,M)\}$ and all $f \in (\mathcal{M}(\beta,\gamma))'$ with $0 < \beta, \gamma < \theta$,*

$$
\begin{aligned}
f(x) = &\sum_{\tau\in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) m_{Q_\tau^{0,\nu}}(D_0(f)) \widetilde{D}_{Q_\tau^{0,\nu}}(x) \\
&+ \sum_{k\in\mathbb{Z}_+} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau,M)} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k(x, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}), \qquad (2.1)
\end{aligned}
$$

where $\operatorname{diam}(Q_\tau^{k,\nu}) \sim 2^{k+M}$ for $k \in \mathbb{Z}_+, \tau \in I_k, \nu \in \{1, \dots, N(k,\tau,M)\}$ and a fixed large $M \in \mathbb{N}$, the series converges in the norm of $L^p(X), 1 < p < \infty$, and $\mathcal{M}(\beta',\gamma')$ for $f \in \mathcal{M}(\beta,\gamma)$ with $\beta' < \beta$ and $\gamma' < \gamma$, and $(\mathcal{M}(\beta',\gamma'))'$ for $f \in (\mathcal{M}(\beta,\gamma))'$ with $\theta > \beta' > \beta$ and $\theta > \gamma' > \gamma$. Moreover, $\widetilde{D}_k(x,y), k \in \mathbb{N}$, satisfies for any given $\epsilon \in (0,\theta)$, all $x, y \in X$ the following conditions:

$$
|\widetilde{D}_k(x,y)| \leq C \frac{2^{-k\epsilon'}}{(2^{-k} + \rho(x,y))^{d+\epsilon'}}; \qquad (2.2)
$$

$$
|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \leq C \left(\frac{\rho(x,x')}{2^{-k} + \rho(x,y)}\right)^{\epsilon'} \frac{2^{-k\epsilon'}}{(2^{-k} + \rho(x,y))^{d+\epsilon'}} \qquad (2.3)
$$

for $\rho(x,x') \le \frac{1}{2A}(2^{-k}+\rho(x,y))$;

$$\int_X \widetilde{D}_k(x,y)d\mu(y) = \int_X \widetilde{D}_k(x,y)d\mu(x) = 0$$

for all $k \in \mathbb{Z}_+$.

$\widetilde{D}_{Q_\tau^{0,\nu}}(x)$ *for* $\tau \in I_0$ *and* $\nu \in \{1,\dots,N(0,\tau,M)\}$ *satisfies*

$$\int_X \widetilde{D}_{Q_\tau^{0,\nu}}(x)d\mu(x) = 1,$$

$$|\widetilde{D}_{Q_\tau^{0,\nu}}(x)| \le \frac{C}{(1+\rho(x,y))^{d+\epsilon}} \tag{2.4}$$

for all $x \in X$ *and* $y \in Q_\tau^{0,\nu}$ *and*

$$|\widetilde{D}_{Q_\tau^{0,\nu}}(x) - \widetilde{D}_{Q_\tau^{0,\nu}}(z)| \le C\left(\frac{\rho(x,z)}{1+\rho(x,y)}\right)^{\epsilon} \frac{1}{(1+\rho(x,y))^{d+\epsilon}} \tag{2.5}$$

for all $x, z \in X$ *and* $y \in Q_\tau^{0,\nu}$ *satisfying* $\rho(x,z) \le \frac{1}{2A}(1+\rho(x,y))$; *the constant* C *in* (2.2) – (2.5) *is independent of* M.

To prove Theorem A and Theorem B, we need the following lemmas. Their proofs are similar to that of Lemma 4.1 in [10].

Lemma 2.2. *With notation as in Lemma* 2.1 *and Theorem A, then*

(i) *for* $k \in \mathbb{Z}_+$, $\tau' \in I_0$ *and* $\nu' \in \{1,\dots,N(0,\tau',M)\}$, $y_{\tau'}^{0,\nu'}$ *is any fixed point of* $Q_{\tau'}^{0,\nu'}$, $x \in X$,

$$|D_k T\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(x)| \le C(1+k)2^{-k\epsilon}\frac{1}{(1+\rho(x,y_{\tau'}^{0,\nu'}))^{d+\sigma'}} \tag{2.6}$$

where $\sigma' = \sigma$ *when* $k = 0$ *and* $\sigma' = \epsilon$ *when* $k \in \mathbb{N}$,

(ii) *for* $k \in \mathbb{Z}_+, k' \in \mathbb{N}$, $x, y \in X$,

$$|D_k T\widetilde{D}_{k'}(x,y)| \le C[1+|k-k'|]\left(2^{(k'-k)\epsilon'} \wedge 1\right)\frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')}+\rho(x,y))^{d+\epsilon'}}. \tag{2.7}$$

Lemma 2.3. *With notation as in Lemma* 2.1 *and Theorem B, then*

(i) *for* $k \in \mathbb{Z}_+$, $\tau' \in I_0$ *and* $\nu' \in \{1,\dots,N(0,\tau',M)\}$, $y_{\tau'}^{0,\nu'}$ *is any fixed point of* $Q_{\tau'}^{0,\nu'}$, $x \in X$,

$$|D_k T\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(x)| \le C\frac{1}{(1+\rho(x,y_{\tau'}^{0,\nu'}))^{d+\sigma'}} \tag{2.8}$$

where $\sigma' = \sigma$ *when* $k = 0$ *and* $\sigma' = \epsilon$ *when* $k \in \mathbb{N}$, *(ii) for $k \in \mathbb{Z}_+, k' \in \mathbb{N}$, $x, y \in X$,*

$$|D_k T\widetilde{D}_{k'}(x,y)| \le C[1+|k-k'|]\left(2^{(k-k')\epsilon'} \wedge 1\right)\frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')}+\rho(x,y))^{d+\epsilon'}}. \tag{2.9}$$

Proof of Theorem A. By Lemma 2.1 and Theorem 1.5 in [6], for $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$, we write

$$\begin{aligned}
\|T(f)\|_{B_p^{\alpha,q}(X)} \le & \Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\Big[m_{Q_\tau^{0,\nu}}\big(\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}\mu(Q_{\tau'}^{0,\nu'}) \\
& |D_0T\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(\cdot)|m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)\big)\Big]^p\Big\}^{\frac{1}{p}} \\
& +\Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\big[m_{Q_\tau^{0,\nu}}\big(\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'}) \\
& |D_0T\widetilde{D}_{k'}(\cdot,y_{\tau'}^{k',\nu'})||D_{k'}(f)(y_{\tau'}^{k',\nu'})|\big)\big]^p\Big\}^{\frac{1}{p}} \\
& +\Big\{\sum_{l=1}^{\infty}\big(\sum_{\tau\in I_l}\sum_{\nu=1}^{N(l,\tau,M)}\big[\inf_{z\in Q_\tau^{l,\nu}}\big(\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}\mu(Q_{\tau'}^{0,\nu'}) \\
& \times\mu(Q_\tau^{l,\nu})^{-\frac{\alpha}{d}+\frac{1}{p}}|D_lT\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(z)|m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)\big)\big]^p\big)^{\frac{q}{p}}\Big\}^{\frac{1}{q}} \\
& +\Big\{\sum_{l=1}^{\infty}\big(\sum_{\tau\in I_l}\sum_{\nu=1}^{N(l,\tau,M)}\big[\inf_{z\in Q_\tau^{l,\nu}}\big(\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'}) \\
& \times\mu(Q_\tau^{l,\nu})^{-\frac{\alpha}{d}+\frac{1}{p}}|D_lT\widetilde{D}_{k'}(z,y_{\tau'}^{k',\nu'})||D_{k'}(f)(y_{\tau'}^{k',\nu'})|\big)\big]^p\big)^{\frac{q}{p}}\Big\}^{\frac{1}{q}} \\
\doteq\ & A_1+A_2+A_3+A_4.
\end{aligned}$$

The estimate of A_4 is similar to Theorem 1 in [5]. It remains to deduce the estimates of A_1, A_2 and A_3.

From (2.6), the Hölder inequality for $p>1$ and $(a+b)^p \le a^p+b^p$ for $p \le 1$, we deduce

$$\begin{aligned}
A_1 \le & C\Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}\Big[\frac{1}{\big(1+\rho(y_\tau^{0,\nu},y_{\tau'}^{0,\nu'})\big)^{d+\sigma'}}\Big]^{p\wedge 1} \\
& [m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)]^p\Big\}^{\frac{1}{p}} \\
\le & C\Big\{\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}[m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)]^p\Big\}^{\frac{1}{p}} \\
\le & C\ \|f\|_{B_p^{\alpha,q}(X)}
\end{aligned}$$

where $y_\tau^{0,\nu}$ is any point of $Q_\tau^{0,\nu}$, $y_{\tau'}^{0,\nu'}$ is any point of $Q_{\tau'}^{0,\nu'}$.

By (2.7), it follows that

$$\begin{aligned}
A_2 \le\ & C\Big\{ \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau',M)} \sum_{\tau\in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \Big[2^{-k'd}2^{-k'\alpha}[1+k'] \\
& \times \frac{1}{\big(1+\rho(y_\tau^{0,\nu}, y_{\tau'}^{k',\nu'})\big)^{d+\epsilon'}}\Big]^{p\wedge 1} \Big[\mu(Q_{\tau'}^{k',\nu'})^{-\frac{\alpha}{d}} |D_{k'}(f)(y_{\tau'}^{k',\nu'})|\Big]^p \Big\}^{\frac{1}{p}} \\
\le\ & C\Big\{ \sum_{k'=1}^{\infty} \Big(\big[2^{-k'd}2^{-k'\alpha}(1+k')\big]^{p\wedge 1} 2^{k'd}\Big)^{\frac{q}{p}\wedge 1} \\
& \times \Big(\sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau',M)} \Big[\mu(Q_{\tau'}^{k',\nu'})^{-\frac{\alpha}{d}+\frac{1}{p}} \sup_{z\in Q_{\tau'}^{k',\nu'}} |D_{k'}(f)(z)|\Big]^p\Big)^{\frac{q}{p}}\Big\}^{\frac{1}{q}} \\
\le\ & C\|f\|_{B_p^{\alpha,q}(X)},
\end{aligned}$$

where these inequalities follow from the fact that

$$\sum_{k'=1}^{\infty} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau',M)} 2^{-k'd}2^{-k'\alpha}[1+k'] \frac{1}{\big(1+\rho(y_\tau^{0,\nu}, y_{\tau'}^{k',\nu'})\big)^{d+\epsilon'}} \le C,$$

$$\sum_{k'=1}^{\infty} \big[2^{-k'd}2^{-k'\alpha}(1+k')\big]^{p\wedge 1} 2^{k'd} + \sum_{k'} \Big(\big[2^{-k'd}2^{-k'\alpha}(1+k')\big]^{p\wedge 1}\Big)^{\frac{q}{p}\wedge 1} \le C$$

and the last inequality follows from the Plancherel-Pôlya characterization of the Besov spaces [6].

By (2.6), it follows that

$$\begin{aligned}
A_3 \le\ & C\Big\{ \sum_{l=1}^{\infty} \Big(2^{-dl} \sum_{\tau\in I_l} \sum_{\nu=1}^{N(l,\tau,M)} \sum_{\tau'\in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'}) \big[m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)\big]^p \\
& \times \Big[2^{l\alpha}(1+l)2^{-l\epsilon} \frac{1}{\Big(1+\rho(y_\tau^{l,\nu}, y_{\tau'}^{0,\nu'})\Big)^{d+\epsilon}}\Big]^{p\wedge 1}\Big)^{\frac{q}{p}}\Big\}^{\frac{1}{q}} \\
\le\ & C\Big\{ \sum_{l=1}^{\infty} \big[2^{l\alpha}(1+l)2^{-l\epsilon}\big]^{(p\wedge 1)\frac{q}{p}} \Big(\sum_{\tau'\in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'}) \big[m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)\big]^p\Big)^{\frac{q}{p}}\Big\}^{\frac{1}{q}} \\
\le\ & C\|f\|_{B_p^{\alpha,q}(X)}.
\end{aligned}$$

Similarly, for $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$, we have

$$
\begin{aligned}
&\|T(f)\|_{F_p^{\alpha,q}(X)} \\
&\leq \Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q_\tau^{0,\nu})\Big[m_{Q_\tau^{0,\nu}}\Big(\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}\mu(Q_{\tau'}^{0,\nu'}) \\
&\qquad |D_0T\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(\cdot)|m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)\Big)\Big]^p\Big\}^{\frac{1}{p}} \\
&\quad+\Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)}\Big[m_{Q_\tau^{0,\nu}}\Big(\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'}) \\
&\qquad |D_0\;T\widetilde{D}_{k'}(\cdot,y_{\tau'}^{k',\nu'})||D_{k'}(f)(y_{\tau'}^{k',\nu'})|\Big)\Big]^p\Big\}^{\frac{1}{p}} \\
&\quad+\Big\|\Big\{\sum_{l=1}^{\infty}\sum_{\tau\in I_l}\sum_{\nu=1}^{N(l,\tau,M)}\Big[\inf_{z\in Q_\tau^{l,\nu}}\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}\mu(Q_\tau^{l,\nu})^{-\frac{\alpha}{d}} \\
&\qquad |D_lT\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(z)|m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)\chi_{Q_\tau^{l,\nu}}\Big]^q\Big\}^{\frac{1}{q}}\Big\|_{L^p(X)} \\
&\quad+\Big\|\Big\{\sum_{l=1}^{\infty}\sum_{\tau\in I_l}\sum_{\nu=1}^{N(l,\tau,M)}\Big[\inf_{z\in Q_\tau^{l,\nu}}|\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'}) \\
&\quad\times\mu(Q_\tau^{l,\nu})^{-\frac{\alpha}{d}}D_lT\widetilde{D}_{k'}(\cdot,y_{\tau'}^{k',\nu'})(z)D_{k'}(f)(y_{\tau'}^{k',\nu'})|\chi_{Q_\tau^{l,\nu}}\Big]^q\Big\}^{\frac{1}{q}}\Big\|_{L^p(X)} \\
&\doteq\; B_1+B_2+B_3+B_4,
\end{aligned}
$$

where $y_{\tau'}^{k',\nu'}$ are any point in $Q_{\tau'}^{k',\nu'}$.

The estimates of B_1 and B_4 are similar to A_1 above and Theorem 2 in [5], respectively. It remains to deduce the estimates of B_2 and B_3.

From (2.7), the Hölder inequality for $q>1$ and $(a+b)^q\leq a^q+b^q$ for $q\leq 1$, Lemma A.2 in [8], the Fefferman-Stein vector-valued inequality in [7], it follows that

$$
\begin{aligned}
B_2\leq\;& C\Big\{\Big[\sum_{k'=1}^{\infty}2^{-k'd}2^{-k'\alpha}[1+k']2^{\frac{k'd}{r}} \\
&\times\Big[\mathrm{M}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})^{-\frac{\alpha}{d}}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|\chi_{Q_{\tau'}^{k',\nu'}}\Big)^r\Big]^{\frac{1}{r}}\Big]^q\Big\}^{\frac{1}{q}}\Big\|_{L^p(X)} \\
\leq\;& C\Big\|\Big\{\sum_{k'=1}^{\infty}\Big[2^{-k'd}2^{-k'\alpha}[1+k']2^{\frac{k'd}{r}}\Big]^{q\wedge 1} \\
&\times\Big[\mathrm{M}\Big(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})^{-\frac{\alpha}{d}}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|\chi_{Q_{\tau'}^{k',\nu'}}\Big)^r\Big]^{\frac{q}{r}}\Big\}^{\frac{1}{q}}\Big\|_{L^p(X)}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\Big\|\Big\{\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\Big[\mu(Q_{\tau'}^{k',\nu'})^{-\frac{\alpha}{d}}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|\chi_{Q_{\tau'}^{k',\nu'}}\Big]^q\Big\}^{\frac{1}{q}}\Big\|_{L^p(X)}\\
&\le C\|f\|_{F_p^{\alpha,q}(X)},
\end{aligned}
$$

where $\frac{d}{d+\alpha} < r < \min(p,q,1)$.

From (2.6), the Hölder inequality for $p > 1$ and $(a+b)^p \le a^p + b^p$ for $p \le 1$, the Lemma A.2 in [7], it follows that

$$
\begin{aligned}
B_3 \le\ & C\Big\{\int\Big(\sum_{l=1}^{\infty}\sum_{\tau\in I_l}\sum_{\nu=1}^{N(l,\tau,M)}\chi_{Q_\tau^{l,\nu}}(x)\Big[2^{l\alpha}(1+l)2^{-l\epsilon}\\
&\times\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}\frac{1}{\Big(1+\rho(x,y_{\tau'}^{0,\nu'})\Big)^{d+\epsilon}}m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)\Big]^q\Big)^{\frac{p}{q}}d\mu(x)\Big\}^{\frac{1}{p}}\\
\le\ & C\Big\{\int\Big(\sum_{l=1}^{\infty}2^{l\alpha q}(1+l)^q 2^{-l\epsilon q}\Big[\mathrm{M}\Big(\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}m_{Q_{\tau'}^{0,\nu'}}\\
&(|D_0(f)|)\chi_{Q_{\tau'}^{0,\nu'}}\Big)^r(x)\Big]^{\frac{q}{r}}\Big)^{\frac{p}{q}}d\mu(x)\Big\}^{\frac{1}{p}}\\
\le\ & C\Big\{\int\Big[\mathrm{M}\Big(\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)\chi_{Q_{\tau'}^{0,\nu'}}\Big)^r(x)\Big]^{\frac{p}{r}}d\mu(x)\Big\}^{\frac{1}{p}}\\
\le\ & C\|f\|_{F_p^{\alpha,q}(X)},
\end{aligned}
$$

where we used the $L^{\frac{p}{r}}(X)$ boundedness of Hardy-Littlewood maximal functions. This proves Theorem A. ∎

Proof of Theorem B. The main difference of proof between Theorem B and Theorem A is that we should replace Lemma 2.2 by Lemma 2.3. We leave the details to the reader. ∎

References

1. M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, *Colloq Math.* **60/61** (1990), 601-628.
2. R. R. Coifman and G. Weiss, *Analyse Harmonique Noncommutative Sur Certains Espaces Homogenes, Lecture Notes in Math, 242*, Springer-Verlag, Berlin and New York, 1971.
3. G. David and J. L. Journé, A boundedness criterion for generalized Calderón-Zygmund operators, *Ann. of Math.* **120** (1984), 371-397.
4. Y. C. Han, New characterizations of inhomogeneous Besov and Triebel-Lizorkin spaces over spaces of homogeneous type, Submitted.

5. D. G. Deng and Y. S. Han, $T1$ Theorem for Besov and Triebel-Lizorkin spaces, *Science in China Ser.A Math.* **34** (2004), 657-664.
6. D. G. Deng, Y. S. Han and D. C. Yang, Inhomogeneous Plancherel-Pôlya type inequality on spaces of homogeneous type and their applications, *Communications in Contemporary Mathematics,* **6** (2004), 221-243.
7. C. Feferman and E. M. Stein, Some maximal inequalities, *Amer. J. Math.,* **93** (1971), 107-115.
8. M. Frazier and B. Jawerth, A discrete transform and decompositions of distribution spaces, *J. Funct. Anal.* **93** (1990), 34-170.
9. Y. S. Han, Plancherel-Pôlya type inequality on spaces of homogeneous type and its applications, *Proc. Amer. Math. Soc.,* **126** (1998), 3315-3327.
10. Y. S. Han, S. Z. Lu and D. C. Yang, Inhomogeneous Besov and Triebel-Lizorkin space on spaces of homogeneous type, *Approx. Theory Appl.* **15** (1999), 37-65.
11. Y. S. Han, S. Z. Lu and D. C. Yang, Inhomogeneous discrete Calderón reproducing formulas for spaces of homogeneous type, *J. Fourier Anal. Appl.*, **7** (2001), 571-600.
12. Y. S. Han and E. T. Sawyer, Littlewood-Paley theory on spaces of homogeneous type and classical function spaces, *Mem. Amer. Math. Soc.,* **110** (1994), 1-126.
13. Y. S. Han and D. C. Yang, New characterizations and applications of inhomogeneous Besov and Triebel-Lizorkin space on spaces of homogeneous type and fractals, *Dissertationes Math. (Rozprawy Mat.)* **403** (2002), 1-102.
14. P. G. Lemarie, Continuite sur les espaces de Besov des operateurs definis par des integrales, *Ann. Inst. Fouier(Grenoble)*, **4** (1985), 175-187.
15. R. A. Macias and C. Segovia, Lipschitz function on spaces of homogeneous type, *Adv. Math.* **33** (1979), 257-270.
16. Y. Meyer, *Wavelets and Operators, (Cambridge Studies in Advanced Mathematics: 37)*, Cambridge University Press, 1992.
17. D. C. Yang, $T1$ Theorem on Besov and Triebel-Lizorkin spaces on spaces of homogeneous type and their applications. *Zeitschrift für Analysis ihre Anwendungen*, **22** (2003), 53-72.