T1 Theorems for Inhomogeneous Besov and Triebel-Lizorkin Spaces over Space of Homogeneous Type

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Abstract. The author establishes T1 theorems for inhomogeneous Besov and Triebel-Lizorkin spaces by discrete Calderón type reproducing formula and the Plancherel-Pôlya characterization for inhomogeneous Besov and Triebel-Lizorkin spaces. These results are new even for \( \mathbb{R}^d \).

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1. Introduction

In the past years, there has been significant progress on the problem of proving the boundedness of generalized Calderón-Zygmund operators on various function spaces. A remarkable result is the famous T1 theorem of David and Journé in [3]. T1 theorem has been extended for Besov and Triebel-Lizorkin spaces. For a broader view of this active area of research, see e.g. [5, 10, 12–14, 16, 17] and references therein.

The main purpose of this paper is to establish T1 theorems for the inhomogeneous spaces \( B_{\alpha,q}^p(X) \) when \( \frac{d}{d+\alpha+\epsilon} < p \leq \infty, 0 < q \leq \infty, 0 < \alpha < \epsilon \) and for \( F_{\alpha,q}^p(X) \) when \( \frac{d}{d+\alpha+\epsilon} < p < \infty, \frac{d}{d+\alpha+\epsilon} < q \leq \infty, 0 < \alpha < \epsilon \), and for \( B_{\alpha,q}^p(X) \) when \( \frac{d}{d+\alpha+\epsilon} < p \leq \infty, 0 < q \leq \infty, -\epsilon < \alpha < 0 \) and for \( F_{\alpha,q}^p(X) \) when \( \frac{d}{d+\alpha+\epsilon} < p < \infty, \frac{d}{d+\alpha+\epsilon} < q \leq \infty, -\epsilon < \alpha < 0 \) for some \( \epsilon > 0 \) by discrete

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Calderón type reproducing formula and Plancherel-Pólya characterization for the inhomogeneous Besov and Triebel-Lizorkin spaces. Roughly speaking, $T$ is bounded on $B^{\alpha,q}_p(X)$ and $F^{\alpha,q}_p(X)$ for the range of $\alpha, p,$ and $q$ indicated above, respectively, if its kernel satisfies only half smoothness and moment conditions. An application of these results is given in [4].

To state main results of this paper, we begin by recalling the definitions necessary for inhomogeneous Besov and Triebel-Lizorkin spaces on spaces of homogeneous type and some basic facts about the Calderón-Zygmund operator theory. A quasi-metric $\rho$ on a set $X$ is a function $\rho : X \times X \to [0, \infty)$ satisfying:

(i) $\rho(x, y) = 0$ if and only if $x = y$;
(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
(iii) There exists a constant $A \in [1, \infty)$ such that for all $x, y, z \in X$,

$$\rho(x, y) \leq A[\rho(x, z) + \rho(z, y)].$$

Any quasi-metric defines a topology, for which the balls $B_\rho(x, r) = \{y \in X : \rho(x, y) < r\}$ for all $x \in X$ and all $r > 0$ form a basis.

The following spaces of homogeneous type are variants of those introduced by Coifman and Weiss in [2].

**Definition 1.1.** [13] Let $d > 0$ and $0 < \theta \leq 1$. A space of homogeneous type $(X, \rho, \mu)_{d, \theta}$ is a set $X$ together with a quasi-metric $\rho$ and a nonnegative Borel measure $\mu$ on $X$ such that $(X, \rho)$ is a space of homogeneous type in the sense of Coifman and Weiss by another quasi-metric $\rho^*$ which yields the same topology on $X$ as $\rho$ such that $(X, \rho, \mu)$ is the space defined by Definition 1.1 with $d = 1$.

Suppose that $T$ is a continuous linear mapping from $C_0^\infty(X)$ to $(C_0^\infty(X))'$, associated to a kernel $K(x, y)$ in the following sense that

$$\langle Tf, g \rangle = \int \int g(x)K(x, y)f(y)d\mu(x)d\mu(y)$$

for all test functions $f$ and $g$ in $C_0^\infty$ with disjoint supports.

Assume that $K(x, y)$ satisfies the pointwise conditions:

$$|K(x, y)| \leq C\rho(x, y)^{-d} \text{ for } \rho(x, y) \neq 0, \quad (1.3)$$

$$|K(x, y)| \leq C\rho(x, y)^{-d-\sigma} \text{ for } \rho(x, y) \geq 1, \quad (1.4)$$

$$|K(x, y) - K(x', y)| \leq C\rho(x, x')^\sigma\rho(x, y)^{-d-\epsilon} \text{ for } \rho(x, x') \leq \frac{\rho(x, y)}{(2A)}, \quad (1.5)$$

$$|K(x, y) - K(x, y')| \leq C\rho(y, y')^\sigma\rho(x, y)^{-d-\epsilon} \text{ for } \rho(y, y') \leq \frac{\rho(x, y)}{(2A)}, \quad (1.6)$$
where \( \epsilon \in (0, \theta) \), \( \sigma > 0 \).

The conditions (1.3)-(1.6) are natural when one considers the boundedness of Calderón-Zygmund operators on inhomogeneous function spaces, which were pointed out by Meyer in [16].

Assume also that \( T \) satisfies the Weak Boundedness Property, denote this by \( T \in \text{WBP} \):

\[
|\langle Tf, g \rangle| \leq C r^{d+2\eta} \|f\|_{C^\eta_0(X)} \|g\|_{C^\eta_0(X)}
\]

for all \( f \) and \( g \) in \( C^\eta_0(X) \) with diameters of supports not greater than \( r \).

To state the definition of the inhomogeneous Besov and Triebel-Lizorkin spaces, we need the following definitions. Let \( Z_+ = \mathbb{N} \cup \{0\} \).

**Definition 1.2.** [9] A sequence \( \{S_k\}_{k \in Z_+} \) of operators is said to be an approximation to the identity if \( S_k(x, y) \), the kernel of \( S_k \), are functions from \( X \times X \) into \( \mathbb{C} \) such that for all \( k \in Z_+ \) and all \( x, x', y \) and \( y' \) in \( X \), and some \( 0 < \epsilon \leq \theta \) and \( C > 0 \),

\[
|S_k(x, y)| \leq C \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x, y))^{d+\epsilon}};
\]

\[
|S_k(x, y) - S_k(x', y)| \leq C \left( \frac{\rho(x, x')}{2^{-k} + \rho(x, y)} \right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x, y))^{d+\epsilon}}
\]

for \( \rho(x, x') \leq \frac{1}{2\lambda} (2^{-k} + \rho(x, y)) \);

\[
|S_k(x, y) - S_k(x, y')| \leq C \left( \frac{\rho(y, y')}{2^{-k} + \rho(x, y)} \right)^\epsilon \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x, y))^{d+\epsilon}}
\]

for \( \rho(y, y') \leq \frac{1}{2\lambda} (2^{-k} + \rho(x, y)) \);

\[
|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]| \leq C \left( \frac{\rho(x, x')}{2^{-k} + \rho(x, y)} \right)^{\epsilon'} \left( \frac{\rho(y, y')}{2^{-k} + \rho(x, y)} \right)^{\epsilon'} \frac{2^{-k\sigma}}{(2^{-k} + \rho(x, y))^{d+\sigma}}
\]

for \( 0 < \epsilon' < \epsilon, \sigma = \epsilon - \epsilon' > 0, \rho(x, x') \leq \frac{1}{2\lambda} (2^{-k} + \rho(x, y)) \) and \( \rho(y, y') \leq \frac{1}{2\lambda} (2^{-k} + \rho(x, y)) \);

\[
\int S_k(x, y) d\mu(x) = 1
\]

(1.10)

for all \( k \in Z_+ \);

\[
\int S_k(x, y) d\mu(y) = 1
\]

(1.11)

for all \( k \in Z_+ \).

**Definition 1.3.** [12] Fix two exponents \( 0 < \beta \leq \theta \) and \( \gamma > 0 \). A function \( f \) defined on \( X \) is said to be a test function of type \((\beta, \gamma)\) centered at \( x_0 \in X \) with width \( d > 0 \) if \( f \) satisfies the following conditions:

\[
|f(x)| \leq C \frac{r^\beta}{(d + \rho(x, x_0))^{d+\gamma}};
\]

(1.12)
\[ |f(x) - f(x')| \leq C \left( \frac{\rho(x, x')}{(d + \rho(x, x_0))^\beta} \right)^{d^\gamma} \] (1.13)

for \( \rho(x, x') \leq \frac{1}{2A}(d + \rho(x, x_0)) \).

If \( f \) is a test function of type \((\beta, \gamma)\) centered at \( x_0 \) with width \( d > 0 \), we write \( f \in \mathcal{M}(x_0, d, \beta, \gamma) \), and the norm of \( f \) in \( \mathcal{M}(x_0, d, \beta, \gamma) \) is defined by

\[ \|f\|_{\mathcal{M}(x_0, d, \beta, \gamma)} = \inf\{C \geq 0 : (1.12) \text{ and } (1.13) \text{ hold}\} . \]

We denote by \( \mathcal{M}(\beta, \gamma) \) the class of all \( f \in \mathcal{M}(x_0, 1, \beta, \gamma) \). It is easy to see that \( \mathcal{M}(x_1, d, \beta, \gamma) = \mathcal{M}(\beta, \gamma) \) with equivalent norms for all \( x_1 \in X \) and \( d > 0 \). Furthermore, it is also easy to check that \( \mathcal{M}(\beta, \gamma) \) is a Banach space with respect to the norm in \( \mathcal{M}(\beta, \gamma) \). We denote by \( (\mathcal{M}(\beta, \gamma))' \) the dual space of \( \mathcal{M}(\beta, \gamma) \) consisting of all linear functionals \( L \) from \( \mathcal{M}(\beta, \gamma) \) to \( \mathbb{C} \) with the property that there exists a constant \( C \) such that for all \( f \in \mathcal{M}(\beta, \gamma) \),

\[ |L(f)| \leq C\|f\|_{\mathcal{M}(\beta, \gamma)} . \]

We denote by \( \langle h, f \rangle \) the natural pairing of elements \( h \in (\mathcal{M}(\beta, \gamma))' \) and \( f \in \mathcal{M}(\beta, \gamma) \). Since \( \mathcal{M}(x_1, d, \beta, \gamma) = \mathcal{M}(\beta, \gamma) \) with the equivalent norms for all \( x_1 \in X \) and \( d > 0 \), thus, for all \( h \in (\mathcal{M}(\beta, \gamma))' \), \( \langle h, f \rangle \) is well defined for all \( f \in \mathcal{M}(x_0, d, \beta, \gamma) \) with \( x_0 \in X \) and \( d > 0 \). In what follows, we let \( \tilde{\mathcal{M}}(\beta, \gamma) \) be the completion of \( \mathcal{M}(\theta, \theta) \) in \( \mathcal{M}(\beta, \gamma) \) when \( 0 < \beta, \gamma < \theta \).

We also need the following construction of Christ in [1], which provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type.

**Lemma 1.4.** Let \( X \) be a space of homogeneous type. Then there exist a collection \( \{Q^k_\alpha \subset X : k \in \mathbb{Z}_+, \alpha \in I_k \} \) of open subsets, where \( I_k \) is some (possible finite) index set, and constants \( \delta \in (0, 1) \) and \( C_1, C_2 > 0 \) such that

(i) \( \mu(X \setminus \cup_\alpha Q^k_\alpha) = 0 \) for each fixed \( k \) and \( Q^k_\alpha \cap Q^k_\beta = \emptyset \) if \( \alpha \neq \beta \);

(ii) for any \( \alpha, \beta, k, l \) with \( l \geq k \), either \( Q^k_\beta \subset Q^l_\alpha \) or \( Q^l_\beta \cap Q^k_\alpha = \emptyset \);

(iii) for each \( (k, \alpha) \) and each \( l < k \) there is a unique \( \beta \) such that \( Q^k_\alpha \subset Q^l_\beta \); 

(iv) \( \text{diam}(Q^k_\alpha) \leq C_1 \delta^k \);

(v) each \( Q^k_\alpha \) contains some ball \( B(z^k_\alpha, C_2 \delta^k) \), where \( z^k_\alpha \in X \).

In fact, we can think of \( Q^k_\alpha \) as being a dyadic cube with diameter roughly \( \delta^k \) and centered at \( z^k_\alpha \). In what follows, we always suppose \( \delta = 1/2 \). See [12] for how to remove this restriction. Also, in the following, for \( k \in \mathbb{Z}_+, \tau \in I_k \), we will denote by \( Q^{k,\tau}_\nu \), \( \nu = 1, \ldots, N(k, \tau, M) \), the set of all cubes \( Q^{k,\tau}_\nu \subset Q^{k,\tau}_1 \), where \( M \) is a fixed large positive integer.

Now, we can introduce the inhomogeneous Besov spaces \( B^{\nu,\eta}_p(X) \) and Triebel-Lizorkin spaces \( F^{\nu,\eta}_p(X) \) via approximations to the identity.

**Definition 1.5.** Suppose that \( -\theta < \alpha < \theta \), and \( \beta \) and \( \gamma \) satisfying

\[ \max(0, -\alpha + \max(0, d(1/p - 1))) < \beta < \theta, 0 < \gamma < \theta. \] (1.14)
Suppose $\{S_k\}_{k \in \mathbb{Z}_+}$ is an approximation to identity and let $D_0 = S_0$, and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$. Let $M$ be a fixed large positive integer, $Q^0_{\nu}$ be as above.

**Inhomogeneous Besov space $B^{\alpha,q}_p(X)$** for $\max\left(\frac{d}{d+\alpha-q}, \frac{d}{d+\alpha+q}\right) < p \leq \infty$, $0 < q \leq \infty$ is the collection of all $f \in (\mathcal{M}(\beta,\gamma))'$ such that

$$\|f\|_{B^{\alpha,q}_p(X)} = \left\{ \sum_{\tau \in J_0} \sum_{\nu = 1}^{N(0,\tau,M)} \mu(Q^0_{\tau}) [m_{Q^0_{\tau}}(|D_0(f)|)]^p \right\}^{\frac{1}{p}}$$

$$+ \left\{ \sum_{k=1}^{\infty} \left( 2^k \|D_k(f)\|_{L^p(X)} \right)^q \right\}^{\frac{1}{q}} < \infty.$$

**Inhomogeneous Triebel-Lizorkin space $F^{\alpha,q}_p(X)$** for $\max\left(\frac{d}{d+\alpha-q}, \frac{d}{d+\alpha+q}\right) < q \leq \infty$ is the collection of $f \in (\mathcal{M}(\beta,\gamma))'$ such that

$$\|f\|_{F^{\alpha,q}_p(X)} = \left\{ \sum_{\tau \in J_0} \sum_{\nu = 1}^{N(0,\tau,M)} \mu(Q^0_{\tau}) [m_{Q^0_{\tau}}(|D_0(f)|)]^p \right\}^{\frac{1}{p}}$$

$$+ \left\{ \left[ \sum_{k=1}^{\infty} \left( 2^k \|D_k(f)\|_{L^p(X)} \right)^q \right]^{\frac{1}{q}} \right\} < \infty,$$

where $m_{Q^0_{\tau}}(D_0(f))$ are averages of $D_0(f)$ over $Q^0_{\nu}$.

**T1 Theorems for Inhomogeneous Besov and Triebel-Lizorkin Spaces**

T1 theorems for inhomogeneous Besov and Triebel-Lizorkin spaces were proved in [10]. Roughly speaking, if $T$ is bounded on $B^{\alpha,q}_p$, $1 \leq p, q \leq \infty$ and $0 < \alpha < \epsilon$, and on $F^{\alpha,q}_p, 1 < p, q \leq \infty$ and $0 < \alpha < \epsilon$, if $T$ has the weak boundedness property, $T1 = 0$ and the conditions (1.3)–(1.5) hold in [10]. In this paper, we will prove the following results.

**Theorem A.** Let $0 < \epsilon \leq \theta, 0 < \alpha < \epsilon$. Suppose that $T(1) = 0, T \in \text{WBP}$, and $K(x,y)$, the kernel of $T$, satisfies (1.3)–(1.5) with $\sigma \geq \max(0, d(\frac{1}{p} - 1))$. Then $T$ is bounded on $B^{\alpha,q}_p(X)$, for $\frac{d}{d+\alpha-q} < p \leq \infty, 0 < q \leq \infty$, and on $F^{\alpha,q}_p(X)$, for $\frac{d}{d+\alpha-q} < p < \infty, \frac{d}{d+\alpha+q} < q \leq \infty$.

**Theorem B.** Let $0 < \epsilon \leq \theta, -\epsilon < \alpha < 0$. Suppose that $T^*(1) = 0, T \in \text{WBP}$, and $K(x,y)$, the kernel of $T$, satisfies (1.3), (1.4) and (1.6) with $\sigma \geq \max(0, d(\frac{1}{p} - 1))$. Then $T$ is bounded on $B^{\alpha,q}_p(X)$, for $\frac{d}{d+\alpha+q} < p \leq \infty, 0 < q \leq \infty$, and on $F^{\alpha,q}_p(X)$, for $\frac{d}{d+\alpha+q} < p < \infty, \frac{d}{d+\alpha+q} < q \leq \infty$.

Theorems A and B are to give a uniform treatment in [10]. To be precise, to deal with the case where $0 < \alpha < \epsilon, p, q > 1$, the main tools used were the continuous Calderón reproducing formula. The proof of the case where
$-\epsilon < \alpha < 0$, and $p, q > 1$ then follows from the duality argument. However, the
continuous Calderón reproducing formula and duality argument do not work for the
cases where either $p$ or $q$, or both $p$ and $q$ are less than or equal to $1$. The
key point of the present paper is to use discrete Calderón reproducing formula
and Plancherel-Pólya characterization of the Besov and Triebel-Lizorkin spaces
developed in [6, 11]. T1 theorems for inhomogeneous Triebel-Lizorkin space
$F_{p, q}^\alpha(X)$ with $-\epsilon < \alpha < \epsilon$, $\max\left\{ \frac{d}{p+\epsilon}, \frac{d}{q+\epsilon} \right\} < p < \infty$ and $\max\left\{ \frac{d}{p+\epsilon}, \frac{d}{q+\epsilon} \right\} < q \leq \infty$ in [17] are also stated, if $T$ has the weak boundedness property, $T(1) = 0$, $T^* (1) = 0$ and the conditions (1.3)–(1.6) hold. Furthermore, by use of the real
interpolation theorems the author obtained the
Besov space $X_{p, q}$ can be stated as follows.

\begin{equation}
\text{Lemma 2.1. Suppose that } \{ S_k \} \text{ is an approximation to the identity as in}
\text{Definition 1.2. Set } D_k = S_k - S_{k-1} \text{ for } k \in \mathbb{N} \text{ and } D_0 = S_0.
\text{Then there exist functions } \tilde{Q}_k^\alpha, \tau \in I_0 \text{ and } \nu \in \{ 1, \ldots, N(0, \tau, M) \}
\text{and } \{ \tilde{D}_k(x, y) \} \text{ such that for any fixed } y_{\nu}^k \in Q^\nu_k, k \in \mathbb{N}, \tau \in I_k \text{ and } \nu \in \{ 1, \ldots, N(k, \tau, M) \}
\text{and all } f \in (M(\beta, \gamma))^\prime \text{ with } 0 < \beta, \gamma < \theta,
\end{equation}

\begin{align}
f(x) &= \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0, \tau, M)} \mu(Q^\nu_\tau \cap D_0(f)) \tilde{Q}^\nu_\tau (x) \\
&+ \sum_{k \in \mathbb{Z}, \tau \in I_k} \sum_{\nu=1}^{N(k, \tau, M)} \mu(Q^\nu_\tau) \tilde{D}_k(x, y_{\nu}^k) D_k(f)(y_{\nu}^k),
\end{align}

where $\text{diam}(Q^k_\tau) \sim 2^{k+M}$ for $k \in \mathbb{Z}$, $\tau \in I_k, \nu \in \{ 1, \ldots, N(k, \tau, M) \}$ and a
fixed large $M \in \mathbb{N}$, the series converges in the norm of $L^p(X), 1 < p < \infty,$
and $M(\beta', \gamma')$ for $f \in M(\beta, \gamma)$ with $\beta' < \beta$ and $\gamma' < \gamma$, and $M(\beta', \gamma')^\prime$ for
$f \in (M(\beta, \gamma))^\prime$ with $\theta > \beta' > \beta$ and $\theta > \gamma' > \gamma$. Moreover, $\tilde{D}_k(x, y), k \in \mathbb{N},$
satisfies for any given $\epsilon \in (0, \theta)$, all $x, y \in X$ the following conditions:

\begin{equation}
|\tilde{D}_k(x, y)| \leq C \frac{2^{-k\epsilon'}}{(2^{-k} + \rho(x, y))^{d+\epsilon'}};
\end{equation}

\begin{equation}
|\tilde{D}_k(x, y) - \tilde{D}_k(x', y)| \leq C \left( \frac{\rho(x, x')}{2^{-k} + \rho(x, y)} \right)^{\epsilon'} \frac{2^{-k\epsilon'}}{(2^{-k} + \rho(x, y))^{d+\epsilon'}}
\end{equation}

2. Proofs of Theorems A and B

The basic tool to show main results is the discrete Calderón reproducing formulae
in [6]. It can be stated as follows.
for \( \rho(x, x') \leq \frac{1}{2}(2^{-k} + \rho(x, y)) \);

\[
\int_X \tilde{D}_k(x, y) d\mu(y) = \int_X \tilde{D}_k(x, y) d\mu(x) = 0
\]

for all \( k \in \mathbb{Z}_+ \).

\( \tilde{D}_{Q_0^{\alpha, \nu}}(x) \) for \( \tau \in I_0 \) and \( \nu \in \{1, \ldots, N(0, \tau, M)\} \) satisfies

\[
\int_X \tilde{D}_{Q_0^{\alpha, \nu}}(x) d\mu(x) = 1,
\]

\[
|\tilde{D}_{Q_0^{\alpha, \nu}}(x)| \leq \frac{C}{(1 + \rho(x, y))^{d+\epsilon}}
\]  

(2.4)

for all \( x \in X \) and \( y \in Q_0^{\alpha, \nu} \) and

\[
|\tilde{D}_{Q_0^{\alpha, \nu}}(x) - \tilde{D}_{Q_0^{\alpha, \nu}}(z)| \leq C \left( \frac{\rho(x, z)}{1 + \rho(x, y)} \right)^\epsilon \frac{1}{(1 + \rho(x, y))^{d+\epsilon}}
\]  

(2.5)

for all \( x, z \in X \) and \( y \in Q_0^{\alpha, \nu} \) satisfying \( \rho(x, z) \leq \frac{1}{2}(1 + \rho(x, y)) \); the constant \( C \) in (2.2) – (2.5) is independent of \( M \).

To prove Theorem A and Theorem B, we need the following lemmas. Their proofs are similar to that of Lemma 4.1 in [10].

**Lemma 2.2.** With notation as in Lemma 2.1 and Theorem A, then

(i) for \( k \in \mathbb{Z}_+ \), \( \tau' \in I_0 \) and \( \nu' \in \{1, \ldots, N(0, \tau', M)\} \), \( y_{\tau', \nu'} \) is any fixed point of \( Q_0^{\alpha', \nu'} \), \( x \in X \),

\[
|D_k \tilde{D}_{Q_0^{\alpha, \nu}}(x)| \leq C(1 + k)2^{-k} \frac{1}{(1 + \rho(x, y_{\tau', \nu}'))^{d+\sigma'}}
\]

(2.6)

where \( \sigma' = \sigma \) when \( k = 0 \) and \( \sigma' = \epsilon \) when \( k \in \mathbb{N} \),

(ii) for \( k \in \mathbb{Z}_+ \), \( k' \in \mathbb{N} \), \( x, y \in X \),

\[
|D_k T \tilde{D}_k(x, y)| \leq C[1 + |k - k'|] \left( 2^{|k-k'|} - 1 \right) \frac{2^{-|k-k'|}}{(1 + \rho(x, y_{\tau', \nu}'))^{d+\epsilon'}}
\]

(2.7)

**Lemma 2.3.** With notation as in Lemma 2.1 and Theorem B, then

(i) for \( k \in \mathbb{Z}_+ \), \( \tau' \in I_0 \) and \( \nu' \in \{1, \ldots, N(0, \tau', M)\} \), \( y_{\tau', \nu'} \) is any fixed point of \( Q_0^{\alpha', \nu'} \), \( x \in X \),

\[
|D_k \tilde{D}_{Q_0^{\alpha, \nu}}(x)| \leq C \frac{1}{(1 + \rho(x, y_{\tau', \nu}'))^{d+\sigma'}}
\]

(2.8)

where \( \sigma' = \sigma \) when \( k = 0 \) and \( \sigma' = \epsilon \) when \( k \in \mathbb{N} \),
(ii) for \( k \in \mathbb{Z}_+, k' \in \mathbb{N}, x, y \in X \),

\[
|D_kT\tilde{D}_{k'}(x,y)| \leq C[1 + |k - k'|]\left(2^{(k-k')'} \wedge 1\right) \frac{2^{-(k \wedge k')'}}{(2^{-(k \wedge k')}) + \rho(x,y)^{d + \frac{d'}{r'}}}.
\]

(2.9)

Proof of Theorem A. By Lemma 2.1 and Theorem 1.5 in [6], for \( f \in \tilde{M}(\beta, \gamma) \), we write

\[
\|T(f)\|_{L^p_{\alpha,q}(X)} \leq \left\{ \sum_{\tau \in I_0} \left[ \sum_{\nu=1}^{N(0,\tau,M)} \left[ m_{Q_\tau}^0 \left( \sum_{\tau' \in I_0} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'}) \right) \right] \right] \right\}^{\frac{1}{p}}
\]

\[
|D_0T\tilde{D}_{Q_\tau^0}^\nu(\cdot)|m_{Q_\tau^0}^0(\|D_0(f)\|)\right]^{\frac{1}{p}}
\]

\[
+ \left\{ \sum_{\tau \in I_0} \left[ \sum_{\nu=1}^{N(0,\tau,M)} \left[ m_{Q_\tau^0}^0 \left( \sum_{k'=1}^{N(k',\tau,M)} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{k',\nu'}) \right) \right] \right] \right\}^{\frac{1}{p}}
\]

\[
|D_0T\tilde{D}_{k'}^\nu(\cdot, y_{k'}^{\nu'})||D_k(f)(y_{k'}^{\nu'})|\right]^{\frac{1}{p}}
\]

\[
+ \left\{ \sum_{l=1}^{\infty} \left[ \sum_{\tau \in I_l} \left[ \sum_{\nu=1}^{N(0,\tau,M)} \left[ m_{Q_\tau^l}^\nu \left( \sum_{k'=1}^{N(k',\tau,M)} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{k',\nu'}) \right) \right] \right] \right] \right\}^{\frac{1}{p}}
\]

\[
\times \mu(Q_{\tau}^{L_\nu})^{-\frac{\alpha}{p} + \frac{1}{p}} |D_0T\tilde{D}_{Q_\tau^0}^\nu(z)|m_{Q_\tau}^0(\|D_0(f)\|)\right]^{\frac{1}{p}}\right\}^{\frac{1}{p}}
\]

\[
+ \left\{ \sum_{l=1}^{\infty} \left[ \sum_{\tau \in I_l} \left[ \sum_{\nu=1}^{N(0,\tau,M)} \left[ m_{Q_\tau^l}^\nu \left( \sum_{k'=1}^{N(k',\tau,M)} \sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{k',\nu'}) \right) \right] \right] \right] \right\}^{\frac{1}{p}}
\]

\[
\times \mu(Q_{\tau}^{L_\nu})^{-\frac{\alpha}{p} + \frac{1}{p}} |D_0T\tilde{D}_{k'}^\nu(z, y_{k'}^{\nu'})||D_k(f)(y_{k'}^{\nu'})|\right]^{\frac{1}{p}}\right\}^{\frac{1}{p}}
\]

\[
\leq A_1 + A_2 + A_3 + A_4.
\]

The estimate of \( A_4 \) is similar to Theorem 1 in [5]. It remains to deduce the estimates of \( A_1, A_2 \) and \( A_3 \).

From (2.6), the Hölder inequality for \( p > 1 \) and \((a + b)^p \leq a^p + b^p \) for \( p \leq 1 \), we deduce

\[
A_1 \leq C \left\{ \sum_{\tau \in I_0} \left[ \sum_{\nu=1}^{N(0,\tau,M)} \left[ m_{Q_\tau^0}^0(\|D_0(f)\|)^p \right] \right] \right\}^{\frac{1}{p}}
\]

\[
\leq C \left\{ \sum_{\tau \in I_0} \left[ \sum_{\nu'=1}^{N(0,\tau,M)} \left[ m_{Q_\tau}^0(\|D_0(f)\|)^p \right] \right] \right\}^{\frac{1}{p}}
\]

\[
\leq C \|f\|_{L^p_{\alpha,q}(X)}
\]
where \( y_0^{0,0} \) is any point of \( Q_0^{0,0} \), \( y_0^{0,0'} \) is any point of \( Q_0^{0,0'} \).

By (2.7), it follows that

\[
A_2 \leq C \left\{ \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{l=1}^{N(0,\tau',M)} \sum_{\tau \in I_0} \sum_{\nu' = 1}^{N(0,\tau,M)} 2^{-k'd_2^{k'}2^{-k}'[1+k']^{p\Lambda_1}} \right. \\
\times \frac{1}{1 + \rho(y_0^{l,\nu'},y_0^{k',\nu'})^{d+\tau}} \left[ \mu(Q_0^{k',\nu'})^{-\frac{d}{p\Lambda_1}} |D_0(f)(y_0^{k',\nu'})| \right]^{\frac{1}{p}} \\
\leq C \left\{ \sum_{k'=1}^{\infty} \left[ 2^{-k'd_2^{k'}2^{-k}'[1+k']} \right]^{p\Lambda_1} 2^{k'd_2} + \sum_{k'} \left[ 2^{-k'd_2^{k'}2^{-k}'[1+k']} \right]^{p\Lambda_1} \right\}^{\frac{1}{p\Lambda_1}} \\
\leq C \|f\|_{B_p^{\alpha}(\mathcal{X})},
\]

where these inequalities follow from the fact that

\[
\sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{l=1}^{N(0,\tau',M)} \sum_{\tau \in I_0} \sum_{\nu' = 1}^{N(0,\tau,M)} 2^{-k'd_2^{k'}2^{-k}'[1+k']} \frac{1}{1 + \rho(y_0^{l,\nu'},y_0^{k',\nu'})^{d+\tau}} \leq C,
\]

\[
\sum_{k'=1}^{\infty} \left[ 2^{-k'd_2^{k'}2^{-k}'[1+k']} \right]^{p\Lambda_1} 2^{k'd_2} + \sum_{k'} \left[ 2^{-k'd_2^{k'}2^{-k}'[1+k']} \right]^{p\Lambda_1} \leq C
\]

and the last inequality follows from the Plancherel-Pólya characterization of the Besov spaces \([6]\).

By (2.6), it follows that

\[
A_3 \leq C \left\{ \sum_{l=1}^{\infty} \sum_{\tau \in I_0} \sum_{\nu = 1}^{N(0,\tau,M)} \sum_{\nu' = 1}^{N(0,\tau',M)} \mu(Q_0^{0,\nu'}) \left[ m_{Q_0^{0,\nu'}} (|D_0(f)|) \right]^p \\
\times \left[ 2^{2l\alpha} (1 + l)^{2^{-l}} \frac{1}{1 + \rho(y_0^{l,\nu'},y_0^{l,\nu''})^{d+\tau}} \right]^{p\Lambda_1} \right\}^{\frac{1}{p}} \\
\leq C \sum_{l=1}^{\infty} \left[ 2^{2l\alpha} (1 + l)^{2^{-l}} \right]^{p\Lambda_1} \left( \sum_{\tau \in I_0} \sum_{\nu' = 1}^{N(0,\tau',M)} \mu(Q_0^{0,\nu'}) \left[ m_{Q_0^{0,\nu'}} (|D_0(f)|) \right]^p \right)^{\frac{1}{p}} \\
\leq C \|f\|_{\widetilde{M}(\beta, \gamma)}.
\]

Similarly, for \( f \in \widetilde{M}(\beta, \gamma) \), we have
that Lemma A.2 in [8], the Fefferman-Stein vector-valued inequality in [7], it follows
\[ \|D_b T \tilde{D}_{q''} (\cdot) m_{Q''} (|D_b (f)|)\|_p^\frac{1}{p} \]
\[ \leq \left\{ \sum_{\tau \in I_0} \sum_{\nu = 1}^{N(0,\tau, M)} \mu(Q''_{\tau, \nu}) \left( \sum_{\tau' \in I'_0} \sum_{\nu' = 1}^{N(0,\tau', M)} \mu(Q''_{\tau', \nu'}) \right) \right\}^\frac{1}{p} \]
\[ + \left\{ \sum_{\tau \in I_0} \sum_{\nu = 1}^{N(0,\tau, M)} m_{Q''_{\tau, \nu}} \left( \sum_{k' = 1}^{\infty} \sum_{\tau' \in I'_0} \sum_{\nu' = 1}^{N(k',\tau', M)} \mu(Q''_{\tau', \nu'}) \right) \right\}^\frac{1}{p} \]
\[ \leq B_1 + B_2 + B_3 + B_4, \]
where \(y_{\tau, \nu}^{k', \nu'}\) are any point in \(Q''_{\tau, \nu, k'}\).

The estimates of \(B_1\) and \(B_4\) are similar to \(A_1\) above and Theorem 2 in [5], respectively. It remains to deduce the estimates of \(B_2\) and \(B_3\).

From (2.7), the Hölder inequality for \(q > 1\) and \((a+b)^q \leq a^q + b^q\) for \(q \leq 1\), Lemma A.2 in [8], the Fefferman-Stein vector-valued inequality in [7], it follows that
\[ B_2 \leq C \left\{ \left[ \sum_{k' = 1}^{\infty} 2^{-k' d_2 - k' \alpha} [1 + k']^{\frac{1}{2} + \frac{d_2}{q}} \right] \right\}^q \]
\[ \times \left[ \mathcal{M} \left( \sum_{\tau' \in I_{k'}} \sum_{\nu' = 1}^{N(k',\tau', M)} \mu(Q''_{\tau', \nu'})^{-\frac{q}{q-1}} |D_{k'} (f)(y_{\tau', \nu'}^{k', \nu'})| \right)^\frac{q}{q-1} \right\}^\frac{1}{q} \]
\[ \leq C \left\{ \sum_{k' = 1}^{\infty} \left[ 2^{-k' d_2 - k' \alpha} [1 + k']^{\frac{1}{2} + \frac{d_2}{q}} \right] \right\}^q \]
\[ \times \left[ \mathcal{M} \left( \sum_{\tau' \in I_{k'}} \sum_{\nu' = 1}^{N(k',\tau', M)} \mu(Q''_{\tau', \nu'})^{-\frac{q}{q-1}} |D_{k'} (f)(y_{\tau', \nu'}^{k', \nu'})| \right)^\frac{q}{q-1} \right\}^\frac{1}{q} \]
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\[ \leq C \left\| \sum_{k=1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu = 1}^{N(k',\tau',M)} \left[ \mu(Q_{k',\nu'})(D_{k'}(f)g_{k',\nu'}(x)\chi_{Q_{k',\nu'}})^\alpha d\nu \right] \right\|_{L_p(X)} \]

\[ \leq C \|f\|_{F_{\alpha,q}^p(X)}, \]

where \( \frac{d}{d+a} < r < \min(p, q, 1) \).

From (2.6), the Hölder inequality for \( p > 1 \) and \( (a + b)^p \leq a^p + b^p \) for \( p \leq 1 \), the Lemma A.2 in [7], it follows that

\[ B_3 \leq C \left\{ \int \left( \sum_{l=1}^{\infty} \sum_{\tau \in I_l} \sum_{\nu = 1}^{N(l,\tau,M)} x_{Q_{l,\nu}}(x) \left[ 2^{la} (1 + l)^{2-r\epsilon} \right] \right) \times \sum_{\tau \in I_0} \sum_{\nu = 1}^{N(0,\tau,M)} \frac{1}{(1 + \rho(x, y_{\nu}))^{d+\epsilon}} m_{Q_{\tau,\nu}}(\chi_{Q_{\tau,\nu}})^\alpha d\mu(x) \right\}^{\frac{1}{p}} \]

\[ \leq C \left\{ \int \left( \sum_{l=1}^{\infty} 2^{la} (1 + l)^{2-r\epsilon} \left[ M(\sum_{\tau \in I_0} \sum_{\nu = 1}^{N(0,\tau,M)} m_{Q_{\tau,\nu}}(\chi_{Q_{\tau,\nu}})^\alpha \right] \right) \right\}^{\frac{1}{p}} d\mu(x) \}

\[ \leq C \left\{ \int M(\sum_{\tau \in I_0} \sum_{\nu = 1}^{N(0,\tau,M)} m_{Q_{\tau,\nu}}(\chi_{Q_{\tau,\nu}})^\alpha \right) \right\}^{\frac{1}{p}} d\mu(x) \}

\[ \leq C \|f\|_{F_{\alpha,q}^p(X)}, \]

where we used the \( L^2(X) \) boundedness of Hardy-Littlewood maximal functions. This proves Theorem A. ■

**Proof of Theorem B.** The main difference of proof between Theorem B and Theorem A is that we should replace Lemma 2.2 by Lemma 2.3. We leave the details to the reader. ■

**References**

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