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An LQ Regularization Method for Pseudomonotone Equilibrium Problems on Polyhedra

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Abstract. We present a new method for solving equilibrium problems on polyhedra. The method is based on the special logarithmic quadratic function which replaces the usual quadratic. We first use this function to solve a pseudomonotone equilibrium problems satisfying a certain Lipschitz condition. Next, to avoid the Lipschitz condition we combine this technique with line search technique to obtain a convergent algorithm for pseudomonotone equilibrium problems. An application to variational inequalities is discussed.

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1. Introduction

Let C be a polyhedral set on $\mathbb{R}^n$ defined by

$$C := \{x \in \mathbb{R}^n | \ Ax \leq b\},$$

where A is an $p \times n$ matrix, $b \in \mathbb{R}^p, p \geq n$. We suppose that the matrix A is of maximal rank, i.e., rank$A = n$ and that int$C = \{x| \ Ax < b\}$ is nonempty. Let $f : C \times C \to \mathbb{R}$ be such that $f(x, x) = 0$ for every $x \in C$. We consider the following equilibrium problem:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \quad \forall y \in C. \qquad \text{(EPP)}$$

It is well known that various classes of optimization problem, variational inequality, saddle point problem, Nash equilibrium problem, minimax problem and others (see e.g. [15, 17, 23]) can be formulated in the form of (EPP).

The logarithmic quadratic regularization technique is a powerful tool for analyzing and solving optimization problems (see e.g. [12, 14]). Recently this technique has been used to develop proximal iterative algorithm for variational inequalities (see e.g. [6, 9, 10]).

In our recent papers [1] we have used the logarithmic quadratic function for pseudomonotone equilibrium on $\mathbb{R}^n_+ := \{x = (x_1, ..., x_n) \in \mathbb{R}^n : \ x_i \geq 0 \ \ \forall i = 1, ..., n\}$ and developed algorithms for solving them.

In this paper we extend our results in [1] to pseudomonotone equilibrium problem (EPP). Namely, we first develop a linearly convergent algorithm for (EPP) with f being pseudomonotone bifunction satisfying a certain Lipschitz type condition on C by using the logarithmic quadratic function. Next, in order to avoid the Lipschitz condition we will use the line search and this function to obtain a convergent algorithm for solving equilibrium problem (EPP) with psedomonotone bifunction f.

The structure of the paper is simple. In Sec. 2, we present a linearly convergent algorithm for pseudomonotone and Lipschitz equilibrium problems. In Sec. 3, we modify the algorithm by combining a line search with the logarithmic quadratic function, which allows avoiding the Lipschitz condition and in Sec. 4 we apply to variational inequalities.

2. The Logarithmic Quadratic Algorithm

We list some well known definitions and the projection under the Euclidean norm which will be required in our following analysis.

Definition 2.1. *Let C be a convex set in $\mathbb{R}^n$, A is a $p \times n$ matrix, and let $f : C \times C \to \mathbb{R} \cup \{+\infty\}$. The bifunction f is said to be*

(i) *monotone on C if for each $x, y \in C$, we have*

$$f(x, y) + f(y, x) \leq 0;$$

(ii) *pseudomonotone on C if for each $x, y \in C$, it holds*

$$f(x, y) \geq 0 \ \ \textit{implies} \ \ f(y, x) \leq 0;$$

(iii) *$A-$Lipschitz with two constants $\bar{c_1} > 0$ and $\bar{c_2} > 0$, if we have*

$$f(x,y)+f(y,z) \geq f(x,z)-\bar{c_1}||A(y-x)||^2-\bar{c_2}||A(z-y)||^2 \ \ \forall x,y,z \in C. \tag{2.1}$$

We note that when $x = z$, $f(x, x) = 0$, this condition deduces to

$$f(x, y) + f(y, x) \geq -(\bar{c_1} + \bar{c_2})||A(y - x)||^2 \ \ \forall x, y \in C.$$

Remark 2.2. Let A be a $p \times n$ matrix, $\text{rank}A = n$, $C := \{x \in \mathbb{R}^n : \ Ax \leq b\}$, and $f : C \times C \to \mathbb{R}$. Suppose that the bifunction f satisfies

$$f(x, y) + f(y, z) \geq f(x, z) - c^0_1||y - x||^2 - c^0_2||z - x||^2 \ \ \forall x, y, z \in C, \tag{2.2}$$

where $c_1^0, c_2^0 > 0$, f is usually said to be Lipschitz with constants $c_1^0, c_2^0 > 0$ [12, 15]. Then f is $A-$Lipschitz with constants

$$\bar{c_1} := c_1^0 ||\bar{A}^{-1}\|^2, \bar{c_2} := c_2^0 \|\bar{A}^{-1}\|^2,$$

where $\bar{A} := (a_{ij})_{n\times n}$ is a submatrix of $A := (a_{ij})_{p\times n}$ such that rank$\bar{A} = n$ and

$$\|\bar{A}^{-1}\| = \sup_{\|x\|=1} \|\bar{A}^{-1}x\|.$$

Indeed, we have

$$\|x\| = \|\bar{A}^{-1}(\bar{A}x)\| \le \|\bar{A}^{-1}\| \ \|\bar{A}x)\| \le \|\bar{A}^{-1}\| \ \|Ax\| \ \ \forall x \in \mathbb{R}^n. \tag{2.3}$$

From (2.2) and (2.3), it follows that

$$\begin{aligned} &f(x,y) + f(y,z) \\ &\quad \ge f(x,z) - c_1^0\|\bar{A}^{-1}\|^2 \ \|A(y-x)\|^2 - c_2^0\|\bar{A}^{-1}\|^2 \ \|A(z-x)\|^2 \ \forall x,y,z \in C. \end{aligned}$$

It means that f is $A-$Lipschitz with constant $\bar{c_1} := c_1^0\|\bar{A}^{-1}\|^2, \bar{c_2} := c_2^0\|\bar{A}^{-1}\|^2$.

Let C be a closed convex set in $\mathbb{R}^n$ with the Euclidean norm $\|.\|$, we denote the projection on C by $P_C(.)$, i.e,

$$P_C(x) = \text{argmin}\{\|y-x\| : \ y \in C\} \ \ \forall x \in \mathbb{R}^n.$$

From the above definition and the convexity of C, it follows that

$$\|P(x) - y\| \le \|x-y\| \ \ \forall y \in C, x \in \mathbb{R}^n.$$

The following lemma can be found, for example in [21].

Lemma 2.3. *Let $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a bifunction. Then the following statements are equivalent:*
(i) *x^* is a solution to* (EPP).
(ii) *$x^* \in C$ is a solution to the problem* $\min\{f(x^*, y) : \ y \in C\}$.

Proof. See e.g. Proposition 1 in [21].

It is well known that a classical variational problem, denoted by (VIP), is to find a vector $x^* \in C$ such that

$$\langle F(x^*), y - x^*\rangle \ge 0 \ \ \forall y \in C,$$

where $C \subseteq \mathbb{R}^n$ is a nonempty closed convex subset of $\mathbb{R}^n$ and F is a continuous mapping from $\mathbb{R}^n$ into itself. Then it can be alternatively formulated as finding the zero point of the operator $T(x) = F(x) + N_C(x)$ where

$$N_C(x) = \begin{cases} \{y \in C : \ \langle y, z-x\rangle \le 0, \ \forall z \in C\} & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases} \tag{2.4}$$

A classical method to solve this problem is the proximal point algorithm [5, 20, 26], which starting with any point $x^0 \in C$ and $\lambda_k \geq \lambda > 0$, iteratively updates x^{k+1} conforming the following problem:

$$0 \in \lambda_k T(x) + \nabla_x h(x, x^k), \tag{2.5}$$

where

$$h(x, x^k) = \frac{1}{2}\|x - x^k\|^2.$$

Motivation for studying the algorithm of problem (2.5) could be found in [2, 3, 27].

Recently, Auslender et al. [7] have proposed a new type of proximal interior method for solving (VIP) on $C := \mathbb{R}^n_+$ through replacing function $h(x, x^k)$ by $d_\phi(x, x^k)$ which is defined as

$$d_\phi(x, y) = \sum_{i=1}^{n} y_i^2 \phi(y_i^{-1} x_i),$$

where

$$\phi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \mu(t - \log t - 1) & \text{if } t > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.6}$$

with $\nu > \mu > 0$. The fundamental difference here is that the term d_ϕ is used to force the iteratives $\{x^{k+1}\}$ to stay in the interior of $\mathbb{R}^n_+$.

Applying to the equilibrium problem (EPP), in this paper we consider another function defined by

$$d(x, y) = \begin{cases} \frac{1}{2}\|x - y\|^2 + \mu \sum_{i=1}^{n} y_i^2 (\frac{x_i}{y_i} \log \frac{x_i}{y_i} - \frac{x_i}{y_i} + 1) & \text{if } x > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.7}$$

with $\mu \in (0, 1)$.

Let a_i denotes the rows of the matrix A, and define the following quantities:

$$\begin{aligned} &l_i(x) = b_i - \langle a_i, x \rangle, \\ &l(x) = (l_1(x), l_2(x), ..., l_p(x)), \\ &D(x, y) = d(l(x), l(y)). \end{aligned}$$

Then we consider the following regularized auxiliary problem:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) + \frac{1}{c} D(y, x^*) \geq 0 \text{ for all } y \in C, \quad (Aux\ EPP)$$

where $c > 0$ is a regularization parameter.

We denote by $\nabla_1 D(x, y)$ the gradient of $D(., y)$ at x for every $y \in C$. It is easy to see that

$$\nabla_1 D(x, y) = -A^T \big(l(x) - l(y) + \mu X_y \log \frac{l(x)}{l(y)} \big),$$

where $X_y = \mathrm{diag}\big(l_1(y), ..., l_p(y)\big)$ and $\log\frac{l(x)}{l(y)} = \Big(\log\frac{l_1(x)}{l_1(y)}, ..., \log\frac{l_p(x)}{l_p(y)}\Big)$.

The equivalence between (EPP) and (Aux EPP) is due to the following lemma.

Lemma 2.4. *Let $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a bifunction and $x^* \in C$. Then x^* is a solution to (EPP) if and only if x^* is a solution to (Aux EPP).*

Proof. See e.g. Proposition 1 in [21]. ∎

Lemma 2.4 shows that the solution of the equilibrium problem (EPP) can be approximated by an iterative procedure $x^{k+1} = h(x^k), k = 0, 1, ...$ where $c > 0$, x^0 is any starting point in C and $h(x^k)$ is the unique solution of the strongly convex program

$$\min\{f(x^k, y) + \frac{1}{c}D(y, x^k) : \ y \in C\}.$$

However, generally, the sequence $\{x^k\}$ does not converge to a solution of the equilibrium [16, 17]. In order to avoid this drawback, the extragradient algorithm has been introduced for monotone equilibrium problems [24].

Algorithm 2.5.
Step 0. *Choose $x^0 \in C, k := 0$, a positive sequence $\{c_k\}$ such that $c_k \to c > 0$ as $k \to +\infty$.*
Step 1. *Solve the strongly convex program:*

$$\min\{f(x^k, y) + \frac{1}{c_k}D(y, x^k) : \ y \in C\} \tag{2.8}$$

to obtain the unique solution y^k.

If $y^k = x^k$, then terminate: x^k is a solution to (EPP). Otherwise go to Step 2.
Step 2. *Find x^{k+1} which is the unique solution to the strongly convex program:*

$$\min\{f(y^k, y) + \frac{1}{c_k}D(y, x^k) : \ y \in C\}.$$

Step 3. *Set $k := k + 1$, and go to Step 1.*

The following theorem establishes convergence of the algorithm.

Theorem 2.6. *Suppose that the bifunction $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ is pseudomonotone, $A-$ Lipschitz on C and $f(x, .)$ is closed convex subdifferentiable on C for each $x \in C$. Let f be lower semicontinuous on $C \times C$ and $f(., y)$ upper semicontinuous on C for each $y \in C$. Then,*
(i) *If Algorithm 2.5 terminates at Step 1, then x^k is a solution to (EPP).*
(ii) *If the algorithm does not terminate, then we obtain*

$$\begin{aligned}\|A(x^{k+1} - x^*)\|^2 \le & \|A(x^k - x^*)\|^2 - \frac{1 - 3\mu - 2\bar{c}_2 c_k}{1 + \mu}\|A(x^{k+1} - y^k)\|^2 \\ & - \frac{1 - 5\mu - 2\bar{c}_1 c_k}{1 + \mu}\|A(x^k - y^k)\|^2 + \frac{2\mu}{1 + \mu}\|A(x^{k+1} - x^k)\|^2.\end{aligned}$$

Moreover, if $0 < \mu < \min\{\frac{1-\epsilon-2\bar{c}_2 c_k}{3}, \frac{1-\epsilon-2\bar{c}_1 c_k}{5}\}$ *where* $\epsilon > 0$*, then the sequence* $\{x^k\}$ *converges to a solution* x^* *of (EPP).*

Proof. (i) If the algorithm terminates at Step 1, then $y^k = x^k$. It means that x^k is a solution of problem (2.8). By Lemma 2.3 and Lemma 2.4 it is a solution of (EPP).

(ii) Since y^k is a solution of problem (2.8), from an optimization result in convex programming [25], we have

$$0 \in \partial_2 f(x^k, y^k) + \frac{1}{c_k}\nabla_1 D(y^k, x^k) + N_C(y^k),$$

where $\partial_2 f(x^k, y^k)$ denotes the subdifferential of $f(x^k, .)$ at y^k and N_C denotes the normal cone. Using Moreau-Rockafellar theorem ([25]), we have

$$0 = w_1 + \frac{1}{c_k}\nabla_1 D(y^k, x^k) + w_2, \tag{2.9}$$

where $w_1 \in \partial_2 f(x^k, y^k), w_2 \in N_C(y^k)$.
Since $w_2 \in N_C(y^k)$ and $y^k \in \text{int}C$, we have

$$w_2 = 0. \tag{2.10}$$

From (2.9) and (2.10) it follows that

$$\langle \frac{1}{c_k}\nabla_1 D(y^k, x^k), y - y^k\rangle = \langle w_1, y^k - y\rangle \quad \forall y \in C.$$

By the definition of subgradient, we have from the the last inequalities that

$$\frac{1}{c_k}\langle \nabla_1 D(y^k, x^k), y - y^k\rangle \geq f(x^k, y^k) - f(x^k, y) \quad \forall y \in C. \tag{2.11}$$

Replacing y by x^*, we obtain

$$\frac{1}{c_k}\langle \nabla_1 D(y^k, x^k), x^* - y^k\rangle \geq f(x^k, y^k) - f(x^k, x^*).$$

Note that, x^* is a solution of (EPP), $f(x^*, y) \geq 0$. By pseudomonotonicity of f, it follows that $f(y, x^*) \leq 0$. Then

$$\frac{1}{c_k}\langle \nabla_1 D(y^k, x^k), x^* - y^k\rangle \geq f(x^k, y^k).$$

On the other hand, since x^{k+1} is a solution to the convex program

$$\min\{f(y^k, y) + \frac{1}{c_k}D(y, x^k) : \ y \in C\},$$

in the same way, we can show that

$$\frac{1}{c_k}\langle \nabla_1 D(x^{k+1}, x^k), x^* - x^{k+1}\rangle \geq f(y^k, x^{k+1}). \tag{2.12}$$

We recall that

$$\nabla_1 D(x^{k+1}, x^k) = -A^T\big(l(x^{k+1}) - l(x^k) + \mu X_{x^k}\log\frac{l(x^{k+1})}{l(x^k)}\big),$$

where $X_k = \text{diag}(l_1(x^k), ..., l_p(x^k))$ and $\log\frac{l(x^{k+1})}{l(x^k)} = \Big(\log\frac{l_1(x^{k+1})}{l_1(x^k)}, ..., \log\frac{l_p(x^{k+1})}{l_p(x^k)}\Big)^T$. Then (2.12) can be written as

$$\begin{aligned}\langle l(x^{k+1}) - l(x^k), A\big(x^{k+1} - x^*\big)\rangle + \mu\langle X_{x_k}\log\frac{l(x^{k+1})}{l(x^k)}, A\big(x^{k+1} - x^*\big)\rangle&\\ \geq c_k f(y^k, x^{k+1}).&\end{aligned} \tag{2.13}$$

From (2.13) and $l(x) = b - Ax$, it follows that

$$\begin{aligned}&\langle A(x^{k+1} - x^k), A(x^* - x^{k+1})\rangle\\ &\quad\geq \mu\langle X_{x^k}\log\frac{l(x^{k+1})}{l(x^k)}, A(x^* - x^{k+1})\rangle + c_k f(y^k, x^{k+1}).\end{aligned} \tag{2.14}$$

Now applying the Lipschitz condition (2.1) of f with $x = x^k, y = y^k, z = x^{k+1}$ we obtain

$$\begin{aligned}&f(x^k, y^k) + f(y^k, x^{k+1})\\ &\quad\geq f(x^k, x^{k+1}) - \bar{c_1}\|A(y^k - x^k)\|^2 - \bar{c_2}\|A(x^{k+1} - y^k)\|^2.\end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we have

$$\begin{aligned}\langle A(x^{k+1} - x^k),&A(x^* - x^{k+1})\rangle\\ &\geq \mu\langle X_{x^k}\log\frac{l(x^{k+1})}{l(x^k)}, A(x^* - x^{k+1})\rangle\\ &+ c_k f(x^k, x^{k+1}) - c_k f(x^k, y^k)\\ &- \bar{c_1}c_k\|A(y^k - x^k)\|^2 - \bar{c_2}c_k\|A(x^{k+1} - y^k)\|^2.\end{aligned} \tag{2.16}$$

If $y = x^{k+1}$, the inequality (2.11) becomes

$$\begin{aligned}f(x^k, x^{k+1}) - f(x^k, y^k)&\\ &\geq \frac{1}{c_k}\langle \nabla_1 D(y^k, x^k), y^k - x^{k+1}\rangle\\ &= -\frac{1}{c_k}\langle A^T\big(l(y^k) - l(x^k) + \mu X_{x^k}\log\frac{l(y^k)}{l(x^k)}\big), y^k - x^{k+1}\rangle\\ &= \frac{1}{c_k}\langle A(y^k - x^k), A(y^k - x^{k+1}\rangle\\ &- \frac{1}{c_k}\mu\langle X_{x^k}\log\frac{l(y^k)}{l(x^k)}, A(y^k - x^{k+1})\rangle.\end{aligned} \tag{2.17}$$

From (2.16) and (2.17), it follows that

$$\begin{aligned}\langle A(x^{k+1}-x^k),&A(x^*-x^{k+1})\rangle\\ &\geq \mu\langle X_{x^k}\log\frac{l(x^{k+1})}{l(x^k)}, A(x^*-x^{k+1})\rangle\\ &+\langle A(y^k-x^k), A(y^k-x^{k+1})\rangle\\ &-\bar{c_2}c_k\|A(x^{k+1}-y^k)\|^2-\bar{c_1}c_k\|A(y^k-x^k)\|^2\\ &-\mu\langle X_{x^k}\log\frac{l(y^k)}{l(x^k)}, A(y^k-x^{k+1})\rangle.\end{aligned}\tag{2.18}$$

Substituting

$$\|A(x^k-x^*)\|^2=\|A(x^k-x^{k+1})\|^2+\|A(x^{k+1}-x^*)\|^2+2\langle A(x^{k+1}-x^k), A(x^*-x^{k+1})\rangle$$

into (2.18), we obtain the estimation

$$\begin{aligned}&\|A(x^k-x^*)\|^2-\|A(x^k-x^{k+1})\|^2\\ &\geq\|A(x^{k+1}-x^*)\|^2-2\bar{c_1}c_k\|A(y^k-x^k)\|^2\\ &\quad+2\langle A(y^k-x^k), A(y^k-x^{k+1})\rangle+2\mu\langle X_{x^k}\log\frac{l(x^{k+1})}{l(x^k)}, A(x^*-x^{k+1})\rangle\\ &\quad-2\mu\langle X_{x^k}\log\frac{l(y^k)}{l(x^k)}, A(y^k-x^{k+1})-2\bar{c_2}c_k\|A(x^{k+1}-y^k)\|^2.\end{aligned}\tag{2.19}$$

Combining the inequality (2.19) with the following equality

$$\|A(x^{k+1}-x^k)\|^2=\|A(x^{k+1}-y^k)\|^2+\|A(x^k-y^k)\|^2+2\langle A(x^{k+1}-y^k), A(y^k-x^k)\rangle,$$

we have

$$\begin{aligned}&\|A(x^{k+1}-x^*)\|^2\\ &\quad\leq\|A(x^k-x^*)\|^2-\|A(x^{k+1}-y^k)\|^2-\|A(x^k-y^k)\|^2\\ &\quad+2\bar{c_2}c_k\|A(x^{k+1}-y^k)\|^2-2\mu\langle X_{x^k}\log\frac{l(x^{k+1})}{l(x^k)}, A(x^*-x^{k+1})\rangle\\ &\quad+2\mu\langle X_{x^k}\log\frac{l(y^k)}{l(x^k)}, A(y^k-x^{k+1})\rangle+2\bar{c_1}c_k\|A(y^k-x^k)\|^2.\end{aligned}\tag{2.20}$$

For each $t>0$ we have $1-\frac{1}{t}\leq\log t\leq t-1$, then we obtain after multiplication by $l_i(x^*)\geq 0$ for each $i=1,...,p$,

$$l_i(x^k)l_i(x^*)\log\frac{l_i(x^{k+1})}{l_i(x^k)}\leq l_i(x^*)\big(l_i(x^{k+1})-l_i(x^k)\big),\tag{2.21}$$

and after multiplication by $-l_i(x^{k+1})\leq 0$ for each $i=1,...,p$,

$$\begin{aligned}&-l_i(x^k)l_i(x^{k+1})\log\frac{l_i(x^{k+1})}{l_i(x^k)}\\ &\leq -l_i(x^k)l_i(x^{k+1})\big(1-\frac{l_i(x^k)}{l_i(x^{k+1})}\big)=l_i(x^k)\big(l_i(x^k)-l_i(x^{k+1})\big).\end{aligned}\tag{2.22}$$

Adding the two inequalities (2.21) and (2.22), we obtain

$$\begin{aligned} 2l_i(x^k)\log\frac{l_i(x^{k+1})}{l_i(x^k)}\big(l_i(x^*) - l_i(x^{k+1})\big) & \\ \leq 2\big(l_i(x^k) - l_i(x^*)\big)\big(l_i(x^k) - l_i(x^{k+1})\big) & \\ = |l_i(x^k) - l_i(x^*)|^2 + |l_i(x^k) - l_i(x^{k+1})|^2 & \\ - |l_i(x^{k+1}) - l_i(x^*)|^2 \quad \forall i = 1, ..., p. & \end{aligned}$$

These inequalities deduce that

$$\begin{aligned} 2\langle X_{x^k}\log\frac{l(x^{k+1})}{l(x^k)}, A(x^{k+1} - x^*)\rangle & \\ \leq \|A(x^k - x^*)\|^2 + \|A(x^k - x^{k+1})\|^2 - \|A(x^{k+1} - x^*)\|^2. & \end{aligned} \tag{2.23}$$

In the same way, we also have

$$\begin{aligned} 2\langle X_{x^k}\log\frac{l(y^k)}{l(x^k)}, A(y^k - x^{k+1})\rangle & \\ \leq \|A(x^k - y^k)\|^2 + \|A(x^k - x^{k+1})\|^2 - \|A(y^k - x^{k+1})\|^2. & \end{aligned} \tag{2.24}$$

Adding the inequalities (2.20), (2.23) and (2.24), we get

$$\begin{aligned} &\|A(x^{k+1} - x^*)\|^2 \\ &\quad \leq \|A(x^k - x^*)\|^2 - \|A(x^{k+1} - y^k)\|^2 - \|A(x^k - y^k)\|^2 \\ &\quad + 2\bar{c_1}c_k\|A(y^k - x^k)\|^2 + 2\bar{c_2}c_k\|A(x^{k+1} - y^k)\|^2 \\ &\quad + \|A(x^k - x^{k+1})\|^2 - \|A(x^{k+1} - x^*)\|^2) + \mu(\|A(y^k - x^k)\|^2 \\ &\quad + \|A(x^{k+1} - x^k)\|^2 - \|A(x^{k+1} - y^k)\|^2) + \mu(\|A(x^k - x^*)\|^2, \end{aligned}$$

and consequently

$$\begin{aligned} (1+\mu)\|A(x^{k+1} - x^*)\|^2 & \\ \leq (1+\mu)\|A(x^k - x^*)\|^2 - (1+\mu-2\bar{c_2}c_k)\|A(x^{k+1} - y^k)\|^2 & \\ - (1-\mu-2\bar{c_1}c_k)\|A(x^k - y^k)\|^2 + 2\mu\|A(x^{k+1} - x^k)\|^2. & \end{aligned} \tag{2.25}$$

Applying the following inequality

$$\|A(x^{k+1} - x^k)\|^2 \leq 2\|A(x^{k+1} - y^k)\|^2 + 2\|A(x^k - y^k)\|^2$$

to the last term in the right hand side of (2.25), we obtain

$$\begin{aligned} (1+\mu)\|A(x^{k+1} - x^*)\|^2 \leq & (1+\mu)\|A(x^k - x^*)\|^2 \\ & - (1-3\mu-2\bar{c_2}c_k)\|A(x^{k+1} - y^k)\|^2 \\ & - (1-5\mu-2\bar{c_1}c_k)\|A(x^k - y^k)\|^2, \end{aligned} \tag{2.26}$$

which proves the first part of (ii).

Now we prove the last part of (ii). The assumptions

$$0 < \mu < \min\{\frac{1-\epsilon-2\bar{c_2}c_k}{3}, \frac{1-\epsilon-2\bar{c_1}c_k}{5}\}$$

and $\epsilon > 0$ imply

$$1 - 2\bar{c_1}c_k > 0 \ \text{ and } \ 1 - 2\bar{c_2}c_k > 0 \ \ \forall k = 0, 1, ...$$

Then, using (2.26) we have

$$\|A(x^{k+1} - x^*)\|^2 \le \|A(x^k - x^*)\|^2 \ \ \forall k = 0, 1, ...$$

This inequality shows that the sequence $\{\|A(x^k - x^*)\|\}$ is nonincreasing. Since it is bounded below by 0, it must be convergent. Since A is of maximal rank the function $u \to \|u\|_A := \|Au\|$ is a norm on $\mathbb{R}^n$ and it follows that the sequence $\{\|x^k - x^*\|\}$ converges. Then the sequence $\{x^k\}_{k\ge 0}$ is bounded and it has a subsequence $\{x^{k_i}\}$ such that $x^{k_i} \to \bar{x}$ as $i \to +\infty$. From (2.26), we get

$$\frac{1-5\mu-2\bar{c_1}c_k}{1+\mu}\|A(x^k - y^k)\|^2 \le \|A(x^k - x^*)\|^2 - \|A(x^{k+1} - x^*)\|^2 \ \ \forall k = 0, 1, ...$$

Applying these inequalities iteratively, we obtain

$$\sum_{k=0}^{n} \frac{1-5\mu-2\bar{c_1}c_k}{1+\mu}\|A(x^k - y^k)\|^2 \le \|A(x^0 - x^*)\|^2 - \|A(x^{n+1} - x^*)\|^2 \ \ \forall k \ge 0.$$

As the sequence $\{\|A(x^{n+1} - x^*)\|\}_{k\ge 0}$ is convergent, passing $n \to +\infty$ we have

$$\lim_{k\to+\infty} \frac{1-5\mu-2\bar{c_1}c_k}{1+\mu}\|A(x^k - y^k)\|^2 = 0.$$

Using this with the assumption $1 - 5\mu - 2\bar{c_1}c_k > \epsilon > 0$, we get

$$\lim_{k\to+\infty} \epsilon\|A(x^k - y^k)\| = 0,$$

which implies

$$\lim_{i\to+\infty} \|A(\bar{x} - y^{k_i})\| = 0.$$

It holds that

$$\lim_{i\to\infty} y^{k_i} = \bar{x}.$$

Recall that y^{k_i} is a solution of the problem

$$\min\{f(x^{k_i}, y) + \frac{1}{c_{k_i}}D(y, x^{k_i}) : \ y \in C\}.$$

Then

$$f(x^{k_i}, y^{k_i}) + \frac{1}{c_{k_i}} D(y^{k_i}, x^{k_i}) \le f(x^{k_i}, y) + \frac{1}{c_{k_i}} D(y, x^{k_i}) \ \ \forall y \in C.$$

Using the semicontinuity of f, the upper semicontinuity of $f(., y)$ and $D(y, .)$, passing to the limit as $i \to +\infty$ we obtain

$$f(\bar{x}, y) + \frac{1}{c} D(y, \bar{x}) \ge 0 \ \ \forall y \in C.$$

So $\bar{x}$ is a solution to (Aux EPP). Then, by the Lemma 2.4, $\bar{x}$ is a solution to (EPP). Replacing x^* by $\bar{x}$ in (2.26) yields

$$\|A(x^{k+1} - \bar{x})\| \le \|A(x^k - \bar{x})\| \ \ \forall k = 0, 1, ...$$

which implies that the sequence $\{\|A(x^k - \bar{x})\|\}$ is convergent. We then have that the sequence $\{\|x^k - \bar{x}\|\}$ is convergent. By the above proof, the sequence $\{x^k\}$ has a subsequense converging to $\bar{x}$, we deduce that the whole sequence $\{x^k\}$ converges to the solution $\bar{x}$ of (EPP). ■

3. An Algorithm Without Lipschitz Condition

In this section, in order to avoid the Lipschitz condition of f on C, we combine the logarithmic quadratic function with line search technique. This technique has been used widely in descent method for solving this problem (see [19, 24]) and variational inequalities (see [4, 13, 15, 22]).

The algorithm then can be described as follows.

Algorithm 3.1.
Step 0. *Take* $x^0 \in C, k := 0$ *and a sequence* $\gamma_k \in (0; 2) \ \ \forall k \ge 0$.
Step 1. *Find* y^k *which is the solution to the strongly convex program:*

$$\min\{f(x^k, y) + \frac{1}{c_k} D(y, x^k) : \quad y \in C\}. \tag{3.1}$$

If $y^k = x^k$*, then stop.*
Otherwise go to Step 2.
Step 2. *Find* $\lambda_k \in (0, 1)$ *as the smallest number such that*

$$f((1 - \lambda_k)x^k + \lambda_k y^k, y^k) + \frac{1}{2c_k} D(y^k, x^k) \le 0. \tag{3.2}$$

Set $z^k := (1 - \lambda_k)x^k + \lambda_k y^k$*, choose* $g^k \in \partial_2 f(z^k, z^k)$.

If $g^k = 0$*, then stop.*
Otherwide go to Step 3.
Step 3. *Set* $\delta_k := \gamma_k \frac{-\lambda_k f(z^k, y^k)}{(1-\lambda_k)\|g^k\|^2}$ *and*

$$x^{k+1} = P_C(x^k - \delta_k g^k),$$

$k := k+1$ and go to Step 1.

Recall that $P_C(x)$ denotes the projection of x on C.

Now we are in a position to prove the following convergence theorem for Algorithm 3.1.

Theorem 3.2. *Suppose that the sequences $\gamma_k \in (0,2), c_k \to \bar{c}$ as $k \to \infty$, and the bifunction $f : C \times C \to \mathbb{R} \cup \{+\infty\}$ satisfies the following conditions:*

(a) $\liminf \gamma_k(2-\gamma_k) > 0$.

(b) *f is pseudomonotone on C.*

(c) *$f(x,.)$ is closed, convex and subdifferentiable on C for each $x \in C$.*

(d) *f is lower semicontinuous on $C \times C$ and $f(.,y)$ be upper semicontinuous on C.*

Then,

(i) *If Algorithm 3.1 terminates at Step 1 or Step 2 then x^k is a solution to (EPP).*

(ii) *For all x^* which is a solution to (EPP), we have*

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \frac{(2-\gamma_k)\delta_k^2}{\gamma_k}\|g^k\|^2.$$

(iii) *If Algorithm 3.1 does not terminate at Step 1 or Step 2, then the sequence $\{x^k\}$ converges to x^* which is a solution to (EPP).*

Proof. First, we have to show that there always exists $\lambda_k \in (0,1)$ as the smallest number satisfying (3.2). We suppose on the contrary that for every $\lambda \in (0,1)$, we have

$$f((1-\lambda)x^k + \lambda y^k, y^k) + \frac{1}{2c_k}D(y^k, x^k) > 0.$$

Passing to the limit in the above inequality (as $\lambda \to 0^+$), by the upper semicontinuity of $f(.,y)$, we obtain

$$f(x^k, y^k) + \frac{1}{2c_k}D(y^k, x^k) \geq 0. \tag{3.3}$$

Since y^k is a solution to (3.1), it follows that

$$f(x^k, y) + \frac{1}{c_k}D(y, x^k) \geq f(x^k, y^k) + \frac{1}{c_k}D(y^k, x^k).$$

Replacing y by x^k in the above inequality, we have

$$0 \geq f(x^k, y^k) + \frac{1}{c_k}D(y^k, x^k). \tag{3.4}$$

Then from (3.3) and (3.4) it follows that $D(x^k, y^k) = 0$, i.e., $d\big(l(x^k), l(y^k)\big) = 0$. Since $l(x) = b - Ax$ and A is of maximal rank, we obtain $x^k = y^k$. This contracdics $x^k \neq y^k$ in Step 1.

To prove part (i), we suppose that Algorithm 3.1 terminates at Step 1, hence $x^k = y^k$. Then

$$f(x^k, y) + \frac{1}{c_k} D(y, x^k) \geq f(x^k, y^k) + \frac{1}{c_k} D(y^k, x^k) = 0 \quad \forall y \in C.$$

This means that x^k is a solution to (Aux EPP). From Lemma lem2, x^k is also a solution to (EPP).

If Algorithm 2 terminates at Step 2, then $g^k = 0$, that means $0 \in \partial_2 f(z^k, z^k)$. It is easy to see that z^k is a solution to the following convex problem:

$$\min_{y \in C} f(z^k, y).$$

Then by virtue of Lemma 2.3, z^k is a solution to (EPP).

Now we prove part (ii). We have

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &= \|P_k\big(x^k - \delta_k g^k\big) - x^*\|^2 \\ &\leq \|x^k - x^* - \delta_k g^k\|^2 \\ &= \|x^k - x^*\|^2 - 2\delta_k \langle g^k, x^k - x^* \rangle + (\delta_k \|g^k\|)^2.\end{aligned} \tag{3.5}$$

Note that, since x^* is a solution to (EPP), $f(x^*, y) \geq 0$. Then by pseudomonotonicity, it follows that $-f(z^k, x^*) \geq 0$. Combining this with

$$\begin{aligned}\langle g^k, x^k - x^* \rangle &= \langle g^k, x^k - z^k \rangle + \langle g^k, z^k - x^* \rangle \\ &\geq \langle g^k, x^k - z^k \rangle + f(z^k, z^k) - f(z^k, x^*),\end{aligned}$$

we obtain

$$\begin{aligned}\langle g^k, x^k - x^* \rangle &\geq \langle g^k, x^k - z^k \rangle \\ &= \frac{\lambda_k}{1 - \lambda_k} \langle g^k, z^k - y^k \rangle \\ &\geq \frac{\lambda_k}{1 - \lambda_k} \big(f(z^k, z^k) - f(z^k, y^k)\big) \\ &= \frac{-\lambda_k}{1 - \lambda_k} f(z^k, y^k). \\ &= \frac{\delta_k}{\gamma_k} \|g_k\|^2.\end{aligned} \tag{3.6}$$

From (3.3) it follows that $f(z^k, y^k) < 0$. Hence

$$\delta_k = \frac{-\gamma_k \lambda_k f(z^k, y^k)}{(1 - \lambda_k)\|g^k\|^2} > 0. \tag{3.7}$$

Then from (3.5), (3.6) and (3.7), we have

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - 2\frac{\delta_k^2}{\gamma_k} \|g^k\|^2 + (\delta_k \|g^k\|)^2 \\ &= \|x^k - x^*\|^2 - \frac{2 - \gamma_k}{\gamma_k} (\delta_k \|g^k\|)^2,\end{aligned} \tag{3.8}$$

which proves part (ii).

Now we rewrite (3.8) as follows

$$\sum_{k=0}^{n} \frac{2-\gamma_k}{\gamma_k}(\delta_k\|g^k\|)^2 \le \sum_{k=0}^{n}(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2)$$
$$= \|x^0 - x^*\|^2 - \|x^{n+1} - x^*\|^2 \quad \forall n \ge 0.$$

On the other hand, since (3.8) deduces that $\{\|x^k - x^*\|\}$ is a decreasing sequence and is lower bounded by $\|x^0 - x^*\|$, then it must converge. It means that

$$\sum_{k=0}^{\infty} \frac{2-\gamma_k}{\gamma_k}(\delta_k\|g^k\|)^2 < +\infty.$$

Hence

$$\lim_{k\to\infty} \frac{2-\gamma_k}{\gamma_k}(\delta_k\|g^k\|)^2 = 0,$$

which together with $\lim_{k\to\infty} \inf(2-\gamma_k)\gamma_k > 0$ implies

$$\lim_{k\to\infty} \frac{\lambda_k f(z^k, y^k)}{(1-\lambda_k)\|g^k\|} = 0.$$

From the convergence of $\{\|x^k - x^*\|\}_{k\ge 0}$, we have that the sequence $\{x^k\}$ is bounded. Then by the maximum theorem [5], we can deduce that the sequence $\{g^k\}$ is bounded too. Thus

$$\lim_{k\to\infty} \frac{\lambda_k f(z^k, y^k)}{(1-\lambda_k)} = 0. \tag{3.9}$$

According to the rule (3.2), it is easy to see that

$$\frac{1}{2c_k} D(y^k, x^k) \le -f(z^k, y^k). \tag{3.10}$$

We consider two cases:

Case 1: If $\lim_{k\to\infty} \sup \lambda_k > 0$, then there exists $\bar{\lambda} \in (0, 1]$ such that $\lambda_k \ge \bar{\lambda}\ \forall k \ge 0$. From (3.8) and inequality (3.10), we have

$$\lim_{k\to\infty} D(y^k, x^k) = 0. \tag{3.11}$$

Since the sequence $\{x^k\}$ is bounded, hence it has a subsequence $\{x^k :\ k \in M\}$ converging to a point $\bar{x}$. Using the limit (3.11) we see that the subsequence $\{y^k : k \in M\}$ also converges to $\bar{x}$. Note that y^k is a solution to (3.1), hence

$$f(x^k, y) + \frac{1}{c_k} D(y, x^k \ge f(x^k, y^k) + \frac{1}{c_k} D(y^k, x^k) \quad \forall y \in C.$$

Passing to the limit as $k \to \infty$ and using the upper semicontinuity of $f(.,y)$, we have

$$f(\bar{x}, y) + \frac{1}{\bar{c}} D(y, \bar{x}) \geq f(\bar{x}, \bar{x}) + \frac{1}{\bar{c}} D(\bar{x}, \bar{x}) = 0 \quad \forall y \in C.$$

By Lemma 2.4, $\bar{x}$ is a solution to (EPP), thus the proof of the theorem in this case is complete.

Case 2: If $\lim_{k\to\infty} \sup \lambda_k = 0$, then since $\{x^k\}$ is bounded, we have some subsequence $\{x^k : k \in M\}$ converging to some point $\bar{x}$ as $k \to \infty$. From Step 1 of Algorithm 3.1, by the lower semicontinuity of $f(x^k, .) + \frac{1}{c_k} D(., x^k)$, the sequence $\{y^k\}$ is bounded too [5]. Thus, by taking a subsequence, if necessary, we may assume that the subsequence $\{y^k : k \in M\}$ also converges to some point $\bar{y}$. From

$$f(x^k, y) + \frac{1}{c_k} D(y, x^k) \geq f(x^k, y^k) + \frac{1}{c_k} D(y^k, x^k) \quad \forall k \in M, y \in C,$$

by the lower semicontinuity of f, D and the upper semicontinuity of $f(.,y), D(y,.)$, taking the limit as $k \to \infty$, we can write

$$f(\bar{x}, y) + \frac{1}{\bar{c}} D(y, \bar{x}) \geq f(\bar{x}, \bar{y}) + \frac{1}{\bar{c}} D(\bar{y}, \bar{x}). \tag{3.12}$$

Substituting $y = \bar{x}$ we then have

$$0 \geq f(\bar{x}, \bar{y}) + \frac{1}{\bar{c}} D(\bar{y}, \bar{x}). \tag{3.13}$$

On the other hand, by Step 2 in Algorithm 3.1, since $\lambda_k \in (0,1)$ is the smallest number satisfying

$$f\big((1-\lambda_k)x^k + \lambda_k y^k, y^k\big) + \frac{1}{2c_k} D(y^k, x^k) \leq 0,$$

and $\lim_{k\to\infty} \sup \lambda_k = 0$ we deduce that

$$f\big((1-\frac{1}{2}\lambda_k)x^k + \frac{1}{2}\lambda_k y^k, y^k\big) + \frac{1}{2c_k} D(y^k, x^k) > 0 \quad \forall k \geq 0.$$

Passing $k \to \infty, k \in M$ in the above inequality, we obtain

$$f(\bar{x}, \bar{y}) + \frac{1}{2\bar{c}} D(\bar{y}, \bar{x}) \geq 0.$$

This together with (3.13) implies $D(\bar{x}, \bar{y}) = 0$, hence $\bar{x} = \bar{y}$. Then replacing $\bar{y}$ in (3.12) by $\bar{x}$, we deduce that

$$f(\bar{x}, y) + \frac{1}{\bar{c}} D(y, \bar{x}) \geq 0 \quad \forall y \in C.$$

The proof is complete. ■

Remark 3.3. The smallest number $\lambda_k \in (0,1)$ in Step 2 of Aglgorithm 3.1 can be replaced by the following: with $\beta \in (0,1)$, we find n as the smallest natural number such that

$$f\big(\beta^n x^k + (1-\beta^n)y^k, y^k\big) + \frac{1}{2c_k} d(y^k, x^k) \leq 0,$$

then set $\lambda_k := 1 - \beta^n$.

4. Application to Variational Inequalities

It is well known [8] that when $C = \mathbb{R}^n_+$ is a closed convex cone, then (VIP) becomes the complementarity problem:

$$\text{Find } x^* \in C \text{ such that } F(x^*) \in C^*, \langle w^*, x^* \rangle = 0, \qquad (NCP)$$

where

$$C^* := \{w : \ \langle w, x \rangle \geq 0 \ \ \forall x \in C\}$$

is the polar cone of C.

We recall the following definitions [11]:

Definition 4.1. *Let $C \subseteq \mathbb{R}^n$ and $F : C \to \mathbb{R}^n$. The function F is said to be*

(a) *pseudomonotone on C if for each $x, y \in C$ the inequality*

$$\langle F(y), x - y \rangle \geq 0$$

implies

$$\langle F(x), x - y \rangle \geq 0.$$

(b) *Lipschitz on C with constant L (shortly $L-$Lipschitz) if for each $x, y \in C$*

$$\|F(x) - F(y)\| \leq L\|x - y\|.$$

For each pair $(x, y) \in C \times C$ by setting

$$f(x, y) := \langle F(x), y - x \rangle, \tag{4.1}$$

we can easily check that x^* is a solution to (VIP) if and only if it is a solution to (EPP). Then the relation between f and F is due to the following lemma.

Lemma 4.2. *Let $C := \{x \in \mathbb{R} : Ax \leq b\}$ and f be defined by (4.1). The following statements hold:*

(a) *If F is pseudomonotone on C, then f is pseudomonotone on C.*

(b) *If F is $L-$Lipschitz continuous on C, then f is $A-$Lipschitz with constant*

$$\bar{c_1} := \frac{1}{2}L\|\bar{A}^{-1}\|, \bar{c_2} := \frac{1}{2}L\|\bar{A}^{-1}\|.$$

Proof. The first statement is immediate from the definition. We have

$$\begin{aligned} f(x,y) + f(y,z) - f(x,z) &= \langle F(x), y-x\rangle + \langle F(y), z-y\rangle - \langle F(x), z-x\rangle \\ &= \langle F(x) - F(y), y-z\rangle \\ &\geq -\|F(x)-F(y)\|\ \|y-z\| \\ &\geq -L\|x-y\|\ \|y-z\| \\ &\geq -\frac{L}{2}\|x-y\|^2 - \frac{L}{2}\|y-z\|^2. \end{aligned}$$

Using this inequality and Remark 2.2, we obtain the proof of the second statement. ■

Now we apply Algorithm 2.5 to the variational inequality (VIP) when F is pseudomonotone and $L-$Lipschitz on C. Note that in this case, the subproblem

$$y^k = \text{argmin}\left\{ f(x^k, y) + \frac{1}{c_k}D(y, x^k) :\ y \in C \right\}$$

takes the form

$$y^k = \text{argmin}\left\{ \langle F(x^k), y - x^k\rangle + \frac{1}{c_k}D(y, x^k) :\ y \in C \right\}.$$

It in turns is

$$\begin{aligned} y^k =&\text{argmin}\Big\{ \langle F(x^k), y - x^k\rangle + \frac{1}{2c_k}\|l(y) - l(x^k)\|^2 \\ &+ \frac{\mu}{c_k}\sum_{i=1}^{p} l_i^2(x^k)\Big(\frac{l_i(y)}{l_i(x^k)}\log\frac{l_i(y)}{l_i(x^k)} - \frac{l_i(y)}{l_i(x^k)} + 1\Big) :\ y \in C\Big\}. \end{aligned} \tag{4.2}$$

For each $x \in C$ we set $t = l(x)$ and $\bar{F}(x) = \big((\bar{A}^{-1})^T F(x), 0, ...0\big) \in \mathbb{R}^p$ where $\bar{A}$ is defined in Remark 2.2. We can easily check that $x \in \{x \in C :\ l(x) > 0\}$ if and only if

$$\begin{aligned} t \in C_+ := \{t \in \mathbb{R}^p :\ t_i > 0,\ \langle a_j, \bar{A}^{-1}\bar{t}\rangle - t_j = \langle a_i, \bar{A}^{-1}\bar{b}\rangle - b_j \\ \forall i = 1, ..., p, j = n+1, ..., p\}, \end{aligned}$$

where $\bar{t} = (t_1, ..., t_n), \bar{b} = (b_1, ..., b_n) \in \mathbb{R}^n$. Hence (4.2) can be written as $y^k = \bar{A}^{-1}(b - t^k)$, $t^{x^k} = l(x^k)$ and t^k is the unique solution to the strongly convex problem

$$\begin{aligned} \min\Big\{ \langle \bar{F}(x^k), t^{x^k} - t\rangle + \frac{1}{2c_k}\|t - t^{x^k}\|^2 + \frac{\mu}{c_k}\sum_{i=1}^{p}(t_i^{x^k})^2\Big(\frac{t_i}{t_i^{x^k}}\log\frac{t_i}{t_i^{x^k}} - \frac{t_i}{t_i^{x^k}} + 1\Big) : \\ t = (t_1, ..., t_p) \in C_+\Big\}. \end{aligned}$$

In the same way, in Algorithm 2.5, we can show that x^{k+1} is the unique solution of the following problem

$$\min\left\{f(y^k,y)+\frac{1}{c_k}D(y,x^k):\ y\in C\right\},$$

which is also defined by $x^{k+1}=\bar{A}^{-1}(\bar{b}-\bar{t}^k)$ where $t^{x^k}=l(x^k)$, $t^{y^k}=l(y^k)$ and t^k is the unique solution to the strongly convex problem

$$\min\Big\{\langle\bar{F}(y^k),t^{y^k}-t\rangle+\frac{1}{2}\|t-t^{x^k}\|^2+\mu\sum_{i=1}^{p}(t_i^{x^k})^2\big(\frac{t_i}{t_i^{x^k}}\log\frac{t_i}{t_i^{x^k}}-\frac{t_i}{t_i^{x^k}}+1\big)$$
$$:\ t=(t_1,...,t_p)\in C_+\Big\}.$$

In the case $n=p$, the algorithm for the variational inequality (VIP) can be detailed in the following.

Algorithm 4.3.

Step 0. *Choose* $x^0\in C, k:=0,$ *a positive sequence* $\{c_k\}$ *such that* $c_k\to c>0$ *as* $k\to+\infty$.

Step 1. *For every* $i=1,...,n$, *solve the strongly convex program:*

$$\min\left\{\frac{1}{2}z^2-\eta_{ki}z-\xi_{ki}\ z\log z:\quad z\in(0,+\infty)\right\}$$

to obtain the unique solution y_i^k, *where*

$$F(x)=(F_1(x),...,F_n(x)),$$
$$\eta_{ki}=t_i^{x^k}+F_i(x^k)+\mu t_i^{x^k}\log t_i^{x^k}+\mu t_i^{x^k},\forall i=1,...,n,$$
$$\xi_{ki}=\mu t_i^{x^k}.$$

We denote $y^k:=(y_1^k,...,y_n^k)$. *If* $y^k=x^k$, *then terminate:* x^k *is a solution to (VIP).*
Otherwise go to Step 2.

Step 2. *For every* $i=1,...,n$, *find*

$$x_i^{k+1}=\text{argmin}\left\{\frac{1}{2}z^2-\bar{\eta}_{ki}z-\bar{\xi}_{ki}\ z\log z:\quad z\in(0,+\infty)\right\},$$

where

$$\bar{\eta}_{ki}=t_i^{y^k}+F_i(y^k)+\mu t_i^{x^k}\text{log}t_i^{x^k}+\mu t_i^{x^k},\ \bar{\xi}_{ki}=\mu t_i^{x^k}\quad\forall i=1,...,n.$$

We set $x^{k+1}:=(x_1^{k+1},...,x_n^{k+1})$.

Step 3. *Set $k := k + 1$, and go to Step 1.*

Validity and linear convergence of this algorithm is immediate from Algorithm 2.5.

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