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Connected Branches of Positive Solutions of Multi-point Boundary Value Problems Depending on Parameters

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Abstract. The aim of this paper is to apply the theory of positive operators to a multi-point boundary value problem depending on parameters. Namely, we establish conditions under which the positive solutions of this problem form a connected boundedly compact branch.

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1. Introduction

The subject of our study is a connected boundedly compact branch of positive solutions for a multi-point boundary value problem (BVP) depending on parameters, and our main tool is the theory of positive operators.

In Sec. 2 we establish a result that positive solutions of a nonlinear problem form a connected boundedly compact branch of infinite length. Section 3 is devoted to the study of the multi-point boundary value problem depending on parameters. Namely, we use the just obtained result to show that the positive solutions of this problem form a connected boundedly compact branch of infinite length which connects the origin and the infinity.

We note that in recent years, the existence of positive solutions for multi-point BVP has attracted much attention (see [3, 4, 9, 10] and the references therein), but the branch of such solutions have not been considered there. In [1], Bakhtin and Huy investigated the continuous branch of solutions for this problem in the sense of Krasnoselskii, but solutions here are not positive. In difference from mentioned works, we will concern not only with positive solutions of the multi-point BVP depending on parameters, but also their connected boundedly compact branch.

2. Connected Branch of Positive Solutions for a Nonlinear Problem

Let $(E, \|.\|_E)$ be a Banach space with an ordering cone K, M a subset of E and u_0 a fixed vector of $K\backslash\{0\}$. An operator $A : M \longrightarrow E$ is called *positive* on M if $AM \subset K$. An operator $A : K \longrightarrow K$ is called

(i) u_0-*measurable* if for each $x \in K \setminus \{0\}$ there exist scalars $\alpha, \beta > 0$ such that $\alpha u_0 \leqslant Ax \leqslant \beta u_0$.

(ii) u_0-*positive* if for each $x \in K \setminus \{0\}$ there exist $n \in \mathbb{N}^*$ and scalars $\alpha, \beta > 0$ such that $\alpha u_0 \leqslant A^n x \leqslant \beta u_0$.

Throughout the paper ∂M is the boundary of the set M. Set

$$M_{r,R} := \{x \in M |\ r \leq \|x\| \leq R\},\ M_r := \{x \in M |\ \|x\| \leq r\}.$$

Let $A(\mu, .)$ be an operator depending on parameter $\mu \in (a, b)$. Consider the following equation

$$A(\mu, x) = x. \tag{1}$$

Definition 2.1. *We say that*

(i) *A vector $x \in K\backslash\{0\}$ satisfying the equation (1) is a positive solution of the equation (1).*

(ii) *Solutions x of the equation (1) form a continuous branch, connecting the spheres $S(0, r)$ and $S(0, R)$ $(0 \leqslant r < R \leqslant \infty)$ if $V \bigcap \partial\Omega \neq \emptyset$, where V is a set of solutions of the equation (1) and Ω is any open set satisfying $B(0, r') \subset \Omega \subset B(0, R')$ $(0 \leqslant r < r' < R' < R \leqslant \infty)$. Here $B(0, \varepsilon) = \{x \mid \|x\| \leqslant \varepsilon\}$.*

If $R = \infty$, then the set V is a continuous branch of infinite length, connecting the spheres $S(0, r)$ with the infinity.

(iii) *A continuous branch V of solutions of the equation (1) is connected if the set $V\backslash\{0\}$ is connected.*

Definition 2.2. *A continuous branch V connecting the spheres $S(0, r)$ and $S(0, R)$ $(0 \leqslant r < R \leqslant \infty)$ is said to be boundedly compact if for any scalars r', R', $r < r' < R' < R$, the set $V_{r',R'}$ is compact.*

We study first the existence of connected branches of positive solutions for a class of nonlinear equations depending on parameters. Let us recall a known result [2, Theorem 1.1].

Theorem 2.1. *Suppose that the following conditions are satisfied.*

1) *The operator $A : (a, b) \times K_{r,R} \rightarrow E$ is positive and completely continuous on each set $[c, d] \times K_{r',R'}$, where $[c, d] \subset (a, b)$, $r < r' < R' < R$ and $r, R \in [0, +\infty]$ are fixed scalars;*

2) *For any scalars* $r', R' \in (r, R), r' < R'$ *there exist scalars* $a', b' \in (a, b)$ *such that if* $x \in K_{r',R'}$ *and* $\mu \in (a, b)$ *satisfy the equation* (1) *then* $\mu \in [a', b']$.
3) *The set* W *of positive solutions of the equation* (1) *is a continuous branch connecting the spheres* $S(0, r)$ *and* $S(0, R)$.

Then the set W *contains a connected boundedly compact branch connecting the spheres* $S(0, r)$ *and* $S(0, R)$.

Let be given a Banach space E_1 with an ordering cone K_1 and let $P : E_1 \to K_1$ $(P0 = 0)$ be a continuous bounded operator, E be a subspace of E_1, $A : E_1 \to E \subset E_1$ a linear operator. Suppose the subspace AE_1 is included in some normed space E_0. A is called E_0-*contractive* if there exist a linear functional $f_0 \in K_1^*$ $(f_0(x) \not\equiv 0, x \in K_1)$ and scalars $\alpha, \beta > 0$ such that

$$f_0(PAx) \geqslant \alpha f_0(x), f_0(PAx) \geqslant \beta \|Ax\|_{E_0} \quad \forall x \in K_1.$$

We refer the reader to [8] for the concept of the rotation of positive fields (the mapping degree) and its properties. Denote by

$$\gamma(I - A, \Omega(K))$$

the rotation of a positive field $I - A$ on $\partial\Omega(K)$. Our main result states as follows.

Theorem 2.2. *Let be given a cone* $K \subset K_1 \cap E$ *and operators* $F : (a, b) \times K \to K_1$, $A : E_1 \to E$. *Suppose that the following conditions are satisfied:*
1) *The operator* F *is continuous and bounded on* $(a, b) \times K$, *and the operator* A *is linear,* E_0*-contractive, completely continuous and* $A(K_1) \subset K$;
2) *If* λ *is a positive eigenvalue of the operator* $AF(\mu, .)$, $\mu \in (a, b)$ *corresponding to an eigenvector* $x \in \partial\Omega(K)$ (*if such a vector exists*), *then*

$$\limsup_{\mu \to a} \lambda < 1,$$

where $\Omega(K) \subset K$ *is an arbitrary bounded open set containing zero;*

3) *There exist a number* $\mu_0 \in (a, b)$, *positive numbers* m, l *and functions* $\varphi : (\mu_0, b) \longrightarrow (0, \infty)$, $\Phi : (0, \infty) \times (\mu_0, b) \longrightarrow (0, \infty)$, *where* $\Phi(., \mu)$ *is increasing such that*

a) $f_0(F(\mu, x)) \geq m f_0(P(\varphi(\mu)x)) - l \quad (\mu \in (\mu_0, b),\ x \in K)$, *where* f_0, P *are the functional and operator mentioned in the definition of an* E_0*-contractive operator;*

b) *The equality*

$$x = tAF(\mu, x) + (1 - t)mAP(\varphi(\mu)x)$$

where $x \in K, t \in [0, 1], \mu \in (\mu_0, b)$ *implies*

$$\|x\| \leq \Phi(\|x\|_{E_0}, \mu);$$

c) *The inequality*

$$\alpha m \varphi(\mu) > 1$$

holds for all $\mu \in (\mu_0, b)$ *sufficiently close to* b *and there exists the finite limit*

$$\lim_{\mu \to b} \Phi(\frac{\alpha l}{\beta(\alpha m \varphi(\mu) - 1)}, \mu) = r_0,$$

where α, β *are numbers mentioned in the definition of the* E_0*-contractiveness of the operator* A.

Then the set W *of positive solutions of the equation*

$$AF(\mu, x) = x. \tag{2}$$

contains a connected boundedly compact branch of the infinite length connecting the sphere $S(0, r_0)$ *with the infinity.*

Proof. First we show that for all $\mu \in (\mu_0, b)$ sufficiently close to b and satisfying the inequality $\alpha m \varphi(\mu) > 1$, the relation

$$x = tAF(\mu, x) + (1 - t)mAP(\varphi(\mu)x), \tag{3}$$

where $x \in K, t \in [0, 1]$ implies

$$\|x\|_{E_0} \leq \frac{\alpha l}{\beta(\alpha m \varphi(\mu) - 1)}. \tag{4}$$

Indeed, from the equality (3) we get

$$P(\varphi(\mu)x) = PA(\varphi(\mu)[tF(\mu, x) + (1 - t)mP(\varphi(\mu)x)]) \tag{5}$$

which together with the E_0-contractiveness of the operator A and the condition 3a) of the theorem yield

$$\begin{aligned} f_0(P(\varphi(\mu)x)) &\geq \alpha f_0(\varphi(\mu)[tF(\mu, x) + (1 - t)mP(\varphi(\mu)x)]) \\ &\geq \alpha \varphi(\mu)(m f_0(P(\varphi(\mu)x)) - l). \end{aligned}$$

Hence,

$$f_0(P(\varphi(\mu)x)) \leq \frac{\alpha l \varphi(\mu)}{\alpha m(\varphi(\mu) - 1)}.$$

On the other hand, the E_0-contractiveness of A and the equality (5) imply

$$\begin{aligned} f_0(P(\varphi(\mu)x)) &\geq \beta \|\varphi(\mu)A(tF(\mu, x) + (1 - t)mP(\varphi(\mu)x))\|_{E_0} \\ &= \beta \varphi(\mu)\|x\|_{E_0}, \end{aligned}$$

and the inequality (4) holds.

Consequently, taking account of the condition 3b) of the theorem we obtain

$$\|x\| \leq \Phi(\|x\|_{E_0}, \mu) \leq \Phi(\frac{\alpha l}{\beta(\alpha m \varphi(\mu) - 1)}, \mu). \tag{6}$$

Next we show that positive solutions of the equation (2) form a connected boundedly compact branch of infinite length connecting the sphere $S(0, r_0)$ with the infinity. Let $\Omega(K) \supset K_r$ $(r > r_0)$ be an arbitrary open set. The condition 3c) and the inequality (6) imply the existence of a scalar $\mu_1 \in (\mu_0, b)$ such that for all $\mu \in (\mu_1, b)$, the fields $I - AF(\mu, .)$ and $I - mAP(\varphi(\mu)I)$ are nonvanishing and positively homotopic on the set $\partial\Omega(K)$. Hence, for all $\mu \in (\mu_1, b)$ one has

$$\gamma(I - AF(\mu, .), \Omega(K)) = \gamma(I - mAP(\varphi(\mu)I), \Omega(K)).$$

Take an arbitrary $\mu_2 \in (\mu_1, b)$ and an element $u_0 \in K$ such that

$$\alpha m\varphi(\mu) > 1 \quad (\mu \in (\mu_2, b))$$

and $f_0(u_0) > 0$. Clearly, $Au_0 \neq 0$ because

$$f_0(PAu_0) \geq \alpha f_0(u_0) > 0$$

and $P0 = 0$. We claim that

$$\gamma(I - mAP(\varphi(\mu)I), \Omega(K)) = 0, \tag{7}$$

where $\mu \in (\mu_2, b)$. It suffices to show that for any fixed $\mu \in (\mu_2, b)$, the relation

$$x = mAP(\varphi(\mu)x) + \xi Au_0 \tag{8}$$

with $x \in \partial\Omega(K)$ implies $\xi < 0$ ([8], Theorem 33.3). Indeed, it follows from (8) that

$$P(\varphi(\mu)x) = PA(\varphi(\mu)[mP(\varphi(\mu)x) + \xi u_0]).$$

By the E_0-contractiveness of the operator A we have

$$f_0(P(\varphi(\mu)x)) \geq \alpha m\varphi(\mu) f_0(P(\varphi(\mu)x)) + \alpha\xi\varphi(\mu) f_0(u_0). \tag{9}$$

Since the field $I - mAP(\varphi(\mu)I)$ is nonvanishing on $\partial\Omega(K)$, it follows from the inequalities

$$\alpha m\varphi(\mu) > 1, \;\; f_0(u_0) > 0$$

and (9) that $\xi < 0$. This means that (7) holds. Further, by the condition 2) of the theorem, for all $\mu \in (a, b)$ sufficiently close to a one has

$$\gamma(I - AF(\mu, .)), \Omega(K)) = 1$$

([8], Theorem 33.1). Hence, for some $\mu \in (a, b)$ and $x \in \partial\Omega(K)$ we get $AF(\mu, x) = x$ ([8], Lemma 47.2).

Then the set W of positive solutions of the equation (2) forms a continuous branch of infinite length connecting the sphere $S(0, r_0)$ with ∞.

Next we show that, for any scalars r', R' satisfying $r_0 < r' < R' < \infty$ there exist scalars $a', b' \in (a, b)$ such that the relation (2) with $x \in K_{r',R'}, \mu \in (a, b)$ implies $\mu \in [a', b']$.

Indeed, the condition 3c) of the theorem implies the existence of a scalar $b' \in (\mu_0, b)$ such that

$$\Phi(\frac{\alpha l}{\beta(\alpha m\varphi(\mu)-1)}, \mu) < r', \ \ \alpha m\varphi(\mu) > 1$$

for all $\mu \in [b', b)$. By an analogous argument we can deduce from the relation (2) with $x \in K\backslash\{0\}, \ \mu \in [b', b)$ that

$$\|x\| \le \Phi(\frac{\alpha l}{\beta(\alpha m\varphi(\mu)-1)}, \mu) < r'.$$

This implies $\mu \in (a, b')$.

Next, suppose to the contrary that a desired scalar $a' \in (a, b')$ does not exist. Then one can find sequences $\{x_n\} \subset K_{r',R'}$ and $\{\mu_n\} \subset (a, b'), \ \mu_n \to a$ such that

$$AF(\mu_n, x_n) = x_n \quad (n \in \mathbb{N}).$$

Taking a subsequence if necessary, we can assume without loss of generality that only three following cases are possible:

a) $x_n = x_0$ for all $n \in \mathbb{N}$;

b) $x_n \ne x_m \ \ (n \ne m)$ and $x_n \to x_0$;

c) there exists a scalar $\varepsilon_0 > 0$ such that

$$\|x_n - x_m\| \ge \varepsilon_0 \quad (n \ne m).$$

Consider the bounded open set $\Omega(K)$ defined by

$$\Omega(K) = \{x \in K, \ \|x\| < R', \ x \notin \{x_1, \ldots, x_n, \ldots\}\}.$$

It is clear that $\Omega(K) \supset K_{r'} \ni 0$ and

$$\partial\Omega(K) = \begin{cases} S_{R'}(K) \cup \{x_0, x_1, \ldots, x_n, \ldots\} & \text{if b) holds,} \\ S_{R'}(K) \cup \{x_1, \ldots, x_n, \ldots\} & \text{if a), c) hold.} \end{cases}$$

Hence by virtue of the condition 2) of the theorem there exists a scalar $a' \in (a, b')$ such that the relation

$$AF(\mu, x) = \lambda x$$

with $x \in \partial\Omega(K), \ \mu \in (a, a']$ implies $\lambda < 1$. It means that the equation

$$AF(\mu_n)x_n = x_n$$

does not hold for n sufficiently large, that is a contradiction.

Thus we have established the existence of scalars $a', b' \in (a, b)$ with desired properties.

Finally, the theorem follows from Theorem 2.1. ■

By an analogous argument one can prove the following

Theorem 2.3. *Suppose that the conditions* 1) - 3) *of Theorem* 2.2 *hold and*
a) $f_0(F(\mu, x)) \geq m\varphi(\mu)f_0(Px) - l \quad (x \in K, \; \mu \in (\mu_0, b))$*;*
b) the equality

$$x = tAF(\mu, x) + (1-t)m\varphi(\mu)APx \quad (x \in K, t \in [0,1], \mu \in (\mu_0, b))$$

implies

$$\|x\| \leq \Phi(\|x\|_{E_0}, \mu).$$

Then the assertion of Theorem 2.2 *is true.*

Remark that, if in hypotheses of Theorem 2.2 we replace K by E, $\Omega(K)$ by Ω and $\partial\Omega(K)$ by $\partial\Omega$, where $\partial\Omega$ is the boundary of an arbitrary bounded open set $\Omega \subset E$, containing the origin, then the set V of solutions of the equation (2) in the space E contains a connected boundedly compact branch of infinite length connecting the sphere $S(0, r_0)$ with the infinity.

3. Connected Branch of Positive Solutions for a Multi-point BVP

Consider the following multi-point BVP defined by the following system of nonlinear differential equations

$$Lx = F(t, \mu^m x, \mu^{m-1}x', \dots, \mu x^{(m-1)}) = F(x, \mu); \tag{10}$$

$$x^{(l_i)}(t_i) = 0 \quad (i = 1, \dots, m), \tag{11}$$

where x is a vector function from $[0,1]$ into $\mathbb{R}^n$, L is a linear differential operator

$$Lx = \sum_{i=0}^{m} a_i(t)x^{(i)}(t),$$

$a_i(t)$ $(i = 0, 1, \dots, m)$ are matrices i-times continuously differentiable on $[0,1]$ satisfying $\det a_m(t) \neq 0$, $F(t, \xi_0, \xi_1, \dots, \xi_{m-1})$ is a continuous function from $[0,1] \times \mathbb{R}^{mn}$ into $\mathbb{R}^n$.

The boundary conditions (11) imply

$$l_i \leq m-1 \quad (i = 0, 1, \dots, m)$$

and

$$0 = t_1 \leq t_2 \leq \ldots \leq t_m = 1,$$

moreover, if $t_i = t_{i+1}$, then $l_i < l_{i+1}$.

The equations of the form (10) - (11) have been investigated by Bahtin and Huy in [1]. In difference from this work we will consider only such solutions of (10) - (11), which are positive in the sense that they belong to some cone.

Let us begin with recalling some auxilliary facts that will be used later.

Let

$$\overline{l} = \max\{l_i|\ t_i = 1\},\ \underline{l} = \max\{l_i|\ t_i = 0\},\ k_0 = \min\{\overline{l}, \underline{l}\},$$

$$\delta = m - \underline{l} - 1,\ \rho = m - \overline{l} - 1.$$

Denote by $\|.\|$ the norm in $\mathbb{R}^n$, by $\mathbb{R}^n_+$ the cone of vectors with nonnegative coordinates and $|x(t)| := (|x_1(t)|, \dots, |x_n(t)|)$.

Let $E_{\delta,\rho}$ be the space of measurable vector functions z from $[0,1]$ into $\mathbb{R}^n$ with the finite norm

$$\|z\|_{\delta,\rho} = \int_0^1 \|z(t)\| t^\delta (1-t)^\rho dt.$$

Let $E_{m,\delta,\rho}$ be the space of vector functions $z \in E_{\delta,\rho}$, which have m-th derivates in the sense of Sobolev with the norm

$$\|z\|_{m,\delta,\rho} = \sum_{i=0}^m \|z^{(i)}\|_{\delta,\rho},$$

and $C^m([0,1], \mathbb{R}^n)$ be the space of m-times continuously differentiable vector functions x from $[0,1]$ into $\mathbb{R}^n$ with the norm

$$\|x\|_{C^m} = \sum_{i=0}^m \max_{t\in[0,1]} \|x^{(i)}(t)\|.$$

Suppose that the problem

$$Lx = z \tag{12}$$

with the boundary conditions (11) for $z = 0$ has in $C^m([0,1], \mathbb{R}^n)$ only a trivial solution zero. Then for any $z \in C([0,1], \mathbb{R}^n)$, this problem has in $C^m([0,1], \mathbb{R}^n)$ a unique solution $x = Az$.

It has been proved in [5] the following theorem of Klimov.

Theorem 3.1.

1) *The mentioned above operator A from $C([0,1], \mathbb{R}^n)$ into $C^m([0,1], \mathbb{R}^n)$ is well defined and continuous and hence, A maps $C([0,1], \mathbb{R}^n)$ into C^{m-1} $([0,1], \mathbb{R}^n)$ and is completely continuous;*
2) *The operator A is E_0-contractive, where $E_0 = E_{m,\delta,\rho}$ and*

$$Px(t) = |x(t)|,\ f_0(x) = (x, y_0) = \int_0^1 \left(\sum_{i=1}^n x_i(t)\right) t^\delta (1-t)^\rho dt$$

with

$$y_0(t) = (\underbrace{t^\delta(1-t)^\rho, \dots, t^\delta(1-t)^\rho}_{n});$$

$$(x \in C([0,1], \mathbb{R}^n)),$$

3) *there exists a scalar* $\gamma > 0$, *such that*

$$\|x\|_{m,\delta,\rho} \geq \gamma \sum_{i=0}^{m-1} \text{vraimax}_t g_i(t) \|x^{(i)}(t)\| \quad (x \in E_{m,\delta,\rho}),$$

where

$$g_i(t) = t^{(i-\underline{l})_+}.(1-t)^{(i-\overline{l})_+}$$

$$((i-\underline{l})_+ = \max(i-\underline{l}, 0), \ (i-\overline{l})_+ = \max(i-\overline{l}, 0)).$$

The following estimate has been established by Kolmogorov and Gorni ([6], p.393).

Theorem 3.2. *There exist constants* θ_1 *and* θ_2 *such that for all* $f \in C^m([0,1], \mathbb{R})$ *and* $k = 0, 1, \ldots, m$, *the following inequality holds*

$$\max_{0 \leq t \leq 1} |f^{(k)}(t)| \leq \theta_1 (\max_{0 \leq t \leq 1} |f(t)|)^{\frac{m-k}{m}}.(M(f))^{\frac{k}{m}},$$

where

$$M(f) = \max\left\{ \max_{0 \leq t \leq 1} |f^{(m)}(t)|, \ \theta_2 \max_{0 \leq t \leq 1} |f(t)| \right\}.$$

Theorem 3.2 implies the existence of constants $\tilde{\theta_1}$ and $\tilde{\theta_2}$ such that, for all vector functions $x \in C^m([0,1], \mathbb{R}^n)$ one has

$$\max_{0 \leq t \leq 1} \|x^{(k)}(t)\| \leq \tilde{\theta_1} (\max_{0 \leq t \leq 1} \|x^{(k_0)(t)}\|)^{\frac{m-k}{m-k_0}}$$
$$.(\max\left\{ \max_{0 \leq t \leq 1} \|x^{(m)}(t)\|, \ \tilde{\theta_2} \max_{0 \leq t \leq 1} \|x^{(k_0)}(t)\| \right\})^{\frac{k-k_0}{m-k_0}} \tag{13}$$

$$(k = k_0 + 1, \ldots, m).$$

Let us recall a known a priori estimate established in [1], that will be used in the proof of the next theorem.

Lemma 3.1. *Suppose that the nonnegative vector function* $F(t, \xi_0, \xi_1, \ldots, \xi_{m-1})$ $(F(t, 0, \ldots, 0) = 0)$ *satisfies the following conditions:*
1)

$$F(t, \xi_0, \xi_1, \ldots, \xi_{m-1}) \geq a|\xi_0| - b, \tag{14}$$

where a *is a fixed positive scalar and* $b \in \mathbb{R}^n_+$ *is a fixed vector;*
2)

$$\|F(t, \xi_0, \xi_1, \ldots, \xi_{m-1})\| \leq A(\|\xi_0\|, \|\xi_1\|, \ldots, \|\xi_{k_0}\|)$$
$$+ B(\|\xi_0\|, \|\xi_1\|, \ldots, \|\xi_{k_0}\|). \sum_{i=k_0+1}^{m} \|\xi_i\|^{p_i}, \tag{15}$$

where A, B are functions , which are increasing in each variables and defined on $\mathbb{R}_+^{k_0+1}$, and the p_i $(i = k_0+1, \ldots, m-1)$ satisfy the inequalities

$$p_i < \frac{m-k_0}{i-k_0} \quad (i = k_0+1, \ldots, m-1). \tag{16}$$

Then there exist scalars $\mu_0, M > 0$ such that the relation

$$x = tAF(x,\mu) + (1-t)aAP(\mu^m x), \quad t \in [0,1], \mu > \mu_0 \tag{17}$$

implies the following a priori estimate

$$\max_{0 \le t \le 1} \|x^{(m)}(t)\| \le M. \tag{18}$$

From now on let $E_1 = C([0,1], \mathbb{R}^n)$, $K_1 \subset E_1$ be the cone of nonnegative vector functions x, $E = C^{m-1}([0,1], \mathbb{R}^n)$, A be the operator mentioned in Theorem 3.1, $K := AK_1$, $E_0 = E_{m,\delta,\rho}$. Observe that for any $x \in E$ and $\mu \in \mathbb{R}$ the vector function defined by

$$F(t, \mu^m x, \mu^{m-1}x', \ldots, \mu x^{(m-1)})$$

belongs to E_1 and is nonnegative. Hence the operator $F(x,\mu) : K \times (-\infty, \infty) \longrightarrow K_1$. Clearly, we have $AK_1 \subset K$, which means that the operator A is positive.

Theorem 3.3. *Suppose that the conditions of Lemma* 3.1 *and the hypotheses formulated in Sec.* 3 *are satisfied.*

Then positive solutions x $(x \in K)$ of the problem (10) $-$ (11) *form in E a connected boundedly compact branch of infinite length, which goes out from the origin and tends to infinity.*

Proof. Suppose that a vector $x \in C^m([0,1], \mathbb{R}^n)$ and scalars $t \in [0,1]$, $\mu > \mu_0$ $\left(> \left(\frac{1}{a\alpha}\right)^{\frac{1}{m}}\right)$ satisfy the relations (17). Then by virtue of Lemma 3.1 we have

$$\max_{0 \le t \le 1} \|x^{(m)}(t)\| \le M,$$

where the constant M does not depend on x and the inequalities

$$\max_{0 \le t \le 1} \|x^{(i)}(t)\| \le \frac{\alpha f_0(b)}{\gamma\beta(a\alpha\mu^m - 1)} = \psi(\mu) \tag{19}$$

$$(i = 0, 1, \ldots, k_0)$$

hold (see the proof of this lemma). Then one can find a scalar $\tilde{M}$ such that

$$\tilde{\theta}_1(\max\left\{\max_{0 \le t \le 1} \|x^{(m)}(t)\|,\ \tilde{\theta}_2 \max_{0 \le t \le 1} \|x^{(k_0)}(t)\|\right\})^{\frac{k-k_0}{m-k_0}} \le \tilde{M}$$

$$(k = k_0 + 1, \dots, m-1).$$

Taking account of the inequalities (13) and (19) gives

$$\begin{aligned}\max_{0\le t\le 1} \|x^{(k)}(t)\| &\le \tilde{M}(\max_{0\le t\le 1} \|x^{(k_0)}(t)\|)^{\frac{m-k}{m-k_0}} \\ &\le \tilde{M}\,[\psi(\mu)]^{\frac{m-k}{m-k_0}} \quad (k = k_0+1, \dots, m-1)\end{aligned}$$

which means that

$$\begin{aligned}\|x\|_{C^{m-1}} &= \sum_{i=0}^{m-1} \max_{0\le t\le 1} \|x^{(i)}(t)\| \\ &= \sum_{i=0}^{k_0} \max_{0\le t\le 1} \|x^{(i)}(t)\| + \sum_{i=k_0+1}^{m-1} \max_{0\le t\le 1} \|x^{(i)}(t)\| \\ &\le (k_0+1)\psi(\mu) + \tilde{M} \sum_{i=k_0+1}^{m-1} [\psi(\mu)]^{\frac{m-i}{m-k_0}}.\end{aligned} \tag{20}$$

Next we claim that the operators A and $F(x,\mu)$ $(\mu \in (0,\infty), x \in E)$ satisfy all the conditions of Theorem 2.2, where we replace K by E, $\Omega(K)$ by Ω and $\partial\Omega(K)$ by $\partial\Omega$. Indeed, let $m = a, \varphi(\mu) = \mu^m, l = f_0(b)$,

$$\begin{aligned}\Phi(t,\mu) &= \Phi(\mu) \\ &= (k_0+1)\frac{\alpha f_0(b)}{\gamma\beta(a\alpha\mu^m - 1)} + \tilde{M} \sum_{k=k_0+1}^{m-1} \left[\frac{\alpha f_0(b)}{\gamma\beta(a\alpha\mu^m - 1)}\right]^{\frac{m-k}{m-k_0}}.\end{aligned}$$

It is easy to see that the operators $F(x,\mu)$ and A satisfy the condition 1) of Theorem 2.2 (see Theorem 3.1). Since for any $R > 0$

$$\lim_{\mu\to 0} \sup_{\|x\|_{C^{m-1}} \le R} \|AF(x,\mu)\|_{C^{m-1}} = 0,$$

then the condition 2) of Theorem 2.2 is also satisfied.

Further, by virtue of the inequality (14) we get

$$f_0(F(x,\mu)) \ge a f_0(P(\mu^m x)) - f_0(b),$$

which means that the condition 3a) of Theorem 2.2 is satisfied.

Next, the condition 3b) of Theorem 2.2 is immediate from the estimate (20) and the definition of the function $\Phi(t,\mu)$.

Finally, the equalities

$$\lim_{\mu\to\infty} \alpha a \mu^m = \infty$$

and

$$\lim_{\mu\to\infty} \Phi(\mu) = 0$$

yield the condition 3c) of Theorem 2.2.

Now applying the remark to the problem (10)-(11) we see that there exists in the set V of solutions to this problem a connected boundedly compact branch $\mathcal{M}$ of infinite length which connects the origin and the infinity.

To complete the proof it suffices to note that the branch $\mathcal{M} \subset V \subset K$. ∎

Remark. Note that applying the existence of the connected boundedly compact branch in the set of solutions one can deduce that the set of μ, for which the problem (10)-(11) has positive solutions contains some interval.

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