Vietnam Journal of Mathematics **36**:2(2008) 191–207

Vietnam Journal
of
MATHEMATICS

Uniqueness Polynomials and bi-URS for p-adic Meromorphic Functions in Several Variables

Vu Hoai An and Tran Dinh Duc

Institute of Mathematics, 18 Hoang Quoc Viet, 10307 Hanoi, Vietnam

Received October 17, 2007
Revised May 11, 2008

Abstract. In this paper we give some cases of uniqueness polynomials for p-adic meromorphic functions in several variables and show the existence of a bi-URS for p-adic meromorphic functions in several variables of the form $(\{a_1, a_2, a_3, a_4\},\ \{u\})$

2000 Mathematics Subject Classification: 11G, 30D35.
Keywords: Uniqueness polynomials, bi-URS, p-adic meromorphic functions in several variables.

1. Introduction

Let f be a non-zero holomorphic function on $D_{r_{(m)}}$, $a = (a_1, \ldots, a_m) \in D_{r_{(m)}}$, and

$$f = \sum_{|\gamma|=0}^{\infty} a_\gamma (z_1 - a_1)^{\gamma_1} \ldots (z_m - a_m)^{\gamma_m}, \quad z_{(m)} \in D_{r_{(m)}}.$$

For each $i = 1, 2, \ldots, m$, write

$$f(z_{(m)}) = \sum_{k=0}^{\infty} f_{i,k}(\widehat{z_i - a_i})(z_i - a_i)^k.$$

Set

$$g_{i,k}(z_1, ..., z_{i-1}, z_{i+1}, ..., z_m) = f_{i,k}(\widehat{z_i - a_i}),$$
$$b_{i,k} = g_{i,k}(a_1, ..., a_{i-1}, a_{i+1}, ..., a_m).$$

Then

$$f_{i,a}(z) = \sum_{k=0}^{\infty} b_{i,k}(z_i - a_i)^k.$$

Set

$$v_{i,f}(a) = \begin{cases} \min \ \{k : b_{i,k} \neq 0\} & \text{if } f_{i,a}(z) \not\equiv 0 \\ +\infty & \text{if } f_{i,a}(z) \equiv 0. \end{cases}$$

If $f(a) = 0$, then a is a zero of $f(z_{(m)})$. Then the number $v_{i,f}(a)$ is called the i^{th} *partial multiplicity* of a.

For a point $d \in \mathbb{C}_p$ we define the function $v_f^d : \mathbb{C}_p^m \to (\mathbb{N} \cup \{+\infty\})^m$ by $v_f^d(a_{(m)}) = (v_{1,f-d}(a_{(m)}), \ldots, v_{m,f-d}(a_{(m)}))$.

Now let $f = \dfrac{f_1}{f_2}$ be a non-constant meromorphic function on $\mathbb{C}_p^m$, where f_1, f_2 are two holomorphic functions on $\mathbb{C}_p^m$ having no common zeros. For a point $d \in \mathbb{C}_p$ we define the function $v_f^d : \mathbb{C}_p^m \to (\mathbb{N} \cup \{+\infty\})^m$ by $v_f^d(a_{(m)}) = v^0_{f_1 - df_2}(a_{(m)})$ and write $v_f^d(a_{(m)}) = (v^d_{1,f}(a_{(m)}), \ldots, v^d_{m,f}(a_{(m)}))$, $v_f^\infty(a_{(m)}) = v^0_{f_2}(a_{(m)})$ and write $v_f^\infty(a_{(m)}) = (v^\infty_{1,f}(a_{(m)}), \ldots, v^\infty_{m,f}(a_{(m)}))$.

For a subset S of $\mathbb{C}_p$ we set

$$E_i(f, S) = \bigcup_{d \in S} \Big\{(q_i, a_{(m)}) \in (\mathbb{N} \cup \{+\infty\}) \times \mathbb{C}_p^m | f(a_{(m)}) - d = 0,\ v^d_{i,f}(a_{(m)}) = q_i\Big\},$$

$$E_i(f, S \cup \{\infty\}) = E_i(f, S) \bigcup \Big\{(q_i, a_{(m)}) \in (\mathbb{N} \cup \{+\infty\}) \times \mathbb{C}_p^m | v^\infty_{i,f}(a_{(m)}) = q_i\Big\},$$

$i = 1, 2..., m$.

A subset S of $\mathbb{C}_p \cup \{\infty\}$ is called a unique range set (URS for short) for p-adic meromorphic functions of several variables if for any pair of non-constant meromorphic functions f and g on $\mathbb{C}_p^m$ the condition $E_i(f, S) = E_i(g, S)$, $i = 1, \ldots, m$, implies $f = g$. Similarly, let S, T be two subsets of $\mathbb{C}_p \cup \{\infty\}$ with $S \cap T = \emptyset$. (S, T) is called a bi-URS for p-adic meromorphic functions of several variables if for any pair of non-constant meromorphic functions f and g on $\mathbb{C}_p^m$ the conditions $E_i(f, S) = E_i(g, S)$ and $E_i(f, T) = E_i(g, T), i = 1, \ \ldots, m$, imply $f = g$.

Several interesting results about URS and bi-URS for entire and meromorphic functions on $\mathbb{C}_p$ have been studied in [6, 9, 11]. In[9], Khoai and An gave sufficient conditions of URS and bi-URS in terms of uniquenees polynomials and strong uniqueness polynomials for non-archimedean meromorphic functions of one variable. The main tool cited in the above papers is the Nevanlinna theory in one-dimensional non-archimedean case. In this paper by using some arguments in [3, 9] and the p-adic Nevanlinna theory in high dimension, developed in [1, 2, 3, 5, 7, 8], we give some cases of uniqueness polynomials for p-adic meromorphic functions in several variables and show the existence of a bi-URS for p-adic meromorphic functions in several variables of the form $(\{a_1, a_2, a_3, a_4\},\ \{u\})$.

2. Height of p-adic Holomorphic Functions of Several Variables

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers and $\mathbb{C}_p$ the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{Q}_p$ is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on $\mathbb{C}_p$ which extends ord_p. We use the notations

$b_{(m)} = (b_1, ..., b_m), \quad b_i(b) = (b_1, ..., b_{i-1}, b, b_{i+1}, ..., b_m),$

$b_{(m,i_s)} = b_i(b_{i_s}),$

$\widehat{(b_i)} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_m),$

$D_r = \{z \in \mathbb{C}_p : |z| \leqslant r, r > 0\},$

$D_{<r>} = \{z \in \mathbb{C}_p : |z| = r, r > 0\},$

$D_{r_{(m)}} = D_{r_1} \times \cdots \times D_{r_m}$, where $r_{(m)} = (r_1, \ldots, r_m)$ for $r_i \in \mathbb{R}^*_+$,

$D_{<r_{(m)}>} = D_{<r_1>} \times \cdots \times D_{<r_m>},$

$|\gamma| = \gamma_1 + \cdots + \gamma_m,$

$z^\gamma = z_1^{\gamma_1} ... z_m^{\gamma_m}$

$r^\gamma = r_1^{\gamma_1} ... r_m^{\gamma_m},$

$\gamma = (\gamma_1, ..., \gamma_m),$

where $\gamma_i \in \mathbb{N}$, $|\ .\ | = |\ .\ |_p$, $\log = \log_p$.

Notice that the set of $(r_1, ..., r_m) \in \mathbb{R}^{*m}_+$ such that there exist $x_1, ..., x_m \in \mathbb{C}_p$ with $|x_i| = r_i, i = 1, ..., m$, is dense in $\mathbb{R}^{*m}_+$. Therefore, without loss of generality one may assume that $D_{<r_{(m)}>} \neq \emptyset$.

Let f be a non-zero holomorphic function in $D_{r_{(m)}}$ and

$$f = \sum_{|\gamma| \geq 0} a_\gamma z^\gamma, \quad |z_i| \leqslant r_i \text{ for } i = 1, \ldots, m.$$

Then we have

$$\lim_{|\gamma| \to \infty} |a_\gamma| r^\gamma = 0.$$

Hence, there exists a $(\gamma_1, \ldots, \gamma_m) \in \mathbb{N}^m$ such that $|a_\gamma| r^\gamma$ is maximal.

Define

$$|f|_{r_{(m)}} = \max_{0 \leqslant |\gamma| < \infty} |a_\gamma| r^\gamma.$$

Lemma 2.1.([8]) *For each $i = 1, \ldots, m$, let $r_{i_1}, \ldots, r_{i_q}$ be positive real numbers such that $r_{i_1} \geq \cdots \geq r_{i_q}$. Let $f_s(z_{(m)}), s = 1, 2, \ldots, q$, be q non-zero holomorphic functions on $D_{r_{(m,i_s)}}$. Then there exists $u_{(m,i_s)} \in D_{r_{(m,i_s)}}$ such that*

$$|f_s(u_{(m,i_s)})| = |f_s|_{r_{(m,i_s)}}, \qquad s = 1, 2, \ldots, q.$$

Definition 2.2. *The height of the function $f(z_{(m)})$ is defined by*

$$H_f(r_{(m)}) = \log |f|_{r_{(m)}}.$$

If $f(z_{(m)}) \equiv 0$, then set $H_f(r_{(m)}) = -\infty$.

Let f be a non-zero holomorphic function in $D_{r_{(m)}}$ and

$$f = \sum_{|\gamma| \geq 0} a_\gamma z^\gamma, \quad |z_i| \leqslant r_i \text{ for } i = 1, \ldots, m.$$

Write

$$f(z_{(m)}) = \sum_{k=0}^{\infty} f_{i,k}\widehat{(z_i)} z_i^k, \quad i = 1, 2, \ldots, m.$$

Set

$$\begin{aligned}
I_f(r_{(m)}) &= \Big\{(\gamma_1, \ldots, \gamma_m) \in \mathbb{N}^m : |a_\gamma| r^\gamma = |f|_{r_{(m)}}\Big\},\\
n_{1i,f}(r_{(m)}) &= \max\Big\{\gamma_i : \exists\ (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m) \in I_f(r_{(m)})\Big\},\\
n_{2i,f}(r_{(m)}) &= \min\Big\{\gamma_i : \exists\ (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m) \in I_f(r_{(m)})\Big\},\\
n_{i,f}(0,0) &= \min\Big\{k : f_{i,k}\widehat{(z_i)} \not\equiv 0\Big\},\\
\nu_f(r_{(m)}) &= \sum_{i=1}^{m} \big(n_{1i,f}(r_{(m)}) - n_{2i,f}(r_{(m)})\big).
\end{aligned}$$

$r_{(m)}$ is called a *critical point* if $\nu_f(r_{(m)}) \neq 0$.

For a fixed $i \quad (i = 1, \ldots, m)$ we set for simplicity

$$n_{i,f}(0,0) = \ell, k_1 = n_{1i,f}(r_{(m)}),\ k_2 = n_{2i,f}(r_{(m)}).$$

Then there exist multi-indices $\gamma = (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m) \in I_f(r_{(m)})$ and $\mu = (\mu_1, \ldots, \mu_i, \ldots, \mu_m) \in I_f(r_{(m)})$ such that $\gamma_i = k_1$, $\mu_i = k_2$.

We consider the following holomorphic functions on $D_{r_{(m)}}$

$$f_\ell(z_{(m)}) = f_{i,\ell}\widehat{(z_i)} z_i^\ell, f_{k_1}(z_{(m)}) = f_{i,k_1}\widehat{(z_i)} z_i^{k_1}, f_{k_2}(z_{(m)}) = f_{i,k_2}\widehat{(z_i)} z_i^{k_2}.$$

The functions are not identically zero.

Set

$$\begin{aligned}
U_{if,r_{(m)}} = \big\{u = u_{(m)} \in D_{r_{(m)}} :& |f_\ell(u)| = |f_\ell|_{r_{(m)}}, |f(u)| = |f|_{r_{(m)}},\\
& |f_{k_1}(u)| = |f_{k_1}|_{r_{(m)}}, |f_{k_2}(u)| = |f_{k_2}|_{r_{(m)}}\big\},
\end{aligned}$$

where $i = 1, \ldots, m$. By Lemma 2.1, $U_{if,r_{(m)}}$ is a non-empty set. For each $u \in U_{if,r_{(m)}}$, set

$$f_{i,u}(z) = f(u_1, \ldots, u_{i-1}, z, u_{i+1}, \ldots, u_m),\ z \in D_{r_i}.$$

Theorem 2.3. *Let $f(z_{(m)})$ be a holomorphic function on $D_{r_{(m)}}$. Assume that $f(z_{(m)})$ is not identically zero. Then for each $i = 1, \ldots, m$, and for all $u \in U_{if,r_{(m)}}$, we have*

1) $H_f(r_{(m)}) = H_{f_{i,u}}(r_i)$,
2) $n_{1i,f}(r_{(m)})$ *is equal to the number of zeros of* $f_{i,u}$ *in* D_{r_i},
3) $n_{1i,f}(r_{(m)}) - n_{2i,f}(r_{(m)})$ *is equal to the number of zeros of* $f_{i,u}$ *on* $D_{<r_i>}$.
For the proof, see [8, Theorem 3.1].

From Theorem 2.3 we see that $f(z_{(m)})$ has zeros on $D_{<r_{(m)}>}$ if and only if $r_{(m)}$ is a critical point.

For a an element of $\mathbb{C}_p$ and f a holomorphic function on $D_{r_{(m)}}$, which is not identically equal to a, define

$$n_{i,f}(a, r_{(m)}) = n_{1i,f-a}(r_{(m)}), \quad i = 1, \dots, m.$$

Fix real numbers $\rho_1, \dots, \rho_m$ with $0 < \rho_i \leqslant r_i,\ i = 1, \dots, m$.
For each $x \in \mathbb{R}$, set

$$A_i(x) = (\rho_1, \dots, \rho_{i-1}, x, r_{i+1}, \dots, r_m),\ i = 1, \dots, m,$$

$$B_i(x) = (\rho_1, \dots, \rho_{i-1}, x, \rho_{i+1}, \dots, \rho_m), i = 1, \dots, m.$$

Define *the counting function* $N_f(a, r_{(m)})$ by

$$N_f(a, r_{(m)}) = \frac{1}{\ln p} \sum_{i=1}^{m} \int_{\rho_i}^{r_i} \frac{n_{i,f}(a, A_i(x))}{x} dx.$$

If a=0, then set $N_f(r_{(m)}) = N_f(0, r_{(m)})$.

Then

$$N_f(a, B_i(r_i)) = \frac{1}{\ln p} \int_{\rho_i}^{r_i} \frac{n_{i,f}(a, B_i(x))}{x} dx.$$

For each $i = 1, 2, ..., m$, set

$$k_{1,i} = n_{1i,f}(A_i(r_i)), k_{2,i} = n_{2i,f}(A_i(r_i)),$$

$$U^i_{if,A_i(r_i)} = \Big\{ u^i = u^i_{(m)} \in D_{A_i(r_i)} : |f_\ell(u^i)| = |f_\ell|_{A_i(r_i)}, |f(u^i)| = |f|_{A_i(r_i)}, |f_{k_{1,i}}(u^i)| = |f_{k_{1,i}}|_{A_i(r_i)}, |f_{k_{2,i}}(u^i)| = |f_{k_{2,i}}|_{A_i(r_i)} \Big\},$$

$$\Gamma_i = \{A_i(x) : A_i(x) \text{ is a critical point}, 0 < x \leqslant r_i\}.$$

By Lemma 2.1 and Theorem 2.3, Γ_i is a finite set. Suppose that $\Gamma_i,\ i = 1, \dots, m$, contains n elements $A_i(x^j),\ j = 1, \dots, n$. From this and Lemma 2.1 it follows that

$$\mathcal{U}^i_{if,A_i(r_i)} = \{u^i = u^i_{(m)} \in U^i_{if,A_i(r_i)} : \exists u^i_i(u^j) \in U^i_{if,A_i(x^j)},\ j = 1, \dots, n\} \neq \emptyset,$$

$i = 1, \dots, m$.

Lemma 2.4. 1) *Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Then for each $i = 1, 2, ..., m$, and for all $u^i \in \mathcal{U}^i_{if,A_i(r_i)}$, we have*

$$n_{f_{i,u^i}}(x) = n_{i,f} \circ A_i(x), \rho_i \leqslant x \leqslant r_i,$$

2) *Let $f_s(z_{(m)}), s = 1, 2, \ldots, q$, be q non-zero holomorphic functions on $D_{r_{(m)}}$. Then for each i = 1,2,..., m, there exists $u^i \in \mathcal{U}^i_{if_s,A_i(r_i)}$ for all $s = 1, \ldots, q$.*

The result can be proved easily by using Lemma 2.1 and Theorem 2.3.

Theorem 2.5. *Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Then*

$$H_f(r_{(m)}) - H_f(\rho_{(m)}) = N_f(r_{(m)}).$$

The proof of Theorem 2.5 follows immediately from [8, Theorem 3.2].

Set

$$\begin{aligned} v =&(u^1, \ldots, u^m), u^i \in \mathcal{U}^i_{if,A_i(r_i)}, \\ N_{f_v}(r_{(m)}) =&N_{f_{1,u^1}}(r_1) + \cdots + N_{f_{m,u^m}}(r_m), \\ V =&\{v : N_{f_v}(r_{(m)}) = N_f(r_{(m)})\}. \end{aligned}$$

By Lemma 2.4 and [6], V is a non-empty set,

$$\begin{aligned} N_{f_v}(r_{(m)}) = &\sum\nolimits_{\rho_1<|a|\leqslant r_1}^{1} (v(a) + \log r_1) + n_{f_1,u^1}(0, \rho_1)(\log r_1 - \log \rho_1) + \ldots \\ &+ \sum\nolimits_{\rho_m<|a|\leqslant r_m}^{m} (v(a) + \log r_m) + n_{f_m,u^m}(0, \rho_m)(\log r_m - \log \rho_m), \end{aligned} \tag{2.1}$$

where

$$\sum\nolimits_{\rho_i<|a|\leqslant r_i}^{i} (v(a) + \log r_i)$$

is taken on all of zeros a of f_{i,u^i} (counting multiplicity) with $\rho_i < |a| \leqslant r_i$, $i = 1, 2, ..., m$. Notice that, the sums in (2.1) are finite sums.

Denote by $\overline{N}_{f_v}(r_{(m)})$ the sum (2.1), where every zero a of the functions f_{i,u^i}, $i = 1, \ldots, m$, is counted ignoring multiplicity. Set

$$\overline{N}_f(r_{(m)}) = \max_{v\in V} \overline{N}_{f_v}(r_{(m)}).$$

From Lemma 2.4 it follows that one can find $u^i \in \mathcal{U}^i_{if,A_i(r_i)}$ and $v = (u^1, \ldots, u^m)$ such that $N_f\big(r_{(m)}\big) = N_{f_v}(r_{(m)})$.

Now let C be some condition. Let $U^{i*}_{i,A_i(r_i)} \subset \mathcal{U}^i_{if,A_i(r_i)}, U^{i*}_{i,A_i(r_i)} \neq \emptyset$. For each $r_{(m)}$ and $u^i \in U^{i*}_{i,A_i(r_i)}$, set

$$\begin{aligned} v_{i,f}(u^i_i(z); C) =&\begin{cases} v_{i,f}(u^i_i(z)) & \text{if } u^i_i(z) \text{ satisfies the condition } C \\ 0 & \text{otherwise} \end{cases} \\ n_{f_{i,u^i}}(r_i; C) =& \sum_{|z|\leqslant r_i} v_{i,f}(u^i_i(z); C), \\ N_f\big(r_{(m)}; C\big) =& \min_{v\in V} \frac{1}{\ln p} \sum_{i=1}^{m} \int_{\rho_i}^{r_i} \frac{n_{f_{i,u^i}}(x; C)}{x} dx, \\ N_{f_v}(r_{(m)}; C) =&N_{f_{1,u^1}}(r_1; C) + \cdots + N_{f_{m,u^m}}(r_m; C). \end{aligned}$$

From Lemma 2.4 it follows that one can find $u^i \in U^{i*}_{i,A_i(r_i)}$ and $v = (u^1, \dots, u^m)$ such that $N_f\big(r_{(m)}; C\big) = N_{f_v}(r_{(m)}; C)$.

If γ is a multi-index and f is a meromorphic function of m variables, then we denote by $\partial^\gamma f$ the partial derivative

$$\frac{\partial^{|\gamma|} f}{\partial z_1^{\gamma_1} \dots \partial z_m^{\gamma_m}}.$$

Theorem 2.6. *Let f be a non-zero entire function on $\mathbb{C}_p^m$ and γ a multi-index with $\mid \gamma \mid > 0$. Then*

$$H_{\partial^\gamma f}(B_e(r_e)) - H_f(B_e(r_e)) \leqslant - \mid \gamma \mid \log r_e + O(1).$$

The proof of Theorem 2.6 follows immediately from [5, Lemma 4.1].

3. Height of $p-$adic Meromorphic Functions of Several Variables

Let $f = \dfrac{f_1}{f_2}$ be a meromorphic function on $D_{r_{(m)}}$ (resp., $\mathbb{C}_p^m$), where f_1, f_2 are two holomorphic functions on $D_{r_{(m)}}$ (resp., $\mathbb{C}_p^m$), have no common zeros, and $a \in \mathbb{C}_p$.

We set

$$H_f(r_{(m)}) = \max_{1 \leqslant i \leqslant 2} H_{f_i}(r_{(m)}),$$

$$N_f(a, r_{(m)}) = N_{f_1 - af_2}(r_{(m)}),$$

$$N_f(\infty, r_{(m)}; C) = N_{f_2}(r_{(m)}; C),$$

and

$$N_f(a, r_{(m)}; C) = N_{f_1 - af_2}(r_{(m)}; C).$$

Lemma 3.1. *Let $f = \dfrac{f_1}{f_2}$ be a non-constant meromorphic function on $\mathbb{C}_p^m$. Then there exists a multi-index $\gamma_1 = (0, \dots, 0, \gamma_{1e}, 0, \dots, 0)$ such that $\gamma_{1e} = 1$ and $\partial^{\gamma_1} f = \dfrac{\partial^{\gamma_1} f_1 . f_2 - \partial^{\gamma_1} f_2 . f_1}{f_2^2}$ and the Wronskian*

$$W = W(f_1, f_2) = \det \begin{pmatrix} f_1 & f_2 \\ \partial^{\gamma_1} f_1 & \partial^{\gamma_1} f_2 \end{pmatrix}$$

is not identically zero.

For the proof, see [5, Lemma 4.2].

Let $a_1, \dots a_q \in \mathbb{C}_p$. Set $G_j = f_1 - a_j f_2, j = 1, \dots q$, and $G_{q+1} = f_2$. In Theorem 3.2 we take C to be the following condition: $G_j(z_{(m)}) \neq 0$ with some $z_{(m)} \in \mathbb{C}_p^m$ and for all $j = 1, \dots, q+1$.

Set

$$N_{0,W}(r_{(m)}) = N_W(0, r_{(m)}; C),$$
$$N_{0,\partial^{\gamma_1} f}(r_{(m)}) = N_{0,W}(r_{(m)}).$$

Theorem 3.2. *Let f be a non-constant meromorphic function on $\mathbb{C}_p^m$ and $a_j \in \mathbb{C}_p, j = 1, \dots, q$. Then*

$$(q-1)H_f(B_e(r_e))$$
$$\leqslant \sum_{j=1}^{q} \overline{N}_f(a_j, B_e(r_e)) + \overline{N}_f(\infty, B_e(r_e)) - N_{0,\partial^{\gamma_1} f}(B_e(r_e)) - \log r_e + O(1).$$

Proof. Set $G = \{G_{\beta_1} \dots G_{\beta_{q-1}}\}$, where $(\beta_1, \dots, \beta_{q-1})$ is taken on all different choices of $q-1$ numbers in the set $\{1, \dots, q+1\}$, and $G_j = f_1 - a_j f_2,\ j = 1, \dots, q$, and $G_{q+1} = f_2$. Set $H_G(r_{(m)}) = \max_{(\beta_1 \dots \beta_{q-1})} H_{G_{\beta_1} \dots G_{\beta_{q-1}}}(r_{(m)})$. ■

We need the following lemma.

Lemma 3.3. *We have $H_G(r_{(m)}) \geq (q-1)H_f(r_{(m)}) + O(1)$, where the $O(1)$ does not depend on $r_{(m)}$.*

Proof. We have

$$H_G(r_{(m)}) = \max_{(\beta_1, \cdots, \beta_{q-1})} H_{G_{\beta_1} \dots G_{\beta_{q-1}}}(r_{(m)})$$
$$= \max_{(\beta_1, \cdots, \beta_{q-1})} \sum_{1 \leqslant j \leqslant q-1} H_{G_{\beta_i}}(r_{(m)}).$$

Assume that for a fixed $r_{(m)}$, the following inequalities hold

$$H_{G_{\beta_1}}(r_{(m)}) \geq H_{G_{\beta_2}}(r_{(m)}) \geq \dots \geq H_{G_{\beta_{q+1}}}(r_{(m)}).$$

Then

$$H_G(r_{(m)}) = H_{G_{\beta_1}}(r_{(m)}) + H_{G_{\beta_2}}(r_{(m)}) + \dots + H_{G_{\beta_{q-1}}}(r_{(m)}). \tag{3.1}$$

Since $a_1, \dots, a_q$ are distinct numbers in $\mathbb{C}_p$, then

$$f_i = b_{i_0} G_{\beta_q} + b_{i_1} G_{\beta_{q+1}},\ i = 1, 2,$$

where b_{i_0}, b_{i_1} are constants, which do not depend on $r_{(m)}$. It follows that

$$H_{f_i}(r_{(m)}) \leqslant \max_{0 \leqslant j \leqslant 1} H_{G_{\beta_{q+j}}}(r_{(m)}) + O(1).$$

Therefore, we obtain

$$H_{f_i}(r_{(m)}) \leqslant H_{G_{\beta_j}}(r_{(m)}) + O(1),$$

for $j = 1, \dots, q-1$ and $i = 1, 2$. Hence,

$$H_f(r_{(m)}) = \max_{1 \leqslant i \leqslant 2} H_{f_i}(r_{(m)}) \leqslant H_{G_{\beta_j}}(r_{(m)}) + O(1), \tag{3.2}$$

for $j = 1, \dots, q-1$. Summarizing $(q-1)$ inequalities (3.2) and by (3.1), we have

$$H_G(r_{(m)}) \geq (q-1)H_f(r_{(m)}) + 0(1).$$

Now we prove Theorem 3.2. Denote by $W(g_1, g_2)$ the Wronskian of the two entire functions g_1, g_2 with respect to the γ_1 as in Lemma 3.1.

Since f is non-constant, we have $W(f_1, f_2) \not\equiv 0$. Let (α_1, α_2) be two distinct numbers in $\{1, \dots, q+1\}$, and $(\beta_1, \dots, \beta_{q-1})$ be the rest. Note that the functions f_i can be represented as linear combinations of $G_{\alpha_1}, G_{\alpha_2}$. Then we have

$$W(G_{\alpha_1}, G_{\alpha_2}) = c_{(\alpha_1, \alpha_2)} W(f_1, f_2),$$

where $c_{(\alpha_1, \alpha_2)} = c$ is a constant, depending only on (α_1, α_2). We denote

$$A = A(\alpha_1, \alpha_2) = \frac{W(G_{\alpha_1}, G_{\alpha_2})}{G_{\alpha_1} G_{\alpha_2}} = \det \begin{pmatrix} 1 & 1 \\ \dfrac{\partial^{\gamma_1} G_{\alpha_1}}{G_{\alpha_1}} & \dfrac{\partial^{\gamma_1} G_{\alpha_2}}{G_{\alpha_2}} \end{pmatrix}.$$

Hence

$$\frac{G_1 \dots G_{q+1}}{W(f_1, f_2)} = \frac{c G_{\beta_1} \dots G_{\beta_{q-1}}}{A}. \tag{3.3}$$

Set $L_i = \frac{\partial^{\gamma_1} G_{\alpha_i}}{G_{\alpha_i}}$, $i = 1, 2$. Then

$$\log |A|_{B_e(r_e)} \leqslant \max_{1 \leqslant i \leqslant 2} \log |L_i|_{B_e(r_e)}.$$

By Theorem 2.6

$$\log |L_i|_{B_e(r_e)} \leqslant -|\gamma_1| \log r_e + 0(1).$$

Because $|\gamma_1| = 1$

$$\log |L_i|_{B_e(r_e)} \leqslant -\log r_e + 0(1). \tag{3.4}$$

By (3.3), we obtain

$$\sum_{i=j}^{q+1} H_{G_j}(B_e(r_e)) - H_W(B_e(r_e)) = H_{G_{\beta_1} \dots G_{\beta_{q-1}}}(B_e(r_e)) - \log |A|_{B_e(r_e)} + O(1).$$

From this and (3.4), we have

$$\begin{aligned} H_G(B_e(r_e)) &= \max_{(\beta_1, \dots, \beta_{q-1})} H_{G_{\beta_1} \dots G_{\beta_{q-1}}}(B_e(r_e)) \\ &\leqslant \sum_{j=1}^{q+1} H_{G_j}(B_e(r_e)) - H_W(B_e(r_e)) - \log r_e + O(1). \end{aligned}$$

By Lemma 3.3

$$(q-1)H_f(B_e(r_e)) \leqslant \sum_{j=1}^{q+1} H_{G_j}(B_e(r_e)) - H_W(B_e(r_e)) - \log r_e + O(1).$$

Thus

$$(q-1)H_f(B_e(r_e)) + H_W(B_e(r_e)) \leqslant \sum_{j=1}^{q+1} H_{G_j}(B_e(r_e)) - \log r_e + O(1). \quad (3.5)$$

By Theorem 2.5

$$H_W(B_e(r_e)) = N_W(B_e(r_e)) + 0(1),$$
$$H_{G_j}(B_e(r_e)) = N_{G_j}(B_e(r_e)) + 0(1).$$

From this and (3.5) we obtain

$$(q-1)H_f(B_e(r_e)) + N_W(B_e(r_e)) \leqslant \sum_{j=1}^{q+1} N_{G_j}(B_e(r_e)) - \log r_e + O(1). \quad (3.6)$$

For a fixed $B_e(r_e)$, we consider non-zero entire functions $W, G_1, \dots, G_q$ on $D_{B_e(r_e)}$. From Lemma 2.4 it follows that one can find $u^e \in \mathcal{U}^e_{G_j,B_e(r_e)}$ and $u^e \in \mathcal{U}^e_{W,B_e(r_e)}$, $j = 1, \dots, q$, such that

$$N_W(B_e(r_e)) = N_{W_{e,u^e}}(r_e), N_{G_j}(B_e(r_e)) = N_{(G_j)_{e,u^e}}(r_e). \quad (*)$$

Assume that $U^{e*}_{e,B_e(r_e)}$ is the set which contains elements u^e with u^e as in the statement by (*). Now let $u^e_e(x)$ be a zero of G_j having the e^{th} partial multiplicity equal to $k, (k \neq +\infty), k \geq 2$. Since $\gamma_1 = (0, \dots, 0, \gamma_{1e}, 0, \dots, 0)$ with $\gamma_{1e} = 1$, $v_{i,\partial^{\gamma_1}G_j}(u^e_e(x)) = k-1$ if $i = e$.

On the other hand,

$$W(G_{\alpha_1}, G_{\alpha_2}) = c_{(\alpha_1,\alpha_2)}W,$$

where (α_1, α_2) are two distinct numbers in $\{1, \dots, q+1\}$. Therefore $u^e_e(x)$ is a zero of W having the e^{th} partial multiplicity at least $k-1$.

Now we consider the function $F = \prod_{j=1}^{q} G_j$.

Because F is not a constant, F has zeros. Let $u^e_e(x)$ be a zero of F. By the hypothesis, $a_1, \dots, a_q$ are distinct numbers, from this it follows that there exists one function G_j such that $G_j(u^e_e(x)) = 0$. Therefore

$$\sum_{j=1}^{q} N_{(G_j)_{e,u^e}}(r_e) - N_{W_{e,u^e}}(r_e) = \sum_{j=1}^{q} \overline{N}_{(G_j)_{e,u^e}}(r_e) - N_{0,W_{e,u^e}}(r_e).$$

Thus

$$\begin{aligned}&\sum_{j=1}^{q} N_{G_j}(B_e(r_e)) - N_W(B_e(r_e))\\ &\leqslant \sum_{j=1}^{q} \overline{N}_{(G_j)_{e,u^e}}(r_e) - N_{0,W}(B_e(r_e))\\ &\leqslant \sum_{j=1}^{q} \overline{N}_{G_j}(B_e(r_e)) - N_{0,W}(B_e(r_e)).\end{aligned}$$

From this and (3.6) the proof of Theorem 3.2 is complete. ∎

4. Uniqueness Polynomials and bi-URS for p-adic Meromorphic Functions in Several Variables

Theorem 4.1. *Let f, g be two non-zero entire funtions on $\mathbb{C}_p^m$ such that $v_f^0 = v_g^0$ on $\mathbb{C}_p^m$. Then $f = cg$ where c is a non-zero constant in $\mathbb{C}_p$.*

Proof. Take $r_1, \ldots, r_m > 0$ such that f, g have no zeros in $D_{<r_{(m)}>}$. If f is a non-zero constant then so is g. Therefore $f = cg$. Assume that f is non-constant. Since $v_f^0 = v_g^0$, g is also non-constant. Let $a = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_m)$ be two any elements of $D_{<r_{(m)}>}$. Set $C_i(b_i) = (b_1, \ldots, b_i, a_{i+1}, \ldots, a_m)$, $i = 1, \ldots, m$, By $v_f^0 = v_g^0$, $v_{i,f}(z_{(m)}) = v_{i,g}(z_{(m)})$, $i = 1, \ldots, m$. Then

$$f_{i,C_i(b_i)} = c_i g_{i,C_i(b_i)},$$

with $c_i = \frac{f(a)}{g(a)} = \frac{f(C_i(b_i))}{g(C_i(b_i))}$ and $c_i = c_{i+1}, i = 1, 2, ..., m-1$. From this we have

$$\frac{f(a)}{g(a)} = \frac{f(b)}{g(b)} \quad \text{for all} \quad a, b \in D_{r_{<m>}}.$$

Set

$$c = \frac{f(a)}{g(a)},\ a \in D_{<r_{(m)}>},\ h = f - cg.$$

Asume that h is not identically zero. Consider h, f, g in $D_{<r_{(m)}>}$. By Lemma 2.2, there exists $u \in D_{<r_{(m)}>}$ such that $h_{i,u}$, $f_{i,u}$, $g_{i,u}$ are not identically zero, $i = 1, 2, \ldots, m$. We have $f_{i,u} = c' g_{i,u}, c' = \frac{f(u)}{g(u)}$. Theorefore $c = c'$ and $h_{i,u} = f_{i,u} - cg_{i,u}$ identically zero. From this we get a contradiction. So, $f = cg$. ∎

Definition 4.2. *We say that a non-constant polynomial $P(x)$ is a strong uniqueness polynomial for p-adic meromorphic functions on $\mathbb{C}_p^m$ if the identity $P(f) = cP(g)$ implies $f = g$ for any pair of p-adic non-constant meromorphic functions f, g on $\mathbb{C}_p^m$ and for any non-zero constant $c \in \mathbb{C}_p$. Similarly, we say* *that a non-constant polynomial $P(x)$ is a uniqueness polynomial for p-adic meromorphic functions in $\mathbb{C}_p^m$ if the identity $P(f) = P(g)$ implies $f = g$. Let $P(x)$ be a polynomial of degree q without multiple zeros and its derivative is given by*

$$P'(x) = a(x - d_1)^{q_1} \dots (x - d_k)^{q_k},$$

where $q_1 + \cdots + q_k = q - 1$ and $d_1, \dots, d_k$ are distinct zeros of P'. The number k is called the derivative index of P.

Definition 4.3. *A non-zero polynomial $P(x)$ is said to satisfy the condition (H) if $P(d_l) \neq P(d_m)$ for $1 \leqslant \ell < m \leqslant k$.* (See [9]).

We may assume that $d_1, \dots, d_k \in \mathbb{C}_p \backslash \{0\}$.

Let $f = \dfrac{f_1}{f_2}$ be a non-constant meromorphic function on $\mathbb{C}_p^m$, where f_1, f_2 are two holomorphic functions on $\mathbb{C}_p^m$ having no common zeros. For a point $a \in \mathbb{C}_p$ we define the function

$$\chi_f^a : \ \mathbb{C}_p^m \to \ \mathbb{N}$$

by

$$\chi_f^a(z_{(m)}) = \begin{cases} 0 & \text{if } f(z_{(m)}) \neq a \\ 1 & \text{if } f(z_{(m)}) = a \end{cases}$$

If $a = 0$, then set $\chi_f^a = \chi_f$.

If $a = \infty$, define $\chi_f^\infty(z_{(m)}) = -1$ if $z_{(m)}$ is a pole of f. For a condition C, we define

$$\chi^*_{\partial^{\gamma_1} f}(z_{(m)}; C) = \begin{cases} \chi_{\partial^{\gamma_1} f}(z_{(m)}) & \text{if } z_{(m)} \text{satisfies the condition } C \text{ and} \\ & f(z_{(m)})) \neq d_j \text{ for any } j, \\ 0 & \text{otherwise.} \end{cases}$$

In Theorem 4.4 and Theorem 4.6 the condition C is the condition $f(z_{(m)}) = d_j$ and the condition C' is the condition $g(z_{(m)}) = d_j$ with $j = 1, 2, ..., k$.

Theorem 4.4. *Let $P(x) \in \mathbb{C}_p[x]$ have no multiple zeros, have derivative index $k \geq 3$, and satisfy the condition (H). Then $P(x)$ is a uniqueness polynomial for p-adic meromorphic functions on $\mathbb{C}_p^m$.*

Proof. Suppose that there are two distinct non-constant meromorphic functions f and g on $\mathbb{C}_p^m$ such that $P(f) = P(g)$. From this and by Lemma 3.1 there exists a multi-index $\gamma_1 = (0, \dots, \gamma_{1e}, 0, \dots, 0)$ with $\gamma_{1e} = 1$ such that $\partial^{\gamma_1} f \not\equiv 0$ and $\partial^{\gamma_1} g \not\equiv 0$.

Set

$$\varphi = \frac{1}{f} - \frac{1}{g}.$$

Then, $\varphi \not\equiv 0$ and $H_\varphi\Big(B_e(r_e)\Big) \leqslant H_f\Big(B_e(r_e)\Big) + H_g\Big(B_e(r_e)\Big)$. From $P(f) = P(g)$ we conclude that if $f(z_{(m)}) = \infty$ then $g(z_{(m)}) = \infty$ and if $g(z_{(m)}) = \infty$ then $f(z_{(m)}) = \infty$. Therefore $\chi_f^\infty\left(z_{(m)}\right) = \chi_g^\infty\left(z_{(m)}\right)$. On the other hand, we have

$$\partial^{\gamma_1} f(z_{(m)}) P'(f(z_{(m)}) = \partial^{\gamma_1} g(z_{(m)}) \ \ P'(g(z_{(m)})).$$

Since P satisfies the condition (H), we obtain

$$\chi_f^{dj}(z_{(m)}) \leqslant \chi_g^{dj}(z_{(m)}) + \chi^*_{\partial^{\gamma_1} g}(z_{(m)}; C).$$

From this we have

$$\begin{aligned}
&\sum_{j=1}^{k} \chi_f^{dj}\left(z_{(m)}\right) - \chi_f^{\infty}\left(z_{(m)}\right)\\
&\leqslant \sum_{j=1}^{k} \left(\chi_g^{dj}(z_{(m)}) + \chi^*_{\partial^{\gamma_1} g}(z_{(m)}; C)\right) - \chi_g^{\infty}(z_{(m)})\\
&\leqslant \chi_\varphi^0(z_{(m)}) + \sum_{j=1}^{k} \chi^*_{\partial^{\gamma_1} g}(z_{(m)}; C).
\end{aligned}$$

Therefore, applying Theorem 3.2 to the function f and values $d_1, \ldots d_k$ we have

$$\begin{aligned}
&(k-1)\, H_f\left(B_e(r_e)\right)\\
&\leqslant \sum_{j=1}^{k} \overline{N}_f\left(d_j, B_e(r_e)\right) + \overline{N}_f\left(\infty,\ B_e(r_e)\right) - N_{0,\partial^{\gamma_1} f}\left(B_e(r_e)\right) - \log r_e + 0(1)\\
&\leqslant \overline{N}_\varphi\left(B_e(r_e)\right) + N_{0,\partial^{\gamma_1} g}\left(B_e(r_e); C\right) - N_{0,\partial^{\gamma_1} f}\left(B_e(r_e)\right) - \log r_e + O(1).
\end{aligned}$$

Similarly

$$\begin{aligned}
&(k-1) H_g\left(B_e(r_e)\right)\\
&\leqslant \overline{N}_\varphi\left(B_e(r_e)\right) + N_{0,\partial^{\gamma_1} f}\left(B_e(r_e); C'\right) - N_{0,\partial^{\gamma_1} g}\left(B_e(r_e)\right) - \log r_e + O(1).
\end{aligned}$$

Summing up these inequalities and using Theorem 2.5, we obtain

$$\begin{aligned}
&(k-1)\left(H_f\left(B_e(r_e)\right) + Hg\left(B_e(r_e)\right)\right)\\
&\leqslant 2\left(H_f\left(B_e(r_e)\right) + H_g\left(B_e(r_e)\right)\right) - N_{0,\partial^{\gamma_1} f}\left(B_e(r_e)\right) - N_{0,\partial^{\gamma_1} g}\left(B_e(r_e)\right)\\
&\quad + N_{0,\partial^{\gamma_1} g}\left(B_e(r_e); C\right) + N_{0,\partial^{\gamma_1} f}\left(B_e(r_e); C'\right) - 2\log r_e + O(1).
\end{aligned}$$

Since

$$N_{0,\partial^{\gamma_1} g}\left(B_e(r_e); C\right) \leqslant N_{0,\partial^{\gamma_1} g}\left(B_e(r_e)\right),$$

and

$$N_{0,\partial^{\gamma_1} f}\left(B_e(r_e); C'\right) \leqslant N_{0,\partial^{\gamma_1} f}\left(B_e(r_e)\right)$$

we have

$$(k-3)\left(H_f\left(B_e(r_e)\right) + H_g\left(B_e(r_e)\right)\right) + 2\log r_e \leqslant O(1).$$

It follows that $k-3<0$ and we get a contradiction. Theorem 4.4 is proved. ■

Definition 4.5.([9]) *A non-zero polynomial $P(x)$ is said to satisfy the condition (G) if* $\sum\limits_{i=1}^{k} P(d_i) \neq 0$.

Theorem 4.6. *Let $P(x) \in \mathbb{C}_p[x]$ be a polynomial having no multiple zeros. Let $P(x)$ satisfy the conditions (H) and (G) and $k \geq 3$ be the derivative index of $P(x)$. Then $P(x)$ is a strong uniqueness polynomial for p-adic meromorphic functions on $\mathbb{C}_p^m$.*

Proof. By Theorem 4.4, $P(x)$ is a uniqueness polynomial. Asume that $P(x)$ is not a strong uniqueness polynomial for p-adic meromorphic functions on $\mathbb{C}_p^m$. Then there exist two distinct non-constant meromorphic functions f and g on $\mathbb{C}_p^m$ such that $P(f) = cP(g)$ for some non-zero constant c. We consider the set

$$A = \Big\{(\ell, h) : \quad P(d_\ell) = cP(d_h)\Big\}$$

and denote the number of elements of A by k_0. We set $k_0 = 0$ if $A = \emptyset$. For the rest of the proof we need three lemmas below.

Lemma 4.7. *In the above situation, if f is not a Mobius transformation of g, then $k_0 = k$.*

Proof. Since $P(x)$ satisfies the condition (H), if $(\ell_1, h_1),\ (\ell_2, h_2)$ are elements of A such that $h_1 = h_2$ or $\ell_1 = \ell_2$, then $(\ell_1, h_1) = (\ell_2, h_2)$. From this $k_0 \leqslant k$.

Consider the possible cases:

Case 1. $k_0 \geq 2$. After a suitable change of indices, we may assume that

$$P(d_1) = cP(d_{t(1)}), \ldots, P(d_{k_0}) = cP(d_{t(k_0)}).$$

Define

$$\varphi = \frac{1}{f} - \frac{d_{t(1)} - d_{t(2)}}{(d_2 - d_1)(g - d_{t(1)}) + d_1(d_{t(2)} - d_{t(1)})}.$$

Then $\varphi \not\equiv 0$. If $f(z_{(m)}) = \infty$ then $g(z_{(m)}) = \infty$. If $f(z_{(m)}) = d_j,\ 1 \leqslant j \leqslant k_0,\ z_{(m)} \in \mathbb{C}_p^m$, then, $g(z_{(m)}) = d_{t_{(j)}}$ or $\partial^{\gamma_1} g(z_{(m)}) = 0$, because $P(x)$ satisfies the condition (H). If $f(z_{(m)}) = d_j,\ k_0 + 1 \leqslant j \leqslant k$, then $P(d_j) \neq cP(d_j)$. Hence $g(z_{(m)}) \neq d_j$ for every $k_0 + 1 \leqslant j \leqslant k$. This implies $\partial^{\gamma_1} g(z_{(m)}) = 0$. Thus

$$\begin{aligned}
&\sum_{j=1}^{k} \chi_f^{dj}\Big(z_{(m)}\Big) - \chi_f^{\infty}\Big(z_{(m)}\Big)\\
&\leqslant \sum_{j=1}^{k_0} \Big(\chi_g^{d_{t(j)}}\big(z_{(m)}\big) + \chi_{\partial^{\gamma_1} g}\big(z_{(m)}; C\big)\Big) + \sum_{j=k_0+1}^{k} \chi^*_{\partial^{\gamma_1} g}\Big(z_{(m)}; C\Big) - \chi_g^{\infty}\Big(z_{(m)}\Big)\\
&\leqslant \chi_\varphi^0\Big(z_{(m)}\Big) + \sum_{j=3}^{k_0} \chi_g^{d_{t(j)}}\Big(z_{(m)}\Big) + \sum_{j=1}^{k} \chi^*_{\partial^{\gamma_1} g}\Big(z_{(m)}; C\Big).
\end{aligned}$$

Applying Theorem 3.2 to the function f and values $d_1, \dots d_k$, we have

$$\begin{aligned}
&(k-1)H_f(B_e(r_e))\\
&\leqslant \overline{N}_f\Big(\infty, B_e(r_e)\Big) + \sum_{j=1}^{k} \overline{N}_f\Big(d_j, B_e(r_e)\Big) - N_{0,\partial^{\gamma_1} f}\Big(B_e(r_e)\Big) - \log r_e + O(1)\\
&\leqslant \overline{N}_\varphi\Big(B_e(r_e)\Big) + \sum_{j=3}^{k_0} \overline{N}_g\Big(d_{t(j)},\ B_e(r_e)\Big)\\
&\quad + N_{0,\partial^{\gamma_1} g}\Big(B_e(r_e); C\Big) - N_{0,\partial^{\gamma_1} f}\Big(B_e(r_e)\Big) - \log r_e + O(1).
\end{aligned}$$

Similarly

$$\begin{aligned}
&(k-1)H_g(B_e(r_e))\\
&\leqslant \overline{N}_\varphi\Big(B_e(r_e)\Big) + \sum_{j=3}^{k_0} \overline{N}_f\Big(d_{t(j)},\ B_e(r_e)\Big)\\
&\quad + N_{0,\partial^{\gamma_1} f}\Big(B_e(r_e); C'\Big) - N_{0,\partial^{\gamma_1} g}\Big(B_e(r_e)\Big) - \log r_e + 0(1).
\end{aligned}$$

Summing up these inequalities and using Theorem 2.5 we get

$$\begin{aligned}
&(k-1)\left(H_f\left(B_e(r_e)\right) + H_g\left(B_e(r_e)\right)\right)\\
&\leqslant 2\left(H_f\left(B_e(r_e)\right) + H_g\left(B_e(r_e)\right)\right)\\
&\quad + (k_0-2)\left(H_f\left(B_e(r_e)\right) + H_g\left(B_e(r_e)\right)\right) - 2\log r_e + O(1).
\end{aligned}$$

So

$$(k-k_0-1)\left(H_f(B_e(r_e)) + H_g\left(B_e(r_e)\right)\right) + 2\log r_e \leqslant O(1).$$

From this we have $k_0 > k-1$. Hence $k_0 = k$.

Case 2. $k_0 = 0$. Set $\varphi = \dfrac{1}{f} - \dfrac{1}{g}$. As in the proof of Theorem 4.4, we obtain $k < 3$, a contradiction. So $k_0 \neq 0$.

Case 3. $k_0 = 1$. Then there exists a unique element (ℓ, h) such that $P(d_\ell) = cP(d_h)$. Set

$$\varphi = \frac{1}{f} - \frac{d_h}{d_\ell g}.$$

Using Theorem 3.2 and by using the same assymptions as in the proof of Theorem 4.4, we obtain $k < 3$, a contradiction. So $k_0 \neq 1$.
Hence, the proof of Lemma 4.7 is complete. ■

Lemma 4.8. *Under the assymptions of Theorem 4.6, we have $k_0 = k$.*

Proof. We consider the following cases:

Case 1. $f = \dfrac{c_0 g + c_1}{c_2 g + c_3}$.
By $P(f) = cP(g)$, and f and g are not constants, $c_2 = 0$ and $c_3 \neq 0$. Then $f = ag + b$ with $a = \dfrac{c_0}{c_3}$, $b = \dfrac{c_1}{c_3}$ and $a \neq 0$. Since $P(f) = cP(g)$, $P(ag+b) = cP(g)$. From this we have

$$aP'(ag+b) = cP'(g).$$

Thus

$$a^q \Big(g - \frac{d_1 - b}{a}\Big)^{q_1} \ldots \Big(g - \frac{d_k - b}{a}\Big)^{q_k} = c\big(g - d_1\big)^{q_1} \ldots \big(g - d_k\big)^{q_k}.$$

This implies that there exists a permutation $\big(t(1), \ldots, t(k)\big)$ of $(1, \ldots, k)$ such that

$$d_{t(1)} = \frac{d_1 - b}{a}, \ldots, d_{t(k)} = \frac{d_k - b}{a}.$$

Then

$$cP\big(d_{t(\ell)}\big) = cP\Big(\frac{d_\ell - b}{a}\Big) = P\Big(a\frac{d_\ell - b}{a} + b\Big) = P(d_\ell)$$

for all $\ell = 1, \ldots, k$. So $k = k_0$.

Case 2. $f \neq \dfrac{c_0 g + c_1}{c_2 + c_3}$.
By Lemma 4.7, $k = k_0$.
Thus Lemma 4.8 is proved. ■

Lemma 4.9. *Let $k \geq 3$ and $P(x)$ satisfy the condition (H). If there are two distinct non-constant meromorphic functions f and g on $\mathbb{C}_p^m$ such that $P(f) = cP(g)$ for some non-zero constant, then there exists a permutation $(t_{(1)}, \ldots, t_{(k)})$ of $(1, \ldots, k)$ such that*

$$c = \frac{P(d_1)}{P(d_{t(1)})} = \cdots = \frac{P(d_k)}{P(d_{t(k)})}.$$

Proof. Lemma 4.9 follows from Lemma 4.8. ■

We now continue to prove Theorem 4.6. Assume $P(f) = cP(g)$. If $c = 1$, then by Theorem 4.4, $f = g$. If $c \neq 1$, by Lemma 4.9 there exists a permutation $\big(t(1), \ldots, t(k)\big)$ of $\big(1, \ldots$, k) such that

$$c = \frac{P(d_1)}{P(d_{t(1)})} = \cdots = \frac{P(d_k)}{P(d_{t(k)})} \neq 1.$$

Since P satisfies the condition (G), we obtain

$$c = \frac{P(d_1) + P(d_2) \cdots + P(d_k)}{P(d_{t(1)}) + P(d_{t(2)}) + \cdots + P(P_{t(k)})} = 1,$$

and we get a contradiction. The proof of Theorem 4.6 is complete. ∎

Theorem 4.10. *Let $P(x) \in \mathbb{C}_p[x]$ be a polynomial having no multiple zero. Let $P(x)$ satisfy the conditions (H) and (G) and $k \geq 3$ be derivative index of $P(x)$. Let S be the set of roots of $P(x) = 0$ and $u \in (\mathbb{C}_p \setminus S), u \neq 0$. Then $(S, \{u\})$ is a bi-URS for p-adic meromorphic functions on $\mathbb{C}_p^m$.*

Proof. Without loss of generality, we may assume that $u = \infty$. Suppose that f and g are two non-constant meromorphic functions on $\mathbb{C}_p^m$ satisfying $E_i(f, S) = E_i(g, S),\ E_i(f, \infty) = E_i(g, \infty)$, for all $i = 1, \dots, m$. By Theorem 4.1, $P(f)/P(g) = c$ for some non-zero constant. By Theorem 4.6, $P(x)$ is a strong uniqueness polynomial for p-adic meromorphic function on $\mathbb{C}_p^m$. Thus $f = g$. So $(S, \{u\})$ is a bi-URS for p-adic meromorphic functions on $\mathbb{C}_p^m$. ∎

Acknowledgments. The authors would like to thank Professor Ha Huy Khoai and the referee for many helpful comments and suggestions.

References

1. Vu Hoai An, p-adic Poisson-Jensen formula in several variables, *Vietnam J. Math.* **30** (2002), 43-54.
2. Vu Hoai An, Height of p-adic holomorphic maps in several variables and applications, *Acta Math. Vietnam.* **27** (2002), 257-269.
3. Vu Hoai An and Doan Quang Manh, p-adic Nevanlinna-Cartan theorem in several variables for Fermat type hypersurfaces, *East-West J. Math.* **4** (2002), 87-99.
4. Vu Hoai An and Doan Quang Manh, The "ABC" Conjecture for p-adic entire function of several variables, *Southeast Asian Bulletin of Mathematics* **27** (2004), 959-972.
5. W. Cherry and Z. Ye, Non-Archimedean Nevanlinna theory in several variables and the non-Archimedean Nevanlinna inverse problem, *Trans. Amer. Math. Soc.* **349** (1997), 5043 - 5071.
6. P. C. Hu and C. C. Yang, *Meromorphic Functions over non-Archimedean Fields*, Kluwer Academic publishers, 2000.
7. Ha Huy Khoai, La hauteur des fonctions holomorphes p-adiques de plusieurs variables, *C.R.A.Sc. Paris* **312** (1991), 751-754.
8. Ha Huy Khoai and Vu Hoai An, Value distribution on p-adic hypersurfaces, *Taiwanese J. Math.* **7** (2003), 51-67.
9. Ha Huy Khoai and Ta Thi Hoai An, On uniqueness polynomials and bi-URS for p-adic meromorphic functions, *J. Number Theory* **87** (2001), 211-221.
10. Ha Huy Khoai and Mai Van Tu, p-adic Nevanlinna-Cartan Theorem, *Internat. J. Math.* **6** (1995), 719-731.
11. J.T-Y. Wang, Uniqueness polynomials and bi-unique range sets for rational functions and non-archimedean meromorphic functions, *Acta Arith.* **104** (2002), 183-200.