Vietnam Journal of Mathematics **36**:2(2008) 183–190

Vietnam Journal of MATHEMATICS

Primary Decomposition for Noncommutative Rings

Hossain Khabazian

Department of Mathematical Sciences
Isfahan University of Technology, Isfahan 84156-83111, Iran

Received August 23, 2007
Revised Febuary 27, 2008

Abstract. A necessary condition for the existence of primary decomposition in noncommutative rings is investigated and its role is explored. Furthermore, the associated prime ideals are determined.

2000 Mathematics Subject Classification: 16D70, 16D25.
Keywords: Prime ideal, primary ideal, primary decomposition, prime right annihilator, annihilator ideal.

1. Introduction

Throughout, ideals are two sided ideals and rings are unitary. An ideal I which is a right annihilator, in other words which satisfies $\mathrm{ann}_r(\mathrm{ann}_l(I)) = I$, is called rA-ideal. An ideal I with $\mathrm{ann}_l(I) \cap I = 0$, is called lS-ideal. By a *prime right annihilator* we mean a prime ideal which also is a right annihilator (see [10, p. 428]).

A ring is called *I-noetherian* if it satisfies ACC on ideals and is called *rAI-noetherian* if it satisfies ACC on rA-ideals. *lAI-noetherian* is defined similarly. Clearly I-noetherian implies both rAI-noetherian and lAI-noetherian. A nonzero ring R is called *left primary* provided every ideal is either left faithful or is contained in $P(R)$, in other words if for every ideals I and J, $IJ = 0$ implies either $I = 0$ or $J \subseteq P(R)$. $P(R)$ is the prime radical of R. Clearly this is an appropriate extension of the standard primary defined for commutative rings. It is easy to see that in a left primary ring R in which $P(R)$ is nilpotent, $P(R)$ is a prime ideal (in fact $P(R)$ is the unique maximal rA-ideal and the unique minimal prime ideal), also for every ideal I there is $n \geq 1$ such that either $\mathrm{ann}_l(I^n) = 0$ or $I^n = 0$. This conclusion suggests us to define *left semiprimary* rings as rings in which for every ideal I, a power of I is a lS- ideal. An ideal U of a ring R is called *left primary* if R/U is a left primary ring, in other words if for every ideals I and J, $IJ \subseteq U$ implies either $I \subseteq U$ or $J \subseteq \mathrm{rad}(U)$. Thus, in I-noetherian rings, if U is a left primary ideal then $\mathrm{rad}(U)$ is a prime ideal and there is $n \geq 1$ such that $\mathrm{rad}(U)^n \subseteq U$. Remember that a ring R is called semiprimary if $J(R)$, the Jacobson radical of R, is nilpotent and $R/J(R)$ is semisimple. (see [5, p. 59]).

The existence of primary decomposition for rings and modules have been investigated in various papers [see for example (1, Theorem 3.10 and Sec. 3.8], [2], [8] or [9 Exercise 9.31]). The uniqueness of the primary decomposition also has been explored (see for example [1, Sec. 3.7], [3], [6] or [10]), but the associated prime ideals are not introduced. In Theorem 16, we show that the associated prime ideals are just the prime right annihilators. A ring has to be left semiprimary in order to have left primary decomposition. In this paper, we also explore this property. As some results, we present new and shorter proofs for some old results and present new results.

For any set T of subgroups of the additive group R, $\cap_{I\in T} I$ is denoted by $\mathrm{Int}(T)$ and $\sum_{I\in T} I$, the subgroup generated by $\cup_{I\in T} I$, is denoted by $\mathrm{Sum}(T)$. A finite set T of left primary ideals is called a *normal set*, if

1. for every $U \in T$, $\mathrm{Int}(T - \{U\}) \nsubseteq U$,
2. for every distinct $U, V \in T$, $\mathrm{rad}(U) \neq \mathrm{rad}(V)$.

A ring has a *primary decomposition* if there is a finite collection of left primary ideals with zero intersection.

For every ideal P, the sum of lA-ideals J in which $\mathrm{ann}_r(J) \nsubseteq P$ is denoted by P^o. It is easy to see that if P is prime, then $P^o \subseteq P$ and

$$P^o = \{a \in R \mid \text{there exists an ideal } I \nsubseteq P \text{ with } aI = 0\}.$$

Lemma 1. *Let R be a rAI-artinian ($\equiv lAI$-noetherian) ring and P be a prime ideal.*

1. $\mathrm{ann}_r(P^o) \nsubseteq P$ *and P^o is a lA-ideal.*
2. *If $\mathrm{rad}(P^o) = P$, then P^o is a left primary ideal and P is a minimal prime ideal.*
3. *If P is a right annihilator, then $\mathrm{ann}_l(P) \nsubseteq P^o$.*

Proof. 1. Let I be a rA-ideal such that $I \nsubseteq P$ and minimal with respect to these conditions. We show that $P^o = \mathrm{ann}_l(I)$. It is obvious that $\mathrm{ann}_l(I) \subseteq P^o$. Suppose that $a \in P^o$. There exists an ideal $J \nsubseteq P$ with $aJ = 0$. Then $a(\mathrm{ann}_r(\mathrm{ann}_l(J))) = 0$. So, we may assume that J is a rA- ideal, thus $I \cap J$ is a rA-ideal. On the other hand $I \cap J \nsubseteq P$, implying $I \cap J = I$, consequently $I \subseteq J$. Thus $a \in \mathrm{ann}_l(I)$. Since $\mathrm{ann}_r(P^o) = I$, $\mathrm{ann}_r(P^o) \nsubseteq P$.

2. Let A and B be ideals with $AB \subseteq P^o$ and $B \nsubseteq P$. Since $B\ \mathrm{ann}_r(P^o) \nsubseteq P$ and $AB\ \mathrm{ann}_r(P^o) = 0$, $A \subseteq P^o$. Now let Q be a prime ideal contained in P. Then $P^o \subseteq Q^o \subseteq Q$, implying $P = \mathrm{rad}(P^o) \subseteq Q$, consequently $P = Q$.

3. If $\mathrm{ann}_l(P) \subseteq P^o$, then $\mathrm{ann}_r(P^o) \subseteq \mathrm{ann}_r(\mathrm{ann}_l(P)) = P$ which is a contradiction. ■

Lemma 2. *Let A and B be ideals. For every ideal J, if $\mathrm{ann}_l(A) \cap J = 0$ and $\mathrm{ann}_l(B) \cap J = 0$, then $\mathrm{ann}_l(AB) \cap J = 0$.*

Proof. Since

$$(\mathrm{ann}_l(AB) \cap J)A \subseteq \mathrm{ann}_l(AB)A \cap J \subseteq \mathrm{ann}_l(B0 \cap J = 0,$$

$$\mathrm{ann}_l(AB) \cap J \subseteq \mathrm{ann}_l(A)$$

implying $\mathrm{ann}_l(AB) \cap J = 0$. ■

Lemma 3. *The intersection of any two lS-ideals is a lS-ideal.*

Proof. Let I and J be lS-ideals. Since $\mathrm{ann}_l(I) \cap (I \cap J) = 0$ and $\mathrm{ann}_l(J) \cap (I \cap J) = 0$, $\mathrm{ann}_l(IJ) \cap (I \cap J) = 0$ by Lemma 2, implying $\mathrm{ann}_l(I \cap J) \cap (I \cap J) = 0$. Thus, $I \cap J$ is a lS-ideal. Also $\mathrm{ann}_l(IJ) \subseteq \mathrm{ann}_l(I \cap J)$, implying $\mathrm{ann}_l(IJ) = \mathrm{ann}_l(I \cap J)$. Therefore IJ is also a lS-ideal. ■

Lemma 4. *Let R be a left semiprimary ring. For every ideals $I_1, I_2, \ldots, I_m$ if $I_1 I_2 \ldots I_m$ is nilpotent, then there exists $n \geq 1$ such that $\cap_{i=1}^m (I_i)^n = 0$.*

Proof. There exists $n \geq 1$ such that $(I_1 I_2 \ldots I_m)^n = 0$, implying $(\cap_{i=1}^m I_i)^{mn} = 0$ because $(\cap_{i=1}^m I_i)^m \subseteq I_1 I_2 \ldots I_m$. On the other hand, there exists $n_i \geq 1$ such that $(I_i)^{n_i}$ is a lS-ideal for all $1 \leq i \leq n$. Then, $\cap_{i=1}^m (I_i)^{n_i}$ is a lS-ideal by Lemma 4 and also is nilpotent. Thus, $\cap_{i=1}^m (I_i)^{n_i} = 0$. Set n as the maximum of $\{n_i \mid 1 \leq i \leq n\}$. ■

Lemma 5. *In every I-noetherian left semiprimary ring R, $P(R)$ is a rA-ideal, also every minimal prime ideal P is a rA-ideal, P^o is a left primary ideal, $\mathrm{rad}(P^o) = P$ and there exists a positive integer n such that $\mathrm{ann}_r(P^n) \nsubseteq P$.*

Proof. Set I as the intersection of all minimal prime ideals different from P. Since $I \cap P = P(R)$ and $P(R)$ is nilpotent, there exists a positive integer n such that $I^n \cap P^n = 0$ by Lemma 4. Thus $\mathrm{ann}_r(P^n) \nsubseteq P$ because $I^n \nsubseteq P$, implying $P^n \subseteq P^o$. Consequently $\mathrm{rad}(P^o) = P$.

Now we show that P is a rA-ideal. A power of $\mathrm{ann}_r(P^n) \cap \mathrm{ann}_r(\mathrm{ann}_l(P))$, denoted by J, is a lS-ideal. Then $J \subseteq \mathrm{ann}_r(P^n) \cap \mathrm{ann}_r(\mathrm{ann}_l(P))$. So, $J \subseteq \mathrm{ann}_r(\mathrm{ann}_l(P))$, thus $\mathrm{ann}_l(P) \subseteq \mathrm{ann}_l(J)$, implying $\mathrm{ann}_l(P) \cap J = 0$. Thus $\mathrm{ann}_l(P^n J) \cap J = 0$ by Lemma 2. On the other hand, $P^n J = 0$ implying $J = 0$. Thus $\mathrm{ann}_r(P^n) \cap \mathrm{ann}_r(\mathrm{ann}_l(P)) \subseteq P$. Consequently $\mathrm{ann}_r(\mathrm{ann}_l(P)) \subseteq P$. Applying Lemma 1 completes the proof. ■

2. Main Results

Theorem 6. *Every I-noetherian ring having a left primary decomposition is left semiprimary.*

Proof. There exists a finite collection A of primary ideals with zero intersection. Let I be an ideal and set $B = \{U \in A \mid I \subseteq \mathrm{rad}(U)\}$. For every $U \in B$, there exists $n_U \geq 1$ such that $\mathrm{rad}(U)^{n_U} \subseteq U$, implying $I^{n_U} \subseteq \mathrm{rad}(U)^{n_U} \subseteq U$. Thus there exists $n \geq 1$ such that $I^n \subseteq \mathrm{Int}(B)$. On the other hand, for every $U \in A - B$, $\mathrm{ann}_l(I) \subseteq U$. Consequently $\mathrm{ann}_l(I) \cap I^n \subseteq \mathrm{Int}(A) = 0$, implying $\mathrm{ann}_l(I^n) \cap I^n = 0$ by Lemma 4. Therefore I^n is a lS-ideal. ∎

As we saw in Theorem 6, being left semiprimary is a necessary condition for a ring to have a left primary decomposition. Lemma 5 also indicates some properties of left semiprimary rings. Below we present some applications.

The following is a handy lemma.

Lemma 7. *A left semiprimary I-noetherian ring in which every maximal ideal is a minimal prime ideal, is the direct product of a finite number of left primary rings S in which $P(S)$ is the unique maximal ideal of S.*

Proof. We may assume that R is indecomposable. Let I be a proper ideal. There exists a maximal ideal P containing I. Since P is a minimal prime ideal, there exists a positive integer n such that $\mathrm{ann}_r(P^n) \nsubseteq P$ by Lemma 5. Now there exists a positive integer m such that $\mathrm{ann}_r(P^n)^m$ is a lS-ideal. Set $J = \mathrm{ann}_r(P^n)^m$. Then $J \subseteq \mathrm{ann}_r(P^n)$, implying $P^n \subseteq \mathrm{ann}_l(J)$. Consequently $J \cap P^n = 0$. On the other hand, $J \nsubseteq P$ implying $R = P + J$. Since $R = R^n = (P + J)^n \subseteq P^n + J$, $R = P^n \oplus J$, implying $P^n = 0$. Thus $I \subseteq P(R)$. ∎

The following is an application of Theorem 6 which is a generalization of [5, Theorem 23.10].

Theorem 8. *A ring R is semiprimary and left semiprimary if and only if $R \cong \prod_{i=1}^{n} M_{m_i}(D_i)$, where D_i is a local ring and its unique maximal ideal is nilpotent.*

Proof. ($\Rightarrow$) We may assume that R is indecomposable. Let I be a proper ideal. There exists an ideal J containing $P(R)$ such that $R/J(R) = I/J(R) \oplus J/J(R)$, implying $I + J = R$ and $I \cap J \subseteq J(R)$. There exists $m \geq 1$ such that $I^m \cap J^m = 0$ by Lemma 4, implying $I^m \oplus J^m = R$. Hence, $I^m = 0$, implying $I \subseteq P(R)$. Thus, $J(R)$ is the unique maximal ideal and is nilpotent because $J(R) = P(R)$. Therefore $R = M_m(D)$ where D is a local ring and its unique maximal ideal is nilpotent by [5, Theorem 23.10].
($\Leftarrow$) Straightforward. ∎

The following is a convergence of Theorem 8.

Corollary 9. *A ring R is artinian left semiprimary if and only if $R \cong \prod_{i=1}^{n} M_{m_i}$*

(D_i) where D_i is an artinian local ring and its unique maximal ideal is nilpotent.

A ring is called *I-uniform* if for every ideals I and J, $I \cap J = 0$ implies either $I = 0$ or $J = 0$. An ideal Q is called *I-irreducible* if for every ideals I and J, $I \cap J = Q$ implies either $I = Q$ or $J = Q$. An ideal is called *I-essential* if it has nonzero interseciton with every nonzero ideal.

It is a well-known fact that every ideal in an I-noetherian ring is the intersection of a finite number of I-irreducible ideals. Also every noetherian commutative I-uniform ring is a primary ring. Thus every ideal in a noetherian commutative ring is the intersection of a finite number of primary ideals. Therefore, commutative noetherian rings are semiprimary in our sense. Applying Corollary 9 implies another well-known fact that a commutative artinian ring is the product of a finite number of local rings.

Corollary 10. *Every principal left ideal ring is a left semiprimary ring.*

Proof. Let I be an ideal. There exists $a \in I$ such that $I = Ra$. Since R is left noetherian, there exists $m \geq 1$ such that $\text{ann}_l(a^{m+1}) = \text{ann}_l(a^m)$. Then $Ra^m \cap \text{ann}_l(a^m) = 0$. Since $I^m = Ra^m$, $I^m \cap \text{ann}_l(a^m)$, implying $I^m \cap \text{ann}_l(I^m) = 0$ because $\text{ann}_l(I^m) \subseteq \text{ann}_l(a^m)$. Thus I^m is a lS-ideal. ∎

There is a lengthly proof in [4] for the following proposition.

Proposition 11. *Every principal left ideal ring is the direct product of a finite number of rings S in which $P(S)$ is a prime ideal.*

Proof. We may assume that the ring, denoted by R, is indecomposable. Let P be a minimal prime ideal. Since $P/P(R)$ is an annihilator ideal, $P/P(R)$ is a direct summand of $R/P(R)$ by [4, Lemma 4.12], so $P/P(R) \oplus J/P(R) = R/P(R)$ for an ideal J, implying $P + J = R$ and $P \cap J \subseteq P(R)$. There exists $m \geq 1$ such that $P^m \cap J^m = 0$ by Lemma 4 and Corollary 10. On the other hand, $P^m + J^m = R$. Thus, $P^m = 0$, implying $P(R) = P$. ∎

In the rest of the paper we pay attention to the primary decomposition. For a semiprime ideal P, a left primary ideal U with $\text{rad}(U) = P$ is called a left P-primary ideal. In I-noetherian rings, for every left primary ideal U, $\text{rad}(U)$ is a prime ideal. It is easy to see that if P is a semiprime ideal and A is a finite collection of left P-primary ideals, then $\text{Int}(A)$ is also a left P-primary ideal because $\text{rad}(\text{Int}(A)) = \cap_{U \in A}\text{rad}(U) = P$. Now we can see that for every finite collection A of left primary ideals, there is a normal set B such that $\text{Int}(B) = \text{Int}(A)$, every element of B is the intersection of some elements of A and $\{\text{rad}(K) \mid K \in B\} \subseteq \{\text{rad}(U) \mid U \in A\}$.

The proof of the next result is a straightforward adaptation of [5, p. 15, Theorem 2].

Lemma 12. *For every normal sets A and B,* $\text{Int}(A) = \text{Int}(B)$ *implies*

$$\{\text{rad}(U) \mid U \in A\} = \{\text{rad}(K) \mid K \in B\}.$$

Proof. By induction on $|A|$. For every $V \in A$, there is $L \in B$ such that $\text{rad}(V) \subseteq \text{rad}(L)$ because there is $L \in B$ such that $\text{Int}(A - \{V\}) \nsubseteq L$, then $V \subseteq \text{rad}(L)$, implying $\text{rad}(V) \subseteq \text{rad}(L)$. We choose $V \in A$ such that $\text{rad}(V)$ is a maximal of $\{\text{rad}(U) \mid U \in A\}$. There is $L \in B$ such that $\text{rad}(V) \subseteq \text{rad}(L)$. On the other hand there is $U \in A$ such that $\text{rad}(L) \subseteq \text{rad}(U)$, implying $\text{rad}(V) = \text{rad}(L)$. $\text{rad}(L)$ is a maximal of $\{\text{rad}(K) \mid K \in B\}$ because otherwise there is $K \in B$ such that $\text{rad}(L) \subset \text{rad}(K)$, on the other hand there is $U \in A$ such that $\text{rad}(K) \subseteq \text{rad}(U)$, implying $\text{rad}(V) \subset \text{rad}(U)$ which is a contradiction. Thus for every $K \in B - \{L\}$, $\text{rad}(V) \nsubseteq \text{rad}(K)$, implying $V \nsubseteq \text{rad}(K)$, then $\text{Int}(A - \{V\}) \subseteq K$. Consequently $\text{Int}(A - \{V\}) \subseteq \text{Int}(B - \{L\})$. Similarly $\text{Int}(B - \{L\}) \subseteq \text{Int}(A - \{V\})$. Thus $\text{Int}(A - \{V\}) = \text{Int}(B - \{L\})$. Applying the induction completes the proof. ∎

Lemma 13. *Let R be an rAI-noetherian ring. For every non-I-essential left primary ideal U, rad(U) is a prime right annihilator.*

Proof. There exists a nonzero ideal J such that $U \cap J = 0$ and $\text{ann}_r(J)$ is maximal under these conditions. It is easy to see that $\text{ann}_r(J)$ is a prime ideal and $U \subseteq \text{arr}_r(J)$. Thus $\text{rad}(U) \subseteq \text{ann}_r(J)$. On the other hand $\text{ann}_r(J) \subseteq \text{rad}(U)$, because $J\,\text{ann}_r(J) = 0 \subseteq U$ and $J \nsubseteq U$. Therefore $\text{rad}(U) = \text{ann}_r(J)$. ∎

Lemma 14. *Let A be a normal set with zero intersection and P a prime ideal. Set $A_P = \{U \in A \mid U \subseteq P\}$. Then $P^o = \text{Int}(A_P)$ and for every $U \in A$, $P^o \subseteq U$ if and only if $U \in A_P$.*

Proof. For every $U \in A_P$, $\text{rad}(U) \subseteq P$, implying $P^o \subseteq \text{rad}(U)^o \subseteq U$. Thus $P^o \subseteq \text{Int}(A_P)$. On the other hand, $\text{Int}(A - A_P) \nsubseteq P$, implying $\text{Int}(A_P) \subseteq P^o$ because $\text{Int}(A_P)\text{Int}(A - A_P) = 0$. Now if $V \in A - A_P$, then $P^o = \text{Int}(A_P) \nsubseteq V$. ∎

Lemma 15. *Let R be an I-noetherian ring. If A is a normal set with zero intersection, then*

1. *For every prime right annihilator P, there is $U \in A$ with $P = \text{rad}(U)$.*
2. *For every minimal prime ideal P, $P^o \in A$ and $\text{rad}(P^o) = P$.*

Proof. 1. $\text{ann}_l(P) \nsubseteq P^o = \text{Int}(A_P)$ by Lemma 1 and Lemma 14, so there is $U \in A_P$ such that $\text{ann}_l(P) \nsubseteq U$, then $P \subseteq \text{rad}(U)$. On the other hand it is clear that $\text{rad}(U) \subseteq P$. Thus $P = \text{rad}(U)$.
2. A_P has only one element because for every $U \in A_P$, $\text{rad}(U) \subseteq P$ and $\text{rad}(U)$ is a prime ideal by Lemma 13, implying $\text{rad}(U) = P$. Thus $A_P = \{U\}$, implying $P^o = U$ and $\text{rad}(U) = P$, consequently $\text{rad}(P^o) = P$. ∎

The following which is immediate, is one of the main results.

Theorem 16. *Let R be an I-noetherian ring. If A is a normal set with zero interseciton, then $\{\mathrm{rad}(U) \mid U \in A\}$ is the set of prime right annihilators.*

Corollary 17. *Let R be an I-noetherian ring having a left primary decompositon. The set of prime right annihilators is finite. Also every minimal prime ideal is a prime right annihilator.*

Theorem 18. *Let R be an I-noetherian left semiprimary ring such that every maximal rA-ideal is a minimal prime ideal.*

1. *Every minimal prime ideal is a maximal rA-ideal.*
2. *$\cap_P P^o = 0$ where P runs over the set of minimal prime ideals.*
3. *The set of P^o where P runs over the set of minimal prime ideals is the unique normal set with zero intersection.*
4. *R has a left primary decomposition.*

Proof. 1. Let P be a minimal prime ideal. P is an rA-ideal by Lemma 5. Then P is contained in a maximal rA-ideal Q, implying $P = Q$ because Q is a minimal prime ideal. Thus P is a maximal rA-ideal.
2. Set $I = \cap_P P^o$. Each P^o is a lA-ideal by Lemma 1, so is I. Temporary suppose that $I \neq 0$. I contains a minimal lA-ideal J. Set $P = \mathrm{ann}_r(J)$. P is a maximal rA-ideal so it is a minimal prime ideal. Clearly $\mathrm{ann}_l(P) = J$. On the other hand, $\mathrm{ann}_l(P) \nsubseteq P^o$ by Lemma 1, so $J \nsubseteq P^o$ which is a contradiction because $J \subseteq I \subseteq P^o$.
3. For each minimal prime ideal P, P^o is a left primary ideal by Lemma 5. Applying Lemma 15 (2) completes the proof. ∎

Acknowledgments. The research is partially supported by a grant from Isfahan University of Technology and partially by IUT (CEAMA).

References

1. D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springeer-Verlag, New York, 1995.
2. W. J. Fisher, The primary decomposition theory for modules, *Pacific J. Math.* **35** (1970), 359 - 367.
3. R. W. Gilmer, Rings in which the unique primary decomposition theorem holds, *Proc. Amer. Math. Soc.* **14** (1963) 777 - 781.
4. I. N. Herstein, *Topics in Ring Theory*, The University of Chicago Press, Chicago, 1969, pp. 61 - 86.
5. T. Y. Lam, *A First Course in Noncommutative Rings*, Springer-Verlag, New York, 1991.
6. R. L. McCasland, J. B. A. Schmidt, P. F. Smith, and E. M. Stark, *Uniform Dimension and Radical Modules*, University of Glassgow 99/19, 1999.
7. D. G. Northcott, *Ideal Theory*, Cambridge Press, Cambridge, 1965.

8. B. L. Ofsky, Noether Laskar primary decomposition revisited, *Amer. Math. Monthly* **101** (1994), 759 - 768.
9. R. Y. Sharp, Steps in Commutative Algebra, *London Math. Soc. Student Texts 19,* Cambridge Univ. Press, Cambridge 1990.
10. P. F. Smith, Uniqueness of Primary Decomposition, *Turk. J. Math.* **27** (2003), 425 - 434.