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Some Homological Properties of Artinian Modules*

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Abstract. In this paper we show that if $(R, \mathfrak{m})$ is a commutative Gorenstein local ring with maximal ideal $\mathfrak{m}$ and M is an Artinian R-module, then $\text{depth}(R) = \text{Width}(M) + \sup\{i \in \mathbb{N}_0 : \text{Ext}^i_R(E(R/\mathfrak{m}), M) \neq 0\}$. Also, we prove that the following statements are equivalent:
(1) R is Gorenstein.
(2) R is Cohen-Macaulay and for any Artinian module M, $\text{fd}(E(M)) \leq \text{fd}(M)$, where $E(M)$ is an injective envelope of M.
(3) R is Cohen-Macaulay and for any finite length module M of finite injective dimension, $\text{id}(F(M)) = \text{id}(M)$, where $F(M)$ is a flat cover of M.

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1. Introduction

Let R be a commutative Noetherian ring with non-zero identity and let M be an R-module. Auslander and Bridger [1] introduced a notion of Gorenstein dimension, denoted by G-dim, of finitely generated modules over the Cohen-Macaulay rings. It seems appropriate to call G-dim 0 modules Gorenstein projective. As

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a dual of Gorenstein projective modules, Enochs and Jenda [6] defined and studied Gorenstein injective modules. Recall that an R-module M is Gorenstein injective if and only if there is an exact sequence

$$\ldots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \ldots$$

of injective R-modules with $M = \mathrm{Ker}(E^0 \longrightarrow E^1)$ such that for any injective R-module E, $\mathrm{Hom}_R(E, -)$ leaves the above complex exact. We say that an R-module M has Gorenstein injective dimension at most n, denoted by $\mathrm{Gid}(M) \leq n$, if there is an exact sequence

$$0 \longrightarrow M \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \ldots \longrightarrow G^n \longrightarrow 0$$

of R-modules with each G^i Gorenstein injective. If there is no shorter sequence like the one mentioned above, we set $\mathrm{Gid}(M) = n$. Also, if there is no such an n, we set $\mathrm{Gid}(M) = \infty$. Enochs and Jenda [7] call M an h-divisible R-module when M is a homomorphic image of an injective R-module and also they call M an Ext-finite R-module if for any finitely generated R-module N, each $\mathrm{Ext}^i_R(N, M)$ is finitely generated for all $i \geq 1$. For a local ring $(R, \mathfrak{m})$, they also defined k-depth of an Ext-finite R-module M as k-depth $= \inf\{i : \mathrm{Ext}^i_R(R/\mathrm{m}, M) \neq 0\}$. Ooishi [10], defined width of an Artinian R-module as $\mathrm{Width}(M) = \inf\{i : \mathrm{Tor}^R_i(R/\mathrm{m}, M) \neq 0\}$, when R is a local ring with maximal ideal $\mathfrak{m}$. Auslander and Bridger in [1] proved that if R is a Gorenstein local ring with maximal ideal m and M is a finitely generated R-module, then

$$\mathrm{depth}(R) = \mathrm{depth}(M) + \sup\{i \in \mathbb{N}_0 : \mathrm{Ext}^i_R(M, R) \neq 0\}.$$

Later, Enochs and Jenda in [7] proved the following result, which is a dual of Auslander and Bridger formula. Let R be a complete Cohen-Macaulay local ring and M be a non-injective Ext-finite R-module such that $\mathrm{Ext}^i_R(E, M) = 0$ for all $i \geq 1$ and all indecomposable injective R-module $E \neq E(R/\mathrm{m})$. If M is an h-divisible R-module of finite Gorenstein injective dimension, then

$$\mathrm{depth}(R) = \mathrm{Gid}(M) + \inf\{i \in \mathbb{N}_0 : \mathrm{Tor}^R_i(R/\mathrm{m}, M) \neq 0\}.$$

Here, we show that if M is an Artinian R-module, then the above formula is true and also, if R is a Gorenstein local ring with maximal ideal m and M is an Artinian R-module, then $\mathrm{depth}(R) = \mathrm{Width}(M) + \sup\{i \in \mathbb{N}_0 : \mathrm{Ext}^i_R(E(R/\mathrm{m}), M) \neq 0\}$, (see Theorem 2.4). As a consequence, if M is a non-zero finitely generated maximal Cohen-Macaulay R-module of finite injective dimension, then the local cohomology module $H^{\dim(R)}_{\mathrm{m}}(M)$ is an injective R-module (see Corollary 2.6).

Xu in [14] showed that R is Gorenstein if and only if for any finitely generated module M, $\mathrm{fd}(E(M)) \leq \mathrm{fd}(M)$. Also, he proved that R is Gorenstein if and only if $\mathrm{id}(F(M)) \leq \mathrm{id}(M)$ for any module M of finite injective dimension. Here, we prove that if R is Cohen-Macaulay and M is an Artinian R-module then the above results are true, too.

Throughout this paper, R is a commutative Noetherian ring with non-zero identity. For any R-module M, $\mathrm{fd}(M)$ stands for the flat dimension of R-module M, $\mathrm{id}(M)$ stands for the injective dimension of M, $E(M)$ stands for its injective envelope, $F(M)$ stands for its flat cover, and $J(R)$ stands for its Jacobson radical of R. For any unexplained notation or terminology, we refer the reader to [2, 8, 15].

2. The Results

We start this section with the following lemmas.

Lemma 2.1. (Nakayama's Lemma) *Let $\mathfrak{a}$ be an ideal of R and M be an Artinian R-module such that $0 :_M \mathfrak{a} = 0$. If $\mathfrak{a} \subseteq J(R)$, then $M = 0$.*

Proof. See [9, § 4]. ∎

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a local ring, and M be a non-zero, Artinian R-module. Then*
(a) $\mathrm{id}(M) = \sup\{i \in \mathbb{N}_0 : \mathrm{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\}$.
(b) $\mathrm{fd}(M) = \sup\{i \in \mathbb{N}_0 : \mathrm{Tor}^R_i(R/\mathfrak{m}, M) \neq 0\}$.

Proof. (a) We can assume that the right-hand side of the above equality is finite. Thus there is an integer n such that $\mathrm{Ext}^n_R(R/\mathfrak{m}, M) \neq 0$ and $\mathrm{Ext}^i_R(R/\mathfrak{m}, M) = 0$ for all $i > n$ and so $\mathrm{id}(M) \geq n$. Let $\mathrm{id}(M) > n$. Then $X = \{\mathfrak{a}$, an ideal of $R : \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \neq 0$ for some $i > n\}$ is non-empty. Let $\mathfrak{a}$ be a maximal element of X and thus by assumption $\mathfrak{a} \neq \mathfrak{m}$. Therefore there exists $x \in \mathfrak{m} \setminus \mathfrak{a}$. From the exact sequence $0 \longrightarrow R/(\mathfrak{a} : x) \xrightarrow{x} R/\mathfrak{a} \longrightarrow R/(\mathfrak{a}, x) \longrightarrow 0$, we obtain the exact sequence

$$\mathrm{Ext}^i_R(R/(\mathfrak{a}, x), M) \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \xrightarrow{x} \mathrm{Ext}^i_R(R/(\mathfrak{a} : x), M)$$

$$\longrightarrow \mathrm{Ext}^{i+1}_R(R/(\mathfrak{a}, x), M).$$

This gives $\mathrm{Ext}^i_R(R/\mathfrak{a}, M) \neq 0$ for some $i > n$ and since $\mathfrak{a} \subsetneqq (\mathfrak{a}, x)$ we have $\mathrm{Ext}^i_R(R/(\mathfrak{a}, x), M) = 0$. Therefore $(0 :_{\mathrm{Ext}^i_R(R/\mathfrak{a}, M)} x) = 0$ and the Nakayama Lemma yields that $\mathrm{Ext}^i_R(R/\mathfrak{a}, M) = 0$, which is a contradiction. Therefore the set $X = \emptyset$ and we must have $\mathrm{id}(M) = n$.

(b) By making straightforward modification to the arguments in the proof of (a), one can obtain (b). ∎

Proposition 2.3. *Let $(R, \mathfrak{m})$ be a local ring, and M be a non-zero, Artinian R-module. Then the following are true:*
(1) *If M is of finite injective dimension, then* $\mathrm{id}(M) \leq \mathrm{depth}(R)$,
(2) *If M is of infinite injective dimension, then* $\mathrm{Ext}^i_R(R/\mathfrak{m}, M) \neq 0$ *for all* $i \geq \mathrm{dim}(R)$.

Proof. (1) Let $x_1, \ldots, x_n \in \mathfrak{m}$ be a maximal regular sequence for R. Then $\mathfrak{m} \in \mathrm{Ass}_R(R/(x_1, \ldots, x_n))$ and hence $R/\mathfrak{m} \subseteq R/(x_1, \ldots, x_n)$. Thus we have the exact sequence

$$\mathrm{Ext}^t_R(R/(x_1, \ldots, x_n),\ M) \longrightarrow \mathrm{Ext}^t_R(R/\mathfrak{m},\ M) \longrightarrow 0,$$

where $t = \mathrm{id}(M)$. Therefore by Lemma 2.2, $\mathrm{Ext}^t_R(R/(x_1, \ldots, x_n),\ M) \neq 0$. Therefore $n \geq t$, since $\mathrm{pd}(R/(x_1, \ldots, x_n)) = n$.

(2) It is clear. ■

Theorem 2.4. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension n, and M be a non-zero, Artinian R-module. Then*

$$\mathrm{depth}(R) = \mathrm{Width}(M) + \sup\{i \in \mathbb{N}_0 : \mathrm{Ext}^i_R(E(R/\mathfrak{m}),\ M) \neq 0\}.$$

Proof. Let $\mathrm{Width}(M) = t$. Then we proceed by induction on t. Suppose $t = 0$. Then there exists an exact sequence $M \longrightarrow R/\mathfrak{m} \longrightarrow 0$ (see [10, Proposition 3.5]) and since $\mathrm{fd}(E(R/\mathfrak{m})) = n = \mathrm{pd}(E(R/\mathfrak{m}))$ by [5, Corollary 3.3], it therefore follows that the induced map

$$\mathrm{Ext}^n_R(E(R/\mathfrak{m}), M) \longrightarrow \mathrm{Ext}^n_R(E(R/\mathfrak{m}), R/\mathfrak{m})$$

is an epimorphism. On the other hand,

$$\mathrm{Ext}^n_R(E(R/\mathfrak{m}), R/\mathfrak{m}) \cong \mathrm{Ext}^n_R(E(R/\mathfrak{m}), \mathrm{Hom}_R(R/\mathfrak{m},\ E(R/\mathfrak{m})))$$

$$\cong \mathrm{Hom}_R(\mathrm{Tor}^R_n(R/\mathfrak{m}, E(R/\mathfrak{m})), E(R/\mathfrak{m})).$$

Therefore by Lemma 2.2, $\mathrm{Ext}^n_R(E(R/\mathfrak{m}), R/\mathfrak{m})$ is non-zero, and hence $\mathrm{Ext}^n_R(E(R/\mathfrak{m}), M) \neq 0$, which gives us our desired result when $t = 0$. Suppose the theorem is true for $0 \leq t < s$ and suppose $\mathrm{Width}(M) = s$. Let $x \in \mathfrak{m}$ be a coregular sequence for M. Then we know that the $\mathrm{Width}(0 :_M x) = s - 1$ (see [10, Proposition 3.15]). Therefore we know by induction hypothesis that $\mathrm{Ext}^{n-s+1}_R(E(R/\mathfrak{m}), (0 :_M x)) \neq 0$ and $\mathrm{Ext}^j_R(E(R/\mathfrak{m}), (0 :_M x)) = 0$ for all $j > n - s + 1$. From the long exact sequence

$$\ldots \longrightarrow \mathrm{Ext}^i_R(E(R/\mathfrak{m}),\ M) \xrightarrow{x} \mathrm{Ext}^i_R(E(R/\mathfrak{m}),\ M) \longrightarrow \mathrm{Ext}^{i+1}_R(E(R/\mathfrak{m}), (0 :_M x)) \longrightarrow \ldots$$

derived from the exact sequence

$$0 \longrightarrow (0 :_M x) \longrightarrow M \xrightarrow{x} M \longrightarrow 0,$$

it follows that

$$\mathrm{Ext}^i_R(E(R/\mathfrak{m}), M) \xrightarrow{x} \mathrm{Ext}^i_R(E(R/\mathfrak{m}), M)$$

is an epimorphism for all $i \geq n-s+1$. Since $x \in \mathfrak{m}$ and the $\mathrm{Ext}^i_R(E(R/\mathfrak{m}),\, M)$ are finitely generated $\hat{R}$-modules by [3, Proposition 9], it follows from Nakayama's Lemma that $\mathrm{Ext}^i_R(E(R/\mathfrak{m}), M) = 0$ for all $i \geq n-s+1$. On the other hand, it follows now that the map

$$\mathrm{Ext}^{n-s}_R(E(R/\mathfrak{m}),\, M) \longrightarrow \; Ext^{n-s+1}_R(E(R/\mathfrak{m}), (0 :_M x))$$

is an epimorphism. Since $\mathrm{Ext}^{n-s+1}_R(E(R/\mathfrak{m}), (0 :_M x)) \neq 0$, we know that $\mathrm{Ext}^{n-s}_R(E(R/\mathfrak{m}), M) \neq 0$, which establishes the theorem for $t = s$ and thus completes the proof of the theorem. ■

Corollary 2.5. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension n, and M be a non-zero, Artinian R-module of finite injective dimension. Then*

$$\mathrm{depth}(R) = \mathrm{Width}(M) + \mathrm{id}(M).$$

Proof. Let $\mathrm{id}(M) = s$. From the long exact sequence

$$\ldots \longrightarrow \mathrm{Ext}^i_R(E(R/\mathfrak{m}),\, M) \longrightarrow \mathrm{Ext}^i_R(R/\mathfrak{m}, M) \longrightarrow \mathrm{Ext}^{i+1}_R(N, M) \longrightarrow \ldots$$

derived from the exact sequence

$$0 \longrightarrow R/\mathfrak{m} \longrightarrow E(R/\mathfrak{m}) \longrightarrow N \longrightarrow 0,$$

it follows that $\mathrm{Ext}^s_R(E(R/\mathfrak{m}),\, M) \neq 0$ by Lemma 2.2. Hence we conclude that $\mathrm{id}(M) = \sup\{i \in \mathbb{N}_0 : \mathrm{Ext}^i_R(E(R/\mathfrak{m}), M) \neq 0\}$ and so by Theorem 2.4 the result follows. ■

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension n, and M be a non-zero finitely generated maximal Cohen-Macaulay R-module of finite injective dimension. Then the local cohomology module $H^n_{\mathfrak{m}}(M)$ is an injective R-module.*

Proof. Consider the Grothendieck spectral sequence [11, Theorem 11.38]

$$E_2^{i,j} := \mathrm{Ext}^i_R(R/\mathfrak{m}, H^j_{\mathfrak{m}}(M)) \underset{i}{\Longrightarrow} \mathrm{Ext}^{i+j}_R(R/\mathfrak{m}, M). \tag{1}$$

Since $\mathrm{depth}(M) = \dim(R)$ and $\mathrm{id}(M) = \mathrm{depth}(R)$, it follows by (1) that

$$\mathrm{Ext}^i_R(R/\mathfrak{m},\, H^n_{\mathfrak{m}}(M)) = \mathrm{Ext}^{i+n}_R(R/\mathfrak{m}, M).$$

By Lemma 2.2, it therefore follows that $\mathrm{id}(H^n_{\mathfrak{m}}(M)) < \infty$. Now, by using [13, Proposition 2.6] and Corollary 2.5, the result follows. ■

Theorem 2.7. *The following statements are equivalent:*
(1) *R is Gorenstein.*
(2) *For any finitely generated module M, $\mathrm{fd}(E(M)) \leq \mathrm{fd}(M)$.*
(3) *For any module M, $\mathrm{fd}(E(M)) \leq \mathrm{fd}(M)$.*

(4) *R is Cohen-Macaulay and for any Artinian module M,* $\mathrm{fd}(E(M)) \leq \mathrm{fd}(M)$.

Proof. The implications (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) follow from [14, Theorem 2.3]. (1) $\Longrightarrow$ (4) is clear.

(4) $\Longrightarrow$ (1). We can assume that R is local with maximal ideal $\mathfrak{m}$. Now, consider a maximal regular sequence $x_1, \ldots, x_t \in \mathfrak{m}$. Then $M = R/(x_1, \ldots, x_t)$ is Artinian of finite flat dimension. On the other hand, $\mathfrak{m} \in \mathrm{Ass}_R(M)$ and so $R/\mathfrak{m} \subseteq M$. This implies that $E(R/\mathfrak{m}) \subseteq E(M)$ and that by assumption $\mathrm{fd}(E(R/\mathfrak{m})) < \infty$, because it is a direct summand of $E(M)$. Hence R is Gorenstein by [14, Proposition 2.1]. ■

Corollary 2.8. *Let $(R, \mathfrak{m})$ be a local ring. Then the following results are equivalent:*
(1) *R is Gorenstein.*
(2) *R is Cohen-Macaulay and for any Artinian module M of finite flat dimension,* $\mathrm{fd}(E(M)) = \mathrm{fd}(M)$.

Proof. (1) $\Longrightarrow$ (2). Since M is Artinian, $E(M) = \oplus_{i=1}^{t} E(R/\mathfrak{m})$. By [5, Corollary 3.3], $\mathrm{fd}(E(M)) = \dim(R)$. Then by the assumption, [8, Theorem 9.1.10], and Theorem 2.7 the result follows.

(2) $\Longrightarrow$ (1). By similar argument as in the proof of Theorem 2.7 ((4) $\Longrightarrow$ (1)), we get the desired result. ■

From Corollary 2.8, we have immediately the following corollary.

Corollary 2.9. *Let $(R, \mathfrak{m})$ be a regular local ring and M be an Artinian R-module. Then* $\mathrm{fd}(E(M)) = \mathrm{fd}(M)$.

Theorem 2.10. *Let $(R, \mathfrak{m})$ be a local ring. Then the following are equivalent:*
(1) *R is Gorenstein.*
(2) *For any module M of finite injective dimension,* $\mathrm{id}(F(M)) \leq \mathrm{id}(M)$*, here $F(M)$ is the flat cover of M.*
(3) *R is Cohen-Macaulay and for any finite length module M of finite injective dimension,* $\mathrm{id}(F(M)) = \mathrm{id}(M)$.

Proof. (1) $\Longleftrightarrow$ (2) follows from [14, Theorem 3.4].

(2) $\Longrightarrow$ (3). It is clear that $\mathrm{id}(F(M)) \leq \mathrm{id}(M)$. On the other hand, $\mathrm{id}(M) \leq \dim(R)$, since R is Gorenstein. Next, $\mathrm{id}(F(M)) = \mathrm{fd}(\mathrm{Hom}_R(F(M), E(R/\mathfrak{m}))) = \mathrm{fd}(E(R/\mathfrak{m})) = \dim(R)$, since $E(R/\mathfrak{m}) \subseteq \mathrm{Hom}_R(F(M), E(R/\mathfrak{m}))$. It therefore follows that $\mathrm{id}(M) = \mathrm{id}(F(M))$.

(3) $\Longrightarrow$ (1). Let $x_1, \ldots, x_n \in \mathfrak{m}$ be a maximal regular sequence on R. Then, $M = R/(x_1, \ldots, x_n)$ is a finite length module of finite flat dimension and so $\mathrm{Hom}_R(M, E(R/\mathfrak{m}))$ is a finite length module of finite injective dimension. By the assumption, its flat cover F has finite injective dimension. It follows that $E(R/\mathfrak{m})$ has finite flat dimension, since $E(R/\mathfrak{m}) \subseteq \mathrm{Hom}_R(F, E(R/\mathfrak{m}))$ and $\mathrm{Hom}_R(F, E(R/\mathfrak{m}))$ has finite flat dimension. Hence, by [14, Proposition 2.1], R is Gorenstein. ■

Lemma 2.11. *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring and* M *be a non-zero, Artinian Gorenstein injective* R*-module. Then* $\mathrm{Width}(M) = \mathrm{depth}(R)$.

Proof. We proceed by induction on $n = \mathrm{depth}(R)$. If $n = 0$, then the maximal ideal $\mathfrak{m}$ is nilpotent. Hence $M/\mathfrak{m}M \neq 0$ and so $\mathrm{Width}(M) = 0$. Let $n \geq 1$. Then there exists a regular element x of $\mathfrak{m}$ and so by [7, Lemma 3.1], $(0 :_M x)$ is Gorenstein injective $\bar{R}$-module (where $\bar{R} = R/xR$). On the other hand, $\bar{R}$ is a Cohen-Macaulay ring of dimension $n-1$ and $\mathrm{Width}_{\bar{R}}(0 :_M x) = n-1$ by induction hypothesis. Since M is a Gorenstein injective module, we have $M = xM$ and so $\mathrm{Width}_{\bar{R}}(0 :_M x) = \mathrm{Width}(M) - 1$ by [7, Proposition 2.3]. Hence $\mathrm{Width}(M) = \mathrm{depth}(R)$. ■

Proposition 2.12. *Let* $(R, \mathfrak{m})$ *be a complete Cohen-Macaulay local ring and* M *be a non-injective Artinian module such that* $\mathrm{Ext}^i_R(E, M) = 0$ *for all* $i \geq 1$ *and all indecomposable injective* R*-module* $E \neq E(R/\mathfrak{m})$*. If* M *has finite Gorenstein injective dimension, then the following are equivalent:*
(1) *M is Gorenstein injective.*
(2) $\mathrm{Ext}^i_R(E(R/\mathfrak{m}), M) = 0$ for all $i \geq 1$.
(3) M is *h-divisible and* $\mathrm{Width}(M) = \mathrm{depth}(R)$.

Proof. (1) $\Longleftrightarrow$ (2) follows from [5, Proposition 4.3].

(1) $\Longrightarrow$ (3) follows from [8, Remark 10.1.5] and Lemma 2.11.

(3) $\Longrightarrow$ (2). Let $n = \mathrm{depth}(R)$ and $t = \mathrm{Gid}(M)$. We proceed by induction on n. Let $n = 0$. We consider an exact sequence

$$0 \longrightarrow M \longrightarrow E(R/\mathfrak{m})^{n_0} \longrightarrow \ldots \longrightarrow E(R/\mathfrak{m})^{n_{t-1}} \longrightarrow C \longrightarrow 0,$$

where C is a Gorenstein injective Artinian module. But then

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_R(C, E(R/\mathfrak{m}) \longrightarrow R^{n_{t-1}} \longrightarrow \ldots \\ \longrightarrow R^{n_0} \longrightarrow \mathrm{Hom}_R(M, E(R/\mathfrak{m})) \longrightarrow 0 \end{aligned}$$

is exact, and $\mathrm{Hom}_R(C, E(R/\mathfrak{m})$ is Gorenstein projective by [4, Theorem 4.8]. Hence $\text{G-dim}(\mathrm{Hom}_R(M, E(R/\mathfrak{m}))) < \infty$ and so $\mathrm{Hom}_R(M, E(R/\mathfrak{m}))$ is Gorenstein projective by [1, Proposition 4.11], since $n = 0$. Therefore, by [4, Theorem 4.8], the result follows. Now, suppose $n \geq 1$. Then there exists a regular sequence $x \in \mathfrak{m}$ on R and so $\text{k-depth}(M) = 0$. Then, by [12, Corollary 6.1.10], $\mathrm{Width}(M) = n$. Since $(0 :_M x) \neq 0$, then $\mathrm{Width}_{\bar{R}}(0 :_M x) = n-1$ by [7, Proposition 2.3]. Therefore $(0 :_M x)$ is a Gorenstein injective $\bar{R}$-module by induction hypothesis (where $\bar{R} = R/xR$). Now, consider an exact sequence

$$0 \longrightarrow (0 :_M x) \longrightarrow M \stackrel{x}{\longrightarrow} M \longrightarrow 0.$$

But then we have the following long exact sequence

$$\begin{aligned} \ldots \longrightarrow \mathrm{Ext}^i_R(E(R/\mathfrak{m}),\, M) \stackrel{x}{\longrightarrow} \mathrm{Ext}^i_R(E(R/\mathfrak{m}),\, M) \\ \longrightarrow \mathrm{Ext}^{i+1}_R(E(R/\mathfrak{m}), (0 :_M x)) \longrightarrow \ldots. \end{aligned}$$

Since $\mathrm{Ext}_R^{i+1}(E(R/\mathfrak{m}),(0:_M x))\cong \mathrm{Ext}_{\bar{R}}^{i}((0:_{E(R/\mathfrak{m})} x),(0:_M x))$ by [7, Lemma 2.1] and $(0:_M x)$ is a Gorenstein injective $\bar{R}$-module by the above argument, we get $\mathrm{Ext}_R^{i+1}(E(R/\mathfrak{m}),(0:_M x))=0$. Now, by [3, Proposition 9] and Nakayama's Lemma, it follows that $\mathrm{Ext}_R^{i}(E(R/\mathfrak{m}),M)=0$. ■

Enochs and Jenda in [7, Theorem 4.8] proved that the following theorem is true when M is an Ext-finite R-module.

Theorem 2.13. *Let $(R,\mathfrak{m})$ be a complete Cohen-Macaulay local ring of dimension n, and M be a non-injective, Artinian R-module such that $Ext_R^i(E,M)=0$ for all $i\geq 1$ and all indecomposable injective R-module $E\neq E(R/\mathfrak{m})$. If M is an h-divisible module of finite Gorenstein injective dimension, then*

$$\mathrm{depth}(R)=\mathrm{Gid}(M)+\mathrm{Width}(M).$$

Proof. We proceed by induction on the Gorenstein injective dimension. If M is Gorenstein injective, then we are done by Lemma 2.11. Now suppose that $\mathrm{Gid}(M)\geq 1$. Then $\mathrm{depth}(R)\geq 1$. For, if $\mathrm{depth}(R)=0$, then $\mathrm{Width}(M)=0$ and $\mathrm{Ext}_R^i(E,M)=0$ for all $i\geq 1$ by hypothesis and Proposition 2.12, and so M is Gorenstein injective. If $\mathrm{Gid}(M)=1$, then $E(M)/M$ is Gorenstein injective. Therefore $\mathrm{Width}(E(M)/M)=\mathrm{depth}(R)$ by Lemma 2.11. On the other hand, $\mathrm{Width}(M)\leq \mathrm{depth}(R)$. Hence $\mathrm{Width}(M)\leq \mathrm{depth}(R)-1$. For if $\mathrm{Width}(M)=\mathrm{depth}(R)$, then $\mathrm{Ext}_R^i(E,M)=0$ for all $i\geq 1$ by hypothesis and Proposition 2.12, and hence M is Gorenstein injective. Therefore $\mathrm{Width}(M)+1=\mathrm{Width}(E(M)/M)$ by [7, Lemma 4.6] and so we are done. If $\mathrm{Gid}(M)=r>1$, then $\mathrm{Gid}(E(M)/M)=r-1\geq 1$. Therefore $r-1=\mathrm{Gid}(E(M)/M)=\mathrm{depth}(R)-\mathrm{Width}(E(M)/M)$ by induction hypothesis. Hence $r-1=\mathrm{depth}(R)-(\mathrm{Width}(M)+1)$ and thus $\mathrm{Gid}(M)+\mathrm{Width}(M)=\mathrm{depth}(R)$. ■

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