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Controllability Radius of Linear Systems with Perturbed Control Sets

Nguyen Khoa Son[1] **and Do Duc Thuan**[2]

[1] *Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam*

[2] *Department of Applied Mathematics and Informatics, Hanoi University of Technology, 1 Dai Co Viet Str., Hanoi, Vietnam*

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Abstract. In this paper we study controllability of linear systems with constrained controls under the assumption that the set of control parameters is subjected to perturbations. The notion of controllability radius is introduced and some formulas for its computation are derived. Examples are given to illustrate the obtained results.

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1. Introduction

One of the fundamental concepts in control theory is that of controllability. The linear control system $\dot{x} = Ax + Bu, x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, or equivalently the pair (A, B) is said to be *controllable* if for any initial state $x(0) = x_0$ and any desired final state x_1, there exist $T > 0$ and a measurable control function $u(t) \in \mathbb{R}^{n\times m}, 0 \le t \le T$ such that $x(T) = x_1$. It is well-known [7] that the pair (A, B) is controllable if and only if

$$\operatorname{rank}[A - \lambda I, B] = n, \quad \forall \lambda \in \mathbb{C}. \tag{1.1}$$

As pointed out by Lee and Markus [12] the set $\Gamma_{\mathbb{K}}$ of all uncontrollable pairs $(A, B) \in \mathcal{L}(\mathbb{K}^{n+m}, \mathbb{K}^n)$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ is closed. Therefore, for a controllable pair (A, B), one can define its *controllability radius* by introducing a number

$$\mu_{\mathbb{K}}(A, B) = \inf_{(C,D)\in\Gamma_{\mathbb{K}}} \|(A, B) - (C, D)\|, \tag{1.2}$$

which can be considered as robustness measure of controllability of the given pair. The problem of estimating and calculating (1.2) is of great interest in research and application of control theory and has attracted a good deal of attention over last decades (see, e.g [3 - 6]). One of the most remarkable results was due Eising [3] which has shown that

$$\mu_{\mathbb{C}}(A,B) := \min\{\|[\Delta A\ \Delta B]\| : (A+\Delta A, B+\Delta B) \in \Gamma_{\mathbb{C}}\} = \sigma_n[A-\lambda I\ B] \quad (1.3)$$

where $\|\cdot\|$ denotes the 2-norm or Frobenius norm and $\sigma_n[A-\lambda I\ B]$ denotes the n^{th} largest singular value of the $n\times(n+m)$ matrix $[A-\lambda I\ B]$. This is a global nonsmooth optimization problem in two real variables Reλ and Imλ, the real and imaginary parts of λ. The formula for calculating *real controllability radius* $\mu_{\mathbb{R}}(A,B)$ have been established by De Carlo-Wicks [2] and Hu-Davison [9]. Recently, the problem of calculation the distance from uncontrollability was considered in [13] for convex processes.

In this paper we shall consider the robustness of controllability of linear systems with constrained controls of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \Omega \subset \mathbb{R}^m, \tag{1.4}$$

under the assumption that the control set Ω is subject to perturbations.

In the next section we shall introduce some preliminary results and notation we need in the paper. In Sec. 3 we derive some formulas for estimating and calculating the controllability radius of the triple (A,B,Ω) when the control set Ω is perturbed. In Sec. 4 we provide some examples to illustrate the obtained results.

2. Preliminaries

We will use the following notations: throughout the paper, $\mathbb{R}^n$ ($\mathbb{C}^n$) denotes the real (resp., complex) Euclidean space with the usual inner product $\langle .,. \rangle$ and with the norm $\|x\| = \langle .,. \rangle^{1/2}$. If $f \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$ we shall write $\langle f,x\rangle = \langle \text{Re}f, x\rangle + \imath\langle \text{Im}f, x\rangle$. We say that vector $f\in\mathbb{R}^n$ (resp., $f\in\mathbb{C}^n$) is supporting (resp., orthogonal) to a set $V\subset\mathbb{R}^n$ if $\langle f,u\rangle \le 0,\ \forall u\in V$ (resp., $\langle f,u\rangle = 0,\ \forall u \in V$). The set of all supporting vectors (resp., orthogonal vectors) of V is denoted by V^o (resp. by $V^{\perp}$). They are cones with the vertices at the origin and are convex if V is a convex set. The set $V^+ = -V^o$ is called the dual cone of V. The subspace spanned by V is denoted by span$V$: span$V = V - V$. If $A\in\mathbb{R}^{n\times n}$, A^* denotes the transpose of A and $\sigma(A)$ the set of all eigenvalues of A. The nullspace of A is Ker$A = \{x\in\mathbb{R}^n : Ax = 0\}$. The norm of matrices $A\in\mathbb{R}^{n\times n}$ is an operator norm induced by Euclidean vector norms on $\mathbb{R}^n$ and $\mathbb{R}^m$: $\|A\| = \max\{\|Ax\| : \|x\| = 1\}$.

Let $C_1, C_2 \subset \mathbb{R}^n$ are closed convex cones with vertices at the origin. Then we define the gap of C_1 w.r.t C_2 as

$$\rho(C_1,C_2) = \sup\{d(x,C_2) :\ x\in C_1, \|x\| = 1\},$$

where $d(x,C)$ denotes the distance from x to C : $d(x,C)=\inf_{c\in C}\|x-c\|$. We put $\rho(0,C_2)=0$. The following properties are immediate from the definition: (i) $\rho(C_1,C_3)\le\rho(C_1,C_2)+\rho(C_2,C_3)$; (ii) $\rho(C_1,C_2)=0\Leftrightarrow C_1\subset C_2$; and (iii) $\rho(C_1,C_2)\le 1$.

We also need the following technical results. Their proofs are straightforward and are therefore omitted.

Lemma 2.1. *Let $C_1,C_2\subset\mathbb{R}^n$ be closed convex cones. Then the following statements are equivalent:*
(i) $\rho(C_1,C_2)=1$;
(ii) $\exists x\in C_1 : d(x,C_2)=1$;
(iii) $\exists x\in C_1, x\neq 0 :\ \langle x,u\rangle\le 0,\forall u\in C_2,$ *or equivalently,* $C_1\cap C_2^o\neq\{0\}$.

Lemma 2.2. *Let $C_1,C_2\subset\mathbb{R}^n$ be closed convex cones with the vertices at the origin. Then*

$$\rho(C_1,C_2)=\rho(C_2^+,C_1^+). \tag{2.1}$$

Lemma 2.3. *Let $x\in\mathbb{R}^n$ and $C\subset\mathbb{R}^n$ be a closed convex cone with the vertex at the origin. Then*

$$\begin{aligned}\min\{d(x,C),d(-x,C)\}&=\inf_{u\in C:\|u\|=1}\sqrt{\|x\|^2-\langle x,u\rangle^2}\\&=\sqrt{\|x\|^2-\Big(\sup_{u\in C:\|u\|=1}|\langle x,u\rangle|\Big)^2}.\end{aligned}$$

Denote

$$\xi_C(x)=\sqrt{\|x\|^2-\Big(\sup_{u\in C:\|u\|=1}|\langle x,u\rangle|\Big)^2}$$

and for $0\le l\le 1$ define

$$\psi_C(x,l)=\max\Big\{0,\xi_C(x)\sqrt{1-l^2}-l\sqrt{\|x\|^2-\xi_C(x)^2}\Big\}.$$

Lemma 2.4. *Let $C\subset\mathbb{R}^n$ be a closed convex cone with the vertex at the origin. Let $a\in\mathbb{R}^n,\|a\|=1$ and $d(a,C)\le l, 0\le l\le 1$. Then, for all $x\in\mathbb{R}^n$, we have*

$$\inf_{t\in\mathbb{R}}\|x-ta\|\ge\psi_C(x,l).$$

3. Formulas for Controllability Radius

Consider the linear control system with constrained controls of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \Omega \subset \mathbb{R}^m, \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $\Omega \subset \mathbb{R}^m$ is a closed convex cone with the vertex at the origin, having non-empty interior: $\text{int}\Omega \neq \emptyset$. In what follows we shall denote the system (3.1) by (A, B, Ω). The controls function $u(t), \quad 0 \le t \le T$ is said to be admissible on $[0, T]$ if $u(\cdot)$ is integrable and $u(t) \in \Omega$ a.e. on $[0, T]$. The system (3.1) is said to be controllable if for any $x \in \mathbb{R}^n$ there exist $T > 0$ and an admissible control $u(t)$ such that the corresponding solution $x(t) := x_u(t)$ of (3.1) satisfy $x(0) = 0, x(T) = x$. It is well-known (see, e.g. [1, 11]) that the system with constrained controls (A, B, Ω) is controllable if and only if the following two conditions are satisfied:

$$\begin{aligned} &\text{(i) span}\{B\Omega, AB\Omega, ..., A^{n-1}B\Omega\} = n \;\text{ and} \\ &\text{(ii) Ker}(A^* - \lambda I) \cap (B\Omega)^+ = \{0\}, \;\forall \lambda \in \mathbb{R}. \end{aligned} \tag{3.2}$$

It is easy to show that (3.2) is equivalent to

$$\begin{aligned} &\text{(i) span}\{B\Omega, AB\Omega, ..., A^{n-1}B\Omega\} = n \;\text{ and} \\ &\text{(ii)' } B^*\text{Ker}(A^* - \lambda I) \cap \Omega^+ = \{0\}, \;\forall \lambda \in \mathbb{R}. \end{aligned} \tag{3.3}$$

Now, assume that system (A, B, Ω) is controllable and the control set Ω is subjected to perturbations as follows

$$\Omega \longrightarrow \tilde{\Omega}, \quad \text{int}\tilde{\Omega} \neq \emptyset. \tag{3.4}$$

where $\tilde{\Omega}$ is a closed convex cone with the vertex at the origin. We note that the gap between Ω and $\tilde{\Omega}$ is measured by $\rho(\Omega, \tilde{\Omega})$ and $\rho(\tilde{\Omega}, \Omega)$. However, if $\rho(\Omega, \tilde{\Omega}) = 0$, which means equivalently $\Omega \subset \tilde{\Omega}$, then all triples $(A, B, \tilde{\Omega})$ are controllable. Thus, the gap $\rho(\Omega, \tilde{\Omega})$ is namely defining the robustness of controllability of the system (A, B, Ω) when the control set Ω is perturbed.

Definition 3.1. *Controllability radius of the system (A, B, Ω) with respect to the perturbation (3.4) is defined as*

$$r_{A,B}(\Omega) = \inf\{\rho(\Omega, \tilde{\Omega}) : \; \text{int}\tilde{\Omega} \neq \emptyset, (A, B, \tilde{\Omega}) \textit{ uncontrollable}\}. \tag{3.5}$$

For the sake of convenience, let us introduce the following notation: $\sigma_{\mathbb{R}}(A^*) = \sigma(A^*) \cap \mathbb{R}$ -the set of all real eigenvalues of A^*, $M_\lambda = \{v \in (\mathbb{R}^m)^* : \|v\| = 1, v \in B^*\text{Ker}(A^* - \lambda I)\}$ and $M_{\Omega^+} = \{v \in (\mathbb{R}^m)^* : \|v\| = 1, v \in \Omega^+\}$.

Theorem 3.2. *Assume $\sigma_{\mathbb{R}}(A^*) \neq \emptyset$. Then the controllability radius of the system (A, B, Ω) with respect to the perturbation (3.4) is given by*

$$\begin{aligned} r_{A,B}(\Omega) &= \inf\{d(v, \Omega^+) : v \in M_\lambda, \lambda \in \sigma_{\mathbb{R}}(A^*)\} \\ &= \min_{\lambda \in \sigma_{\mathbb{R}}(A^*)} \inf_{v \in M_\lambda, u \in M_{\Omega^+}} \sqrt{\|v\|^2 - \langle v, u \rangle^2} \\ &= \min_{\lambda \in \sigma_{\mathbb{R}}(A^*)} \inf_{v \in M_\lambda} \sqrt{\|v\|^2 - \big(\sup_{u \in M_{\Omega^+}} |\langle v, u \rangle|\big)^2}. \end{aligned}$$

If $\sigma_{\mathbb{R}}(A^*) = \emptyset$, the perturbation (3.4) of the control set does not affect controllability of the system (3.1) and we put $r_{A,B}(\Omega) = 1$.

Proof. Assume $\sigma_{\mathbb{R}}(A^*) \neq \emptyset$. Let $(A, B, \tilde{\Omega})$ be uncontrollable. Since $\mathrm{int}\tilde{\Omega} \neq \emptyset$, the condition (i) of (3.3) holds and therefore the condition (ii) is violated, which means

$$\exists \lambda_0 \in \mathbb{R} \text{ s.t. } \; B^*\mathrm{Ker}(A^* - \lambda_0 I) \cap \tilde{\Omega}^+ \neq \{0\}$$

It follows $\lambda_0 \in \sigma_{\mathbb{R}}(A^*)$ and there exists $v_0 \in B^*\mathrm{Ker}(A^* - \lambda_0 I) \cap \tilde{\Omega}^+$ such that $\|v_0\| = 1$. Thus, taking into account (2.1), we can write, for all convex cones $\tilde{\Omega} \subset \mathbb{R}^m$ (with $\mathrm{int}\tilde{\Omega} \neq \emptyset$) such that the triple $(A, B, \tilde{\Omega})$ is uncontrollable,

$$\begin{aligned} \rho(\Omega, \tilde{\Omega}) &= \rho(\tilde{\Omega}^+, \Omega^+) \\ &= \sup\big\{d(v, \Omega^+) : v \in \tilde{\Omega}^+, \|v\| = 1\big\} \\ &\geq d(v_0, \Omega^+) \\ &\geq \inf\big\{d(v, \Omega^+) : v \in B^*\mathrm{Ker}(A^* - \lambda I), \|v\| = 1, \lambda \in \sigma_{\mathbb{R}}(A^*)\big\}. \end{aligned}$$

Consequently, by the definition of $r_{A,B}(\Omega)$, we get

$$r_{A,B}(\Omega) \geq \inf\big\{d(v, \Omega^+) : v \in B^*\mathrm{Ker}(A^* - \lambda I), \|v\| = 1, \lambda \in \sigma_{\mathbb{R}}(A^*)\big\}. \quad (3.6)$$

To prove the converse inequality of (3.6), noticing that that the set $M_\lambda = \{v \in B^*\mathrm{Ker}(A^* - \lambda I) : \|v\| = 1\}$ is compact, $\sigma_{\mathbb{R}}(A^*)$ is finite and $d(v, \Omega^+)$ is a continuous function of v, we obtain that $\exists \lambda_0 \in \sigma_{\mathbb{R}}(A^*)$, $\exists v_0 \in M_{\lambda_0}$ such that

$$d(v_0, \Omega^+) = \min\{d(v, \Omega^+) : v \in M_\lambda, \lambda \in \sigma_{\mathbb{R}}(A^*)\}.$$

Consider the convex cone $K := \{tv_0 : t \in \mathbb{R}, t \geq 0\}$. Then we have $\rho(K, \Omega^+) = d(v_0, \Omega^+)$ and

$$(K^+)^+ \cap B^+(\mathrm{Ker}(A^* - \lambda_0 I)) = K \neq \{0\}.$$

Moreover, clearly $\mathrm{int}K^+ \neq \emptyset$. Therefore, the triple (A, B, K^+) is uncontrollable. This implies, by definition,

$$\begin{aligned} r_{A,B}(\Omega) &= \inf\big\{\rho(\widetilde{\Omega}^+, \Omega^+) : \exists \lambda \in \mathbb{R} \text{ s.t.} B^*\mathrm{Ker}(A^* - \lambda I) \cap \widetilde{\Omega}^+ \neq \{0\}\big\} \\ &\leq \rho((K^+)^+, \Omega^+) = \rho(K, \Omega^+) = d(v_0, \Omega^+) \\ &= \inf\big\{d(v, \Omega^+) : v \in B^*\mathrm{Ker}(A^* - \lambda I)), \|v\| = 1, \lambda \in \sigma_{\mathbb{R}}(A^*)\big\}. \end{aligned}$$

Therefore, by definition and using Lemma 2.3, we have

$$\begin{aligned} r_{A,B}(\Omega) &= \inf\big\{d(v,\Omega^+) : v \in B^*(\mathrm{Ker}(A^* - \lambda I), \|v\| = 1, \lambda \in \sigma(A^*) \cap \mathbb{R}\big\} \\ &= \inf\big\{d(v,\Omega^+) : v \in M_\lambda, \lambda \in \sigma_{\mathbb{R}}(A^*)\big\} \\ &= \min_{\lambda\in\sigma_{\mathbb{R}}(A^*)} \ \inf_{v\in M_\lambda} \min\big\{d(v,\Omega^+), d(-v,\Omega^+)\big\} \\ &= \min_{\lambda\in\sigma_{\mathbb{R}}(A^*)} \ \inf_{v\in M_\lambda, u\in M_{\Omega+}} \sqrt{\|v\|^2 - \langle v,u\rangle^2} \\ &= \min_{\lambda\in\sigma_{\mathbb{R}}(A^*)} \ \inf_{v\in M_\lambda} \sqrt{\|v\|^2 - \big(\sup_{u\in M_{\Omega+}} |\langle v,u\rangle|\big)^2}. \end{aligned}$$

The proof is complete. ■

For a given number l, $0 \le l \le 1$, we define the set of convex cones in $\mathbb{R}^{n\times m}$ with vertices at the origin which are obtained from the original control set Ω by *perturbations with tolerance level* less than l, by setting

$$\mathcal{G}_l = \mathcal{G}_l(\Omega) = \{\tilde\Omega : \tilde\Omega \ \text{ is convex cone in } \mathbb{R}^m, \mathrm{int}\tilde\Omega \neq \emptyset,\ \rho(\Omega,\tilde\Omega) \le l\}.$$

We consider now the problem of calculating the controllability radius of linear system (A, B, Ω) under the assumption that the system matrices A, B and the control set Ω are perturbed as follows

$$\begin{aligned} &A \longrightarrow A + \Delta_1, \quad \Delta_1 \in \mathbb{R}^{n\times n}, \\ &B \longrightarrow B + \Delta_2, \quad \Delta_2 \in \mathbb{R}^{n\times m}, \\ &\Omega \longrightarrow \tilde\Omega, \tilde\Omega \in \mathcal{G}_l(\Omega). \end{aligned} \tag{3.7}$$

Definition 3.3. *Choose a tolerance level $l \in [0,1]$. The controllability radius of the system (A, B, Ω) described by (3.1) with respect to perturbations of the form (3.7) is defined by*

$$r_l(A,B,\Omega) = \inf\big\{\|[\Delta_1,\Delta_2]\| : \exists\tilde\Omega \in \mathcal{G}_l \ \textit{s.t.}\ \ (A+\Delta_1, B+\Delta_2, \tilde\Omega) \ \textit{uncontrollable}\big\}.$$

Define

$$H^*_\lambda = \begin{bmatrix} A^* - \lambda I \\ B^* \end{bmatrix} (\mathbb{R}^n)^* \to (\mathbb{R}^{n+m})^*, \quad \hat\Omega^+ = \left\{ \begin{bmatrix} 0 \\ v \end{bmatrix} : v \in \Omega^+ \right\} \subset (\mathbb{R}^{n+m})^*.$$

The following result can be easily verified by using the controllability criteria (1.1) and (3.2).

Proposition 3.4. *The system (A, B, Ω) is controllable if and only if*
(i) $\mathrm{span}\{B\Omega, AB\Omega, \ldots, A^{n-1}B\Omega\} = \mathbb{R}^n$;
(ii) $H^*_\lambda(N) \cap \hat\Omega^+ = \emptyset, \qquad \forall\lambda \in \mathbb{R}$,
where $N = \{f \in (\mathbb{R}^n)^* : \|f\| = 1\}$ -the unit sphere in $(\mathbb{R}^n)^*$.

Let $\Omega \subset \mathbb{R}^m$ be a closed convex cone with the vertex at the origin and $\text{int}\Omega \neq \emptyset$. Then, it is clear that the condition (i) in Proposition 3.4 is equivalent to controllability of the pair (A, B) or, equivalently, the Kalman rank condition holds: $\text{rank}[B, AB, \dots, A^{n-1}B] = n$. Let us denote by $r"(A, B, l)$ the radius of property (ii) in Proposition 3.4 under the perturbations of the form (3.7):

$$r"(A, B, l) = \inf\{\|[\Delta_1, \Delta_2]\| : \exists\Omega \in \mathcal{G}_l, \exists\lambda \in \mathbb{R} \text{ s.t. } H^*_{\lambda,\Delta}(N) \cap \hat{\Omega}^+ \neq \emptyset\}$$

where $H_{\lambda,\Delta} = \begin{bmatrix} A^* + \Delta_1^* - \lambda I \\ B^* + \Delta_2^* \end{bmatrix}$. Then, it is obvious that for a given tolerance level l, the controllability radius of the system (A, B, Ω) under the perturbations of the form (3.7) can be calculated by the following formula:

$$r_l(A, B, \Omega) = \min\{\mu_{\mathbb{R}}(A, B); r"(A, B, l)\}. \tag{3.8}$$

Since the formula for computing $\mu_{\mathbb{R}}(A, B)$ have been established in the previous works (see, e.g. [2, 9]), the problem is reduced to deriving the calculation of $r"(A, B, l)$. To this end, let us define, as in Sec. 2,

$$\xi_{\Omega^+}(v) = \sqrt{\|v\|^2 - \big(\sup_{u \in M_{\Omega^+}} |\langle v, u\rangle|\big)^2}$$

(where $M_{\Omega^+} = \{v \in \Omega^+ : \|v\| = 1\}$) and for $0 \leq l \leq 1$ define

$$\psi_{\Omega^+}(v, l) = \max\Big\{0;\ \ \xi_{\Omega^+}(v)\sqrt{1 - l^2} - l\sqrt{\|v\|^2 - \xi_{\Omega^+}(v)^2}\Big\}.$$

Theorem 3.5. *Choose a tolerance level $l \in [0, 1]$. The controllability radius of the system (A, B, Ω) with respect to the perturbations of the form (3.7) is given by (3.8) where*

$$r"(A, B, l) = \inf_{f \in N} \sqrt{\psi_{\Omega^+}(B^* f, l)^2 + \|A^* f\|^2 - \langle A^* f, f\rangle^2},$$

(N denoting the unit sphere in $(\mathbb{R}^n)^*$).

Proof. Let the perturbation $(\Delta_1, \Delta_2, \tilde{\Omega})$ with $d(\Omega, \tilde{\Omega}) \leq l$ violates the property (ii) in Proposition 3.4, that is, for some $\lambda \in \mathbb{R}, f \in N, v \in \tilde{\Omega}^+$ we have

$$H_{\lambda,\Delta} f = \begin{bmatrix} A^* + \Delta_1^* - \lambda I \\ B^* + \Delta_2^* \end{bmatrix} f = \begin{bmatrix} 0 \\ v \end{bmatrix}.$$

It implies

$$\begin{aligned} \|[\Delta_1, \Delta_2]\| &= \left\| \begin{bmatrix} \Delta_1^* \\ \Delta_2^* \end{bmatrix} \right\| \geq \left\| \begin{bmatrix} \Delta_1^* \\ \Delta_2^* \end{bmatrix} f \right\| \\ &= \left\| \begin{bmatrix} (A^* - \lambda I) f \\ B^* f - v \end{bmatrix} \right\| \\ &= \sqrt{\|(A^* - \lambda I) f\|^2 + \|B^* f - v\|^2}. \end{aligned}$$

We have

$$\begin{aligned}\|(A^* - \lambda I)f\| &= \sqrt{\|A^*f\|^2 - 2\lambda\langle A^*f, f\rangle + \lambda^2} \\ &= \sqrt{\|A^*f\|^2 - \langle A^*f, f\rangle^2 + (\lambda - \langle A^*f, f\rangle)^2} \\ &\geq \sqrt{\|A^*f\|^2 - \langle A^*f, f\rangle^2}.\end{aligned}$$

If $v = 0$ then $\|B^*f - v\| = \|B^*f\| \geq \xi_{\Omega^+}(B^*f) \geq \psi_{\Omega^+}(B^*f, l)$. Assume $v \neq 0$. Since $v \in \tilde{\Omega}^+$ and by Lemma 2.2, $d(\tilde{\Omega}^+, \Omega^+) = d(\Omega, \tilde{\Omega}) \leq l$, we have $d(\frac{v}{\|v\|}, \Omega^+) \leq l$. It implies, by Lemma 2.4, $\|B^*f - v\| \geq \psi_{\Omega^+}(B^*f, l)$ and thus

$$\|[\Delta_1, \Delta_2]\| = \left\|\begin{bmatrix}\Delta_1^* \\ \Delta_2^*\end{bmatrix}\right\| \geq \inf_{f\in N} \sqrt{\psi_{\Omega^+}(B^*f, l)^2 + \|A^*f\|^2 - \langle A^*f, f\rangle^2}.$$

Noticing that the above inequality has been established for all perturbations (Δ_1, Δ_2) for which there exists $\tilde{\Omega}$, $d(\Omega, \tilde{\Omega}) \leq l$ and the property (ii) in Proposition 3.4 is violated, we get, by definition,

$$r''(A, B, l) \geq \inf_{f\in N} \sqrt{\psi_{\Omega^+}(B^*f, l)^2 + \|A^*f\|^2 - \langle A^*f, f\rangle^2}.$$

To prove the converse inequality, we define a function on $(R^n)^*$ be setting for each $f \in (\mathbb{R}^n)^*$,

$$g(f) = \inf_{f\in N} \sqrt{\psi_{\Omega^+}(B^*f, l)^2 + \|A^*f\|^2 - \langle A^*f, f\rangle^2}.$$

Since $g(\cdot)$ is continuous, there exists $f_0 \in N$ such that $g(f_0) = \inf_{f\in N} g(f)$. Suggest first $\psi_{\Omega^+}(B^*f_0, l) = 0$. If $B^*f_0 = 0$ the we set

$$\tilde{\Omega} = \Omega, \lambda_0 = \langle A^*f_0, f_0\rangle, \Delta_2^* \equiv 0,$$

$$\Delta_1^*f = \langle f, f_0\rangle \left(\langle A^*f_0, f_0\rangle - A^*f_0\right), \quad \forall f \in (\mathbb{R}^n)^*.$$

Then, for the perturbations $\Delta_1 = (\Delta_1^*)^*, \Delta_2 = (\Delta_2^*)^*$ we have obviously $H^*_{\lambda_0,\Delta}f_0 = \begin{bmatrix}0 \\ 0\end{bmatrix} \in \hat{\Omega}^+$ and

$$\|[\Delta_1, \Delta_2]\| = \left\|\begin{bmatrix}\Delta_1^* \\ \Delta_2^*\end{bmatrix}\right\| = \|\Delta_1^*\| = g(f_0).$$

If $B^*f_0 \neq 0$ then $\dfrac{\xi_{\Omega^+}(B^*f_0)}{\|B^*f_0\|} \leq l$. By Lemma 2.3, we have

$$\min\left\{d(B^*f_0, \Omega^+), d(-B^*f_0, \Omega^+)\right\} = \xi_{\Omega^+}(B^*f_0) \leq l\|B^*f_0\|.$$

Since $g(f_0) = g(-f_0)$ it implies $d\left(\dfrac{B^*f_0}{\|B^*f_0\|}, \Omega^+\right) \leq l$. We set

$$\tilde{\Omega}^+ = \left\{t\frac{B^*x_0}{\|B^*x_0\|} : t \geq 0\right\}, \lambda_0 = \langle A^*x_0, x_0\rangle, \qquad \Delta_2^* \equiv 0,$$

$$\Delta_1^* f = \langle f, f_0 \rangle \left(\langle A^* f_0, f_0 \rangle - A^* f_0 \right), \quad \forall f \in (\mathbb{R}^n)^*.$$

Then, it is easily to verify that for the triple $(\Delta_1, \Delta_2, \tilde{\Omega})$ the property (ii) is violated and

$$d(\Omega, \tilde{\Omega}) \leq l \quad \|[\Delta_1, \Delta_2]\| = g(f_0).$$

Finally, consider the case $\psi_{\Omega^+}(B^* f_0, l) > 0$. Then $\dfrac{\xi_{\Omega^+}(B^* f_0)}{\|B^* f_0\|} > l$ and we have, by Lemma 2.3,

$$\begin{aligned} \xi_{\Omega^+}(B^* f_0) &= \sqrt{\|B^* f_0\|^2 - \left(\sup_{u \in M_{\Omega^+}} |\langle B^* f_0, v \rangle| \right)^2} \\ &= \min \left\{ d(B^* f_0, \Omega^+), d(-B^* f_0, \Omega^+) \right\}. \end{aligned}$$

Let $e \in M_{\Omega^+}$ such that

$$\left| \langle B^* x_0, e \rangle \right| = \sup_{v \in M_{\Omega^+}} \left| \langle B^* f_0, v \rangle \right|$$

We assume $\langle B^* f_0, e \rangle = \left| \langle B^* f_0, e \rangle \right| \geq 0$ (otherwise one can take $B^* f_0 = -B^* f_0$) and we define a vector $w \in (\mathbb{R}^m)^*$ by setting

$$w = \frac{l}{\xi_{\Omega^+}(B^* f_0)} B^* f_0 + \frac{\psi_{\Omega^+}(B^* f_0, l)}{\xi_{\Omega^+}(B^* f_0)} e.$$

Then, by a direct calculation, it can be verified that

$$\|w\| = 1, \qquad \|B^* f_0 - \langle B^* f_0, w \rangle w\| = \psi_{\Omega^+}(B^* f_0, l).$$

We define

$$\tilde{\Omega}^+ = \{tw : t \geq 0\}, \; \lambda_0 = \langle A^* x_0, x_0 \rangle,$$

$$\Delta_1^* f = \langle f, f_0 \rangle (\langle A^* f_0, f_0 \rangle - A^* f_0), \;\; \Delta_2^* f = \langle f, f_0 \rangle \langle B^* f_0, w \rangle w, \quad \forall f \in (\mathbb{R}^n)^*$$

and we put

$$\tilde{\Omega} := \tilde{\Omega^+}, \qquad \Delta_1 := (\Delta_1^*)^*, \qquad \Delta_2 := (\Delta_2^*)^*.$$

Then, it is obvious that the triple $(\Delta_1, \Delta_2, \tilde{\Omega})$ is violating the property (ii) and

$$d(\Omega, \tilde{\Omega}) = l, \quad \|[\Delta_1, \Delta_2]\| = g(f_0).$$

Consequently, we have

$$r''(A, B, l) = g(x_0) = \inf_{f \in N} \sqrt{\psi_{\Omega^+}(B^* f, l)^2 + \|A^* f\|^2 - \langle A^* f, f \rangle^2}.$$

The proof is complete. ■

Corollary 3.6. *Assume that the control set is not perturbed, that is $l = 0$. Then the real controllability radius of the system (A, B, Ω) where A, B are subject to perturbations*

$$\begin{aligned} A &\longrightarrow A + \Delta_1, \quad \Delta_1 \in \mathbb{R}^{n\times n}, \\ B &\longrightarrow B + \Delta_2, \quad \Delta_2 \in \mathbb{R}^{n\times m}, \end{aligned} \tag{3.9}$$

is given by

$$r(A,B) = \min\left\{\mu_{\mathbb{R}}(A,B); \inf_{f\in N} \sqrt{\xi_{\Omega^+}(B^*f)^2 + \|A^*f\|^2 - \langle A^*f, f\rangle^2}\right\},$$

with $\xi_{\Omega^+}(B^*f) = \sqrt{\|B^*f\|^2 - \left(\sup_{v\in M_{\Omega^+}} |\langle B^*f, u\rangle|\right)^2}$, *where* $M_{\Omega^+} = \{v \in (\mathbb{R}^m)^* : v \in \Omega^+, \|v\| = 1\}$.

4. Examples

In this section, for illustration, we shall apply the obtained results in Sec. 3 to a particular case when the system matrix A is symmetric and the control set Ω is a polyhedral cone in $\mathbb{R}^m$.

It is well-known that if $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix ($A = A^*$), then for the system without constraints on controls $\dot{x} = Ax + Bu$, the real controllability radius is equal to the complex one and therefore, by Eising's result [3],

$$\mu(A,B) = \mu_{\mathbb{R}}(A,B) = \mu_{\mathbb{C}}(A,B) = \inf_{\lambda\in\mathbb{C}} \sigma_{\min}[A - \lambda I, B].$$

Let the control set Ω be a polyheral cone defined by

$$\Omega = \left\{x \in \mathbb{R}^m : \langle x, e_i\rangle \geq 0, \forall i = 1, \ldots, r\right\}, \tag{4.1}$$

where $e_1, e_2, \ldots, e_r \in \mathbb{R}^m$, $\|e_i\| = 1, \forall i = 1, \ldots, r$. Then, it is clear that

$$\Omega^+ = \left\{\sum_{i=1}^{r} t_i e_i : \ t_i \geq 0, \forall i = 1, \ldots, r\right\},$$

and therefore

$$\xi_{\Omega^+}(z) = \begin{cases} \sqrt{\|z\|^2 - \max\limits_{1\leq i\leq r} \langle z, e_i\rangle^2} & \text{if } z \notin -\Omega^+ \cup \Omega^+, \\ 0 & \text{if} z \in -\Omega^+ \cup \Omega^+. \end{cases}$$

By Theorem 3.2 we get

Corollary 4.1. *Let the control set Ω be a polyhedral cone defined by (4.1). If $\sigma_{\mathbb{R}}(A^*) \neq \emptyset$ then the controllability radius of the system (A, B, Ω) with respect to the perturbation (3.4) is given by*

$$r_{A,B}(\Omega) = \min_{\lambda\in\sigma_{\mathbb{R}}(A^*)} \inf_{v\in M_\lambda} \sqrt{\|v\|^2 - \max_{1\leq i\leq r} \langle v, e_i\rangle^2}.$$

If $\sigma_{\mathbb{R}}(A^*) = \emptyset$ then the perturbation (3.4) does not affect the controllability of (A, B, Ω) and we write $r_{A,B}(\Omega) = 1$.

By Theorem 3.5 we get

Corollary 4.2. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $\Omega \subset \mathbb{R}^m$ a polyhedral cone defined by (4.1). Then, for a chosen tolerance level $l \in [0,1]$, the controllability radius of the system (A, B, Ω) with respect to the perturbations of the form (3.7) is given by*

$$r_l(A, B, \Omega) = \min\{\mu_{\mathbb{R}}(A, B); r''(A, B, l)\},$$

with

$$\mu_{\mathbb{R}}(A, B) = \inf_{\lambda\in\mathbb{C}} \sigma_{\min}[A - \lambda I, B],$$

and

$$r''(A, B, l) = \inf_{x\in N} \sqrt{\psi_{\Omega^+}(B^* f, l)^2 + \|A^* f\|^2 - \langle A^* f, f\rangle^2},$$

where $\psi_{\Omega^+}(B^* f, l) = 0$ *if* $B^* f \in -\Omega^+ \cup \Omega^+$ *or* $\max_{1\le i\le r} |\langle B^* f, e_i\rangle| \ge \sqrt{1 - l^2}\|B^* f\|$, *and*

$$\psi_{\Omega^+}(B^* f, l) = \sqrt{(1 - l^2)\left(\|B^* f\|^2 - \max_{1\le i\le r} \langle B^* f, e_i\rangle^2\right)} - l \max_{1\le i\le r} |\langle B^* f, e_i\rangle|$$

if otherwise.

Consider the system (A, B, Ω) with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix},$$

and Ω is a positive cone in $\mathbb{R}^2$:

$$\Omega = \mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : \langle x, e_1\rangle \ge 0, \langle x, e_2\rangle \ge 0\}$$

with

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then the system satisfies the rank condition of Kalman:

$$\operatorname{rank}[B, AB] = \operatorname{rank} \begin{bmatrix} 2 & -3 & 0 & 2 \\ 0 & 2 & 2 & -3 \end{bmatrix} = 2. \tag{4.2}$$

Moreover, $\sigma(A^*) = \sigma(A) = \{1; -1\}$ and $\operatorname{Ker}(A^* \pm I) = \{\alpha[\pm 1 \;\; 1]^\top : \alpha \in \mathbb{R}\}$. It follows that $B^*\operatorname{Ker}(A^* \pm I) \cap \Omega^+ = \{0\}$ so that the system (A, B, Ω) is controllable, due to [11], [1]. Using Corollary 4.1, we can calculate the controllability radius of the system (A, B, Ω) with respect to perturbation (3.4) of the control set Ω and we get

$$r_{A,B}(\Omega) = 2/\sqrt{29} \approx 0.37.$$

Furthermore, since A is symmetric, we have

$$\begin{aligned}\mu_{\mathbb{R}}(A,B) &= \mu_{\mathbb{C}}(A,B)\\ &= \inf_{\lambda\in\mathbb{C}} \sigma_{\min}[A-\lambda I, B]\\ &= \inf_{\lambda\in\mathbb{C}} \sqrt{\sigma_{\min}\begin{bmatrix} |\lambda|^2+14 & -2Re\lambda-6\\ -2Re\lambda-6 & |\lambda|^2+5\end{bmatrix}} \qquad (4.3)\\ &= \inf_{\lambda\in\mathbb{C}} \sqrt{(\lambda)^2+(Re\lambda)^2+19/2-\sqrt{4(Re\lambda+3)^2+81/4}}\\ &= \inf_{t\in\mathbb{R}} \sqrt{t^2+19/2-\sqrt{4t^2+24t+225/4}}\\ &\approx 1.14\end{aligned}$$

Taking the tolerance level $l = 0.25 < r_{A,B}(\Omega) = 0.37$ and using Corollary 4.2, we get, by a simple calculation, $r''(A, B, 0.25) \approx 0.18$. Therefore, the controllability radius of the system (A, B, Ω) with respect to perturbations of the form (3.7), with the tolerance level $l = 0.25$, is given by

$$r_{0.25}(A,B,\Omega) = \min\{\mu_{\mathbb{R}}(A,B); r''(A,B,0.25)\} \approx \min\{1.14; 0.18\} = 0.18.$$

References

1. R. Brammer, Controllability of linear autonomous systems with positive controllers, *SIAM J. Control Optim.* **10** (1972), 339–353.
2. R. A. DeCarlo and M. Wicks, Computing the distance to an uncontrollable system, *IEEE Trans. Automat. Control*, **36** (1991), 39–49.
3. R. Eising, Between controllable and uncontrollable, *Systems Control Lett.* **5** (1984), 263–264.
4. P. Gahinet and A. J. Laub, Algebraic Riccati equations and the distance to the nearest uncontrollable pair, *SIAM J. Control Optim.* **4** (1992), 765–786.
5. M. Gu, New methods for estimating the distance to uncontrollability, *SIAM J. Matrix Anal. Appl.* **21** (2000), 989–1003.
6. M. Gu, E. Mengi et al., Fats methods for estimating the distance to uncontrollability, *SIAM J. Matrix Anal. Appl.* **28** (2006), 447–502.
7. M. L. J. Hautus, Controllability and observability conditions of linear autonomous systems, *Nederl. Acad. Wetensch. Proc. Ser. A72*, **31** (1969), 443-448.
8. D. Hinrichsen and A. J. Pritchard, Stability radii of linear systems, *Systems and Control Letters*, **26** (1986), 1–10.
9. G. Hu and E. J. Davison, Real controllability/stabilizability radius of LTI systems, *IEEE Trans. Automat. Control*, **49** (2004), 254–257.

10. T. Kato, *Perturbation Theory for Linear Operator*, Springer - Verlag, Berlin-Heidelberg- New York, 1976.
11. V. I. Korobov et al., Controllability of linear autonomous systems with restraints on control, *Differenfsiafnye Urauneniya*, **11** (1975), 1967–1979.
12. E. B. Lee and L. Markus, *Foundations of Optimal Control Theory*, John Willey, 1967.
13. A. Lewis, R. Henrion, and A. Seerger, Distance to uncontrollability for convex processes, *SIAM Control Optim.* **45** (2006), 26–50.