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On Harada Rings and Serial Artinian Rings

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Abstract. A ring R is called a right Harada ring if it is right Artinian and every non-small right R-module contains a non-zero injective submodule. The first result in our paper is the following: Let R be a right perfect ring. Then R is a right Harada ring if and only if every cyclic module is a direct sum of an injective module and a small module; if and only if every local module is either injective or small. We also prove that a ring R is QF if and only if every cyclic module is a direct sum of a projective injective module and a small module; if and only if every local module is either projective injective or small. Finally, a right QF-3 right perfect ring R is serial Artinian if and only if every right ideal is a direct sum of a projective module and a singular uniserial module.

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1. Introduction and Preliminaries

Throughout this paper, all rings are associative rings with identity and all right R-modules are unitary. For a right R - module M, we denote by $E(M), J(M)$

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and $Z(M)$ the injective hull, radical and the singular submodule of M, respectively. Especially, $J(R)$ is the Jacobson radical of the ring R. A right R-module M is called *uniserial* if the lattice of its submodules is linear. Call M a *serial* module if it is a direct sum of uniserial modules. A ring R is right serial if R_R is serial as a right R-module. A ring R is serial if it is right and left serial. Call a ring R *serial Artinian* if it is serial and two-sided Artinian.

Call M a *small* module if M is small in $E(M)$ otherwise, we call it a *non-small* module. Dually, a right R-module M is called a *co-small* module if for any epimorphism f from P to M with P is projective, $\ker(f)$ is essential in P. A non-cosmall module is defined as a *not co-small* module. A right R-module is a *local module* if it has the greatest proper submodule. A ring R is called a *Quasi-Frobenius* ring (briefly QF-ring) if it is right self-injective right Artinian. Call a ring R a *right QF-3* ring if $E(R_R)$ is projective (see [14, 21]).

Manabu Harada [9, 10, 11] has studied some generalizations of QF-rings by introducing the following two conditions:

(*) Every non-small right R-module contains a non-zero injective submodule;

(**) Every non-cosmall right R-module contains a non-zero projective direct summand.

It should be noted that right perfect rings with (*) and semiperfect rings with (**) are characterized in terms of ideals in [11, 12, 13]. Oshiro [16] gave some characterizations of these kinds of rings and introduced the definitions of right Harada and right co-Harada rings as follows.

A ring R is called a *right Harada* ring if it is right Artinian and (*) holds. A ring R is a right *co-Harada* ring if it satisfies (**) and ACC on right annihilators.

In this paper, we give some characterizations of right Harada rings and serial Artinian rings. In Sec. 1, we recall some well-known results which will be used in this paper. We characterize the classes of right Harada rings and QF-rings by right perfect rings and cyclic or local modules. Section 3 is concerned with serial Artinian rings. For convenience, we list some well-known results here to use in this paper.

Theorem A. ([16, Theorem 2.11]) *For a ring R, the following conditions are equivalent:*

(*1*) *R is a right Harada ring;*

(*2*) *R is right perfect and satisfies the condition that the family of all projective modules is closed under taking small covers, i.e., for any exact sequence $P \xrightarrow{\varphi} E \to 0$, with E is injective and P is projective,* $\ker(\varphi)$ *is small in* P;

(*3*) *Every right R-module is a direct sum of an injective module and a small module;*

(*4*) *Every injective module is a lifting module.*

Theorem B. ([11, Theorem 3.6]) *Let R be a semiperfect ring and $\{e_i\}_{i=1}^n \cup \{f_j\}_{j=1}^m$ a complete set of orthogonal primitive idempotents of R, where each e_iR is non-small, $i = 1, \ldots n$, and f_jR is small, $j = 1, \ldots m$. Then $(**)$ holds if and only if:*

(*a*) *$n \geq 1$ and e_iR is injective, $i = 1, \ldots, n$;*

(b) *For each f_j, there exists e_i such that f_jR can be embedded in e_iR;*

(c) *For each f_j, there exists an integer n_j such that e_iJ^t is projective for $0 \leq t \leq n_i$ and $e_iJ^{n_i+1}$ is a singular module, where $J = J(R)$;*
Further in this case, it is shown that every submodule e_iB in e_iR is either contained in $e_iJ^{n_i+1}$ or equal to some e_iJ^i.

Theorem C. ([16]) *Let R be a right Artinian ring. Then R is a serial ring if and only if for any primitive idempotent f of R, the injective hull $E(fR)$ is a uniserial module.*

2. Right Harada Rings

In this section, we will prove that the classes of right Harada rings and QF-rings are both characterized by perfect rings and cyclic (or local) modules. The proof of the following lemma is routine and is therefore omitted.

Lemma 1. *If $\{X_i, i = 1, \ldots, n\}$ is a family of small modules, then $X = \sum_i X_i$ is also small.*

Proposition 2. *Let R be a right perfect ring. Then the following conditions are equivalent:*

(*1*) *The condition $(*)$ holds;*

(*2*) *Every non-small cyclic module contains a non-zero injective module;*

(*3*) *Every local module is either injective or small.*

Proof. (1) $\Rightarrow$ (2) is clear. We now prove (2) $\Rightarrow$ (3). Let M be a local module. Since R is right perfect, it follows from [5, Proposition 18.23] that there exists a primitive idempotent g of R such that $M \cong gR/G,\ G \subset gR$. Hence M is an indecomposable cyclic module. Therefore M either injective or small by (2).

(3) $\Rightarrow$ (1) Let M be a non-small module and $E = E(M)$. Since R is right perfect, it follows that $M \not\subset EJ$ (since EJ is small in E). Take any $m \in M \setminus EJ$. Then mR is a non-small module. Let $\{e_i | i = 1, 2, \ldots, n\}$ be an orthogonal system of primitive idempotents of R. Then $mR = \sum_{i=1}^{n} me_iR$. By Lemma 1, there exists an idempotent e_i such that me_iR is non-small. Since $me_iR \neq 0$ and $me_iR \cong e_iR/H,\ H \subset e_iR$, the module me_iR is local. It follows from (3) that me_iR is injective. Thus M contains a non-zero me_iR, hence $(*)$ holds. ■

Theorem 3. *Let R be a right perfect ring. Then the following conditions are equivalent:*

(*1*) *R is a right Harada ring;*

(*2*) *Every cyclic module is a direct sum of an injective module and a small module;*

(*3*) *Every local module is either injective or small.*

Proof. (1) $\Rightarrow$ (3). This part follows from Theorem A.

(2) $\Rightarrow$ (3) By the same argument as that of (2) $\Rightarrow$ (3) in the proof of Proposition 2.

(3) $\Rightarrow$ (1) By Proposition 2, R satisfies $(*)$. Hence by [12, Theorem 5], R is a right Artinian ring, and therefore it is a right Harada ring. ■

Theorem 4. *Let R be a right perfect ring. Then the following conditions are equivalent:*

(1) *R is a QF-ring;*

(2) *Every cyclic module is a direct sum of a projective injective module and a small module;*

(3) *Every local module is either projective injective or small.*

Proof. (1) $\Rightarrow$ (2) Since R is QF, it is a right Harada ring by [16, Theorem 4.3]. For a right R-module M, we have $M = I \oplus S$, with I is injective and S is small by Theorem A. It is clear that I is projective, proving (2).

(2) $\Rightarrow$ (3) It follows from Proposition 2.

(3) $\Rightarrow$ (1) It suffices to show that every injective module is projective. Since R is a right perfect ring satisfying (3), in view of Theorem 3, R is a right Harada ring. Let Q be an injective module. Then Q has a decomposition $Q = \bigoplus_I Q_i$, where each Q_i is a non-zero indecomposable module. We will show that each Q_i is projective for each $i \in I$. Since R is right Artinian, Q_i contains a maximal submodule Q_i' (see [1, Theorem 28.4]). Let X be a proper submodule of Q_i such that $X \not\subset Q_i'$. Then $Q_i' + X = Q_i = E(Q_i')$. Hence Q_i' is a non-small module. Since R is a right Harada ring, it implies that Q_i' contains a proper direct summand, a contradiction. Thus Q_i' is the greatest proper submodule of Q_i, i.e., Q_i is a local module. Hence, by (3), Q_i is projective. It follows that Q is a projective module, proving that R is QF. ■

3. Characterizations of Serial Artinian Rings

First we recall a remark due to M. Harada in [11] as folllows:

Remark 5. Let R be a right perfect ring. Then R has a decomposition of the form

$$R = \bigoplus_{i=1}^{n} e_i R \oplus \bigoplus_{m=1}^{m} f_j R$$

where $\{e_i, i = 1, \ldots, n\} \cup \{f_j, j = 1, \ldots, m\}$ is the set of mutually orthogonal idempotents with each $e_i R$ non-small and $f_j R$ being a small module, and always we have $n \geq 1$.

Lemma 6. *Let R be a right perfect ring and M a uniserial right R-module. Then every submodule of M is cyclic and hence M is a Noetherian module.*

Proof. Let N be a non-zero submodule of M. Since R is right perfect, it follows from [1, Theorem 28.4] that N contains a maximal submodule N'. Take any

$x \in N \setminus N'$. Then $xR \not\subset N'$. Since M is uniserial, we must have $N' \subset xR$. Hence $N = xR$, proving that M is a Noetherian module. ∎

Lemma 7. *Let R be a right perfect ring. If $E(gR)$ is uniserial for any primitive idempotent g of R, then R is serial Artinian.*

Proof. By Theorem C, it is enough to show that R is right Artinian. Since R is right perfect, we can write

$$R = \bigoplus_{i=1}^{n} g_i R$$

where $\{g_i, i = 1, \dots n\}$ is a system of orthogonal primitive idempotents. For each i, $E(g_i R)$ is uniserial, therefore by Lemma 6, it is Noetherian and hence R is right Noetherian. Combining with the assumption that R is right perfect, it follows that R is right Artinian by [1, Corollary 15.23] and this completes our proof. ∎

Theorem 8. *Let R be a right perfect ring. Then the following conditions are equivalent:*

(*1*) *R is a serial Artinian ring;*

(*2*) *Every cyclic module is a direct sum of an injective module and a uniform small module;*

(*3*) *Every local module is either injective or uniform small.*

Proof. (1) ⇒ (2) Let R be a right perfect serial ring. Then by [16, Theorem 4.5], R is a right Harada ring. For a cyclic module M, we have $M = I \oplus S$, where I is injective and S is small (see Theorem A). Since R is serial Artinian, $S = \bigoplus_j S_j$ with S_j uniserial. It follows from [16, Lemma 1.1] that S_j is small, proving (2).

(2) ⇒ (3) by Proposition 2.

(3) ⇒ (1) Supppose that R has a decomposition as in Remark 5. Then by (3), $e_i R$ is injective for each $e_i \in \{e_i, i = 1, \dots, n\}$. Hence by [11, Theorem 1.3], the ring R is right QF-3. It follows that $E(f_k R)$ is projective for each $f_k \in \{f_j, j = 1, \dots, m\}$. Therefore $E(f_k R) = \bigoplus_{i \in I, j \in J} e_{ij} R$, $e_{ij} R \cong e_i R$, $I \subset \{1, \dots .n\}$. Since $f_k R$ is uniform by (3), it implies $E(f_k R) \cong e_i R$ for some idempotent e_i. Next, we will show that $e_i R$ is uniserial for all e_i, $i = 1. \dots, n$.

Let U, V be non-zero submodules of eR, where $e \in \{e_i, 1 \le i \le n\}$. Put $I = U \cap V$. Assume that $I \neq U$ and $I \neq V$. Then the module $B = eR/I$ is not uniform, and hence B is not injective because it is indecomposable. Thus the local module B is neither uniform, nor injective, this contradicts the assumption (3). Therefore, either $I = U$ or $I = V$, proving that eR is a uniserial module. In view of Lemma 7, it follows that R is a serial Artinian ring. ∎

Lemma 9. ([11, 19]) *The following statements hold for non-cosmall modules:*

(*1*) *A right R-module M is non-cosmall if it does not coincide with its singular submodule;*

(*2*) *If an R-module M contains a non-zero projective submodule, then it is non-cosmall.*

Lemma 10. *Let R be a right QF-3 semiperfect ring. If every uniform principal right ideal of R is either projective or singular as a right R-module, then R satisfies two conditions (a) and (b) of Theorem B.*

Proof. Since R is semiperfect, it has a decomposition of the form

$$R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR,$$

where $\{e_i \ 1 \leq i \leq n\}$ is the set of mutually orthogonal primitive idempotents. Since R is right QF-3 (i.e., $E(R_R)$ is projective), it follows that there exists at least one e_i such that e_iR is injective. Without loss of generality we may assume that e_iR is injective for $1 \leq i \leq k$ and e_jR is not injective with $k+1 \leq j \leq n$. Take $e = e_j, k+1 \leq j \leq n$. Since $E(R_R)$ is projective, so is $E(eR)$. Hence

$$E(eR) = \bigoplus_{i=1}^{k} \bigoplus_{t=1}^{t(i)} e_{it}R$$

where $e_{it}R \cong e_iR$ for $1 \leq t \leq t(i)$. Put

$$Q = \bigoplus_{i=1}^{k} \bigoplus_{t=1}^{t(i)} e_{it}R$$

and let $\alpha : E(eR) \to Q$ be an isomorphism and $\pi_{it} : \bigoplus_{i=1}^{k} \bigoplus_{t=1}^{t(i)} e_{it}R \longrightarrow e_{it}R$ the projections. Let $F = \alpha(eR)$ and $F_{it} = \pi_{it}(F) \subset e_{it}R \cong e_iR$. Then F_{it} is cyclic. It is easy to see that F_{it} is isomorphic to a principal right ideal of R. Moreover, F_{it} is uniform. Hence by hypothesis, F_{it} is either projective or singular for all pairs (i, t).

Suppose that F_{it} are singular for all pairs (i, t). It follows that F is singular, since $F \subset \bigoplus_i F_{it}$, which is a contradiction to the fact that $F \cong eR$. Therefore there exists a pair (i_0, t_0) such that $F_{i_0t_0}$ is non-zero projective. Let $p = \pi_{i_0t_0}|_F$, induced by the projection, and consider the exact sequence $F \xrightarrow{p} F_{i_0t_0} \to 0$ with $F_{i_0t_0}$ projective. Then $F = \ker p \oplus F'$, for some $F' \cong F_{i_0t_0}$. Since $F(\cong eR)$ is indecomposable and $F' \neq 0$, it implies that $\ker p = 0$, hence $F \cap \bigoplus_{i\neq i_0, t\neq t_0} e_{it}R = 0$, because $F \subset eQ$ (i.e., F is essential in Q). Therefore $Q = e_{i_0t_0}R$. Thus $eR \cong F \subset e_{i_0t_0}R = e_{i_0}R$. Hence R satisfies both conditions (a) and (b) of Theorem B and our Lemma has been proved. ■

Lemma 11. ([11, Theorem 3.6]) *Let R be a semiperfect ring and e a primitive idempotent of R such that eR is injective. Suppose that every submodule of eR is either projective or singular. Then there exists an integer n such that eJ^t is projective for $0 \leq t \leq n$ and eJ^{n+1} is singular, where J is the Jacobson radical of R.*

Lemma 12. *Let R be a right QF-3 right perfect ring and $e \in R$ a primitive idempotent of R such that eR is injective. If every 2-generated right submodule of*

eR is either projective or singular, then every submodule of eR is either projective or singular.

Proof. It is clear that eR is uniform. Let $Z(eR)$ be the singular submodule of eR. We first prove that $eR/Z(eR)$ is uniserial. Suppose on the contrary that there are submodules U and V of eR such that $Z(eR) \subset U \cap V$ and $U \not\subset V$, $V \not\subset U$. Take $u \in U \setminus V$ and $v \in V \setminus U$. Consider the module $X = uR + vR$. Since X is a 2-generated right submodule of eR, by assumption, it is projective and singular. But both uR and vR are not singular, X must be not singular. Hence X is projective. Let X_1 and X_2 be maximal submodules of uR and vR respectively. Then $X_1 + vR$ and $X_2 + uR$ are distinct maximal submodules of X. This contradicts the fact that X is a projective indecomposable module on a right perfect ring. It would imply that $eR/Z(eR)$ is a uniserial module. We now show that $eR/Z(eR)$ has finite length. Let X_1 be the largest submodule of eR. Since R is right perfect, it follows from [1, Theorem 28.4] that X_1 (here, $X_1 = \mathrm{rad}(eR)$) contains a maximal submodule X_2. Continueing this process, we get a strictly descending chain

$$eR \supset X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots \supset Z(eR).$$

We will prove that this chain is stationary. It is clear that each X_i is a local module, hence each X_i is an epimorphism image of fR for some primitive idempotent f of R. Since R is right QF-3, it follows that fR is uniform, and hence X_i is projective or singular. But $X_i \supset Z(eR)$, and $X_i \neq Z(eR)$, it follows that each X_i is projective. We claim that for $i \neq j$, $X_i \ncong X_j$.

Suppose on the contrary that there is an isomorphism $\varphi : X_i \to X_j$. Since eR is injective, φ can be extended to $\bar{\varphi} : eR \to eR$ and $\bar{\varphi}$ is also an isomorphism. From this we obtain $eR/X_i \cong eR/X_j$, and this contradicts the fact that length(eR/X_i) $\neq$length (eR/XC_j). Since the representative set of R is finite, it implies that the chain $X_1 \supset X_2 \supset \ldots$ must be stationary. Therefore the condition (c) of Theorem B is satisfied, proving our Lemma. ■

Theorem 13. *Let R be a semiperfect ring. Then the following conditions are equivalent:*

(*1*) $(**)$ *holds;*

(*2*) *R is right QF-3 and every right ideal is a direct sum of a projective module and a singular module;*

(*3*) *R is right QF-3 and every uniform right ideal is either projective or singular.*

Proof. (1) $\Rightarrow$ (2). In view of Theorem B, we see that $E(R_R)$ is projective, i.e., R is right QF-3. Moreover, R has finite right Goldie dimension. Let B be a right ideal of R. If B is non-cosmall, then by (**), we have $B = B_1 \oplus B_1'$ with B_1 is non-zero and projective.

Again, if B_1' is non-cosmall, then $B_1' = B_2 \oplus B_2'$, with B_2 being non-zero and projective. Since R_R has finite Goldie dimension, we get that B_i is finite dimensional and therefore after a finite number of steps, we get $B = B_1 \oplus \cdots \oplus B_k \oplus B_k'$, where $B_1, \ldots, B_k$ are projective and B_k' is cosmall, i.e., singular (Lemma 9). Hence (2) holds.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) by Lemmas 10 and 11. ∎

Theorem 14. *Let R be a right perfect ring. The following conditions are equivalent:*
(1) R satisfies $(**)$;
(2) R is right QF-3 and every 2-generated right ideal is a direct sum of a projective module and a singular module;
(3) R is right QF-3 and every uniform 2-generated right ideal is either projective or singular. ∎

Proof. The proof of (1) $\Rightarrow$ (2) is similar to that of Theorem 13. The implication (2) $\Rightarrow$ (3) is obvious. From Lemmas 10 and 12, it follows that R satisfies three conditions (a), (b) and (c) of Theorem B. Therefore, R satisfies (**), proving (3) $\Rightarrow$ (1). ∎

Theorem 15. *The following conditions are equivalent for a right QF-3 semiperfect ring R.*
(1) R is a serial Artinian ring;
(2) Every right ideal B of R has a decomposition of the form $B = B_0 \bigoplus_{1\leq i\leq n} B_i$, with B_0 projective and each $B_i (1 \leq i \leq n)$ being a singular uniserial module of finite length;
(3) Every right ideal of R is either a projective module or a singular uniserial module of finite length.

Proof. (1) $\Rightarrow$ (2). Let R be a serial Artinian ring. Then R is a right co-Harada ring by [16, Theorem 4.5]. Hence for any right ideal B of R, $B = B_0 \oplus B'$, where B_0 is projective and B' is singular by [16, Theorem 3.18]. Then $B' = \bigoplus_{i\in I} B_i$, where each B_i is uniserial with finite length, proving (2).

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Since R is right QF-3 and semiperfect, it follows from Theorem 13 that R satisfies the condition (**). By applying Theorem B, R has a decomposition

$$R = \bigoplus_{i=1}^{n} e_i R \oplus \bigoplus_{j=1}^{n} f_j R,$$

where e_iR is injective and f_jR is small. Moreover for each j we have $E(f_jR) \cong e_iR$ for some i. By Theorem C, in order to prove that R is serial, it suffices to show that R is right Artinian and each e_iR is uniserial, $i \in \{1, 2, \ldots, n\}$. Take any $e \in \{e_i, i = 1, \ldots n\}$ and consider the module eR with its singular submodule $Z = Z(e)$. By Theorem B and (3), Z is a uniserial module with finite length. Therefore, eR is an Artinian module. On the other hand, again by Theorem B, we have either $eB \subset Z$ or $Z \subset eB$. Since Z and eR/Z are both uniserial, it follows that eR is a uniserial module. Moreover, eR/Z is of finite length. It is now clear that R is right Artinian and $E(eR)$ is uniserial for any primitive idempotent e of R. The proof is now complete. ∎

Theorem 16. *Let R be a right QF-3 right perfect ring. The following conditions are equivalent:*

(*1*) *R is serial Artinian;*

(*2*) *Every right ideal of R is a direct sum of a projective module and a singular uniserial module;*

(*3*) *Every uniform 2-generated right ideal is either projective or singular uniserial.*

Proof. The proof of (1) ⇒ (3) is similar to that of Theorem 15 and (2) ⇒ (3) is obvious. We now prove that (3) ⇒ (1). Clearly, R satisfies (**) by Theorem 14. By Remark 5, we can write R in the form

$$R = \bigoplus_{i=1}^{n} e_i R \oplus \bigoplus_{j=1}^{m} f_j R$$

with properties in the Remark 5.

By Theorem B, we can see that each e_iR is injective and for each j, we have $E(f_jR) \cong e_iR$ for some $i = 1, \ldots, n$. Using Lemma 7, we now show that each e_iR is a uniserial module, $i = 1, \ldots, n$. Put $Z = Z(e_iR)$. Since R satisfies (**), in the same way as in Theorem 15, we can see that e_iR/Z is a uniserial module with finite length, and for every right ideal B of R, we have either $Z \subset e_iB$ or $e_iB \subset Z$. Therfore, it suffices to show that Z is also a uniserial module. We can suppose that $Z \neq 0$.

Let U, V be non-zero submodules of Z. If $U \not\subset V$ and $V \not\subset U$, we can take any $u \in U \setminus V$ and $v \in V \setminus U$, and consider the module $C = uR + vR$. Then C is a uniform 2-generated right ideal of R and C is singular. However C is not uniserial, since $uR \not\subset vR$ and $vR \not\subset uR$, and this contradicts the hypothesis (3). Hence, either $U \subset V$ or $V \subset U$, proving that $Z(eR)$ is uniserial. The proof is now complete. ■

References

1. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Math. No.13, Springer-Verlag, New York, Heidelberg, Berlin, 1974.
2. H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, *Trans. Amer. Math. Soc.* **95** (1960), 466-488.
3. R. Colby and E. A. Rutters, Jr, Semiperfect QF-3 and p.p rings, *Osaka J. Math.* **5** (1968), 99-102.
4. R. Colby and E. A. Rutters, Jr, Generalizations of QF-3 algebras, *Trans. Amer. Math. Soc.* **153** (1971), 37-386.
5. C.Faith, *Algebra II. Ring Theory*, Springer-Verlag, 1976.
6. C. Faith and E.A. Walker, Direct sum presentations of injective modules, *J.Algebra* **5** (1967), 203-221.
7. A. W. Fuller, On direct representations of quasi-injectives and quasi-projectives, *Arch. Math.* **20** (1969), 495-502 (Correction **21** (1970), 478).
8. A. W. Goldie, Torsion-free modules and rings, *J. Algebra* **1** (1964), 268-287.

9. M. Harada, QF-3 and semiprimary p.p rings I, *Osaka J.Math.* **2** (1965), 357-368.
10. M. Harada, A note on hollow modules, *Rev. Un. Matx. Argentina*, **28** (1978), 186-194.
11. M. Harada, Non-small modules and non-cosmall modules, *Proc. of the 1978 Antw. Conf.*, F. van Ostaen, Ed., Marcel Dekker, New York, pp. 669-689.
12. M. Harada, On one-sided QF-2 rings I, *Osaka J. Math.* **17** (1980), 421-431.
13. M. Harada, On one-sided QF-2 rings II, *Osaka J. Math.* **17** (1980), 433-438.
14. J. P. Jans, Projective injective modules, *Pacific J. Math.* **9**(1959), 1103-1108.
15. D. Jonah, Rings with minimum condition for principal right ideals have the maximum condition for principal left ideals, *Math. Z.* **113** (1965) 106-112.
16. K. Oshiro, Lifting modules, extending modules and their applications to QF-rings, *Hokkaido Math. J.* **13** (1984), 310-338.
17. D. V.Huynh and P.Dan, Some characterizations of right co-H-rings, *Math. J. Okayama Univ.* **34** (1992), 165-174.
18. P. Dan, Right perfect rings with the extending property on finitely generated free modules, *Osaka J. Math.* **26** (1989), 226-273.
19. M. Rayar, Small and Cosmall Modules, Ph. D. Dissertation, Indiana Univ., 1971.
20. F.L. Sandomierski, Non-singular rings, *Proc. Amer. Math. Soc.* **19** (1968), 225-230.
21. R.M. Thrall, Some generalizations of quasi-Frobenius Algebras, *Trans. Amer. Math. Soc.* **64** (1984), 173-183.
22. C. I. Vinsonhaler, Orders in QF-3 rings, *J. Algebra* **14** (1970), 83-90.