Vietnam Journal of Mathematics **36**:3(2008) 353–372

Vietnam Journal of MATHEMATICS

Local Cohomology Modules with Support in 2-regular Monomial Ideals

Dao Thanh Ha[1] **and Marcel Morales**[2]

[1] *Vinh University, Nghe An, Viet Nam*
[2] *Université de Grenoble I, Institut Fourier, UMR 5582, B.P.74, 38402 Saint-Martin D'Hères Cedex, and IUFM de Lyon, 5 rue Anselme, 69317 Lyon Cedex, France*

Received October 9, 2007
Revised May 18, 2008

Abstract. In this paper $S := K[V]$ will be a polynomial ring on a finite set of variables V, and $\mathcal{Q} \subset S$ be a 2-regular monomial ideal. We will compute the local cohomology module $H^i_{\mathcal{Q}}(S)$ and describe the set of its associated primes.

2000 Mathematics Subject Classification: 13D45, 14B15, 32S22.
Keywords: Local cohomology, square free monomial ideals, arrangements of linear spaces, Castelnuovo- Mumford regularity.

1. Introduction

In the whole paper we consider the polynomial ring $S := K[V]$ on a finite set of variables V. Let $\mathcal{Q} \subset S$ be a monomial ideal, that is an ideal generated by monomials. Our main result is to compute the local cohomology modules

$$H^i_{\mathcal{Q}}(S), \quad \text{where} \quad \text{reg}\,(\mathcal{Q}) = 2.$$

Here reg stands for the Castelnuovo Mumford regularity. For short we will say that $\mathcal{Q}$ is a 2−regular monomial ideal.

We recall that any non trivial ideal $\mathcal{Q} \subset S$ has a finite free resolution:

$$0 \to F_s \overset{M_s}{\to} F_{s-1} \to .... \to F_1 \overset{M_1}{\to} \mathcal{Q} \to 0.$$

The number s is called the projective dimension of $S/\mathcal{Q}$ and the Betti numbers are defined by $\beta_i(\mathcal{Q}) = \beta_{i+1}(S/\mathcal{Q}) = \text{rank} F_{i+1}$. We will say that the ideal $\mathcal{Q}$

is 2−regular if $\mathcal{Q}$ is generated by elements in degree 2, and the minimal free resolution of $\mathcal{Q}$ is linear, i.e. for $i \geq 2$ the matrices M_i in the minimal free resolution of $\mathcal{Q}$ have linear entries.

The study of 2−regular ideals were considered for the first time by Bertini, Castelnuovo and Del Pezzo. They have shown that for all projective algebraic (irreducible) varieties $X \subset \mathbb{P}^n$

$$\deg(X) \geq \operatorname{codim}(X) + 1,$$

and they have classified all projective algebraic varieties that satisfy the equality. Actually these varieties are known as varieties of minimal degree. They are some kind of ruled varieties called also scrolls.

Later Joe Harris (see for example [13]), reconsidered the proof of Bertini in terms of modern algebraic geometry. Xambo [20] classified all projective algebraic sets $X \subset \mathbb{P}^n$ (reduced non irreducible) that satisfy the equality

$$\deg(X) = \operatorname{codim}(X) + 1$$

under the additional hypothesis that X is connected in codimension 1. In fact the ideals $I \subset S$ defining algebraic sets of minimal degree are exactly the 2−regular ideals such that S/I is Cohen-Macaulay. 2−regular ideals were studied in a series of papers, see for example [9, 4, 5, 10, 12, 16].

On the other hand, there are many papers on the local cohomology modules with support on monomial ideals, see for example [11, 17, 18, 1, 3, 19]. In this paper we want to give very effective proofs and more precise results in the case of monomial 2−regular ideals. In fact for monomial 2−regular ideals important information is collected by inspection of the minimal prime decomposition.

Let $\mathcal{Q} := \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_l \subset K[V]$ be a 2-regular reduced monomial ideal, that is $\mathcal{Q}_i$ is generated by a set of variables $Q_i \subset V$ in the polynomial ring $S := K[V]$. In [16] it was proved that we have a decomposition $Q_i = D_i \cup P_i$ satisfying some properties (see Theorem 3). Our main result is the following.

Theorem 1. *Let $\mathcal{Q}^{(i)} := \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_i$. Then for any $i = 2, \ldots, l$ and a natural number j we have the following short exact sequence:*

$$0 \to H^j_{\mathcal{Q}^{(i-1)}}(S) \oplus H^j_{\mathcal{Q}_i}(S) \to H^j_{\mathcal{Q}^{(i)}}(S) \to H^{j+1}_{\mathcal{Q}^{(i-1)}+\mathcal{Q}_i}(S) \to 0.$$

Moreover $H^j_{\mathcal{Q}}(S) \neq 0$ if and only if either

1. $j =$ card (Q_k) *for some $k = 1, ..., l$, and in this case* $\dim_K(H^j_{\mathcal{Q}}(S))_{-\alpha(Q_k)} = 1$,

2. $j+1 =$ card $(D_{k-1} \cup P_k)$ *for some $k = 2, ..., l$, and in this case* $\dim_K(H^j_{\mathcal{Q}}(S))_{-\alpha(D_{k-1}\cup P_k)}$ *is the number of occurrences of the set $D_{k-1} \cup P_k$ among the sets $D_1 \cup P_2, ..., D_{l-1} \cup P_l$.*

Here $()_{-\alpha(Q)}$ means the $\mathbb{Z}^n-$graded component, see Sec. 2.

As a corollary we can describe the associate primes of the local cohomology modules.

Theorem 2. *Let $\mathcal{Q}$ be a 2-regular square free monomial ideal. We set*

$$\mathcal{A}_{i,\mathcal{Q}} = \{(Q_j)/ \text{ card}\, Q_j = i\},$$
$$\mathcal{B}_{i,\mathcal{Q}} = \{(D_{k-1}, P_k), \text{ card}\, D_{k-1} + \text{ card}\, P_k = i+1\}.$$

We will say that an element $(D_{k-1}, P_k) \in \mathcal{B}_{i,\mathcal{Q}}$ belongs to the set $\widetilde{\mathcal{B}_{i,\mathcal{Q}}}$ if $(D_{s-1}, P_s) = (Q_j) + (Q_k)$, for some j, k such that $\text{card}\, Q_j, \text{ card}\, Q_k < i$.

With these notations, we have

$$\text{Ass}\,(H^i_{\mathcal{Q}}(S)) = \mathcal{A}_{i,\mathcal{Q}} \cup \widetilde{\mathcal{B}_{i,\mathcal{Q}}}.$$

2. Preliminaries on Local Cohomology

For a good reading on local cohomology we invite the reader to see [14] or [17]. In this section $I \subset R$ is an ideal, R is a noetherian ring.

Proposition 1. *Let x be a transcendental element over R, we have*

1. $H^i_I(R[x]) \simeq (H^i_I(R))[x]$.
2. $H^0_{(I,x)}(R)[x] = 0$ *and for any $j \geq 0$, we have a short exact sequence*

$$0 \to (H^j_I(R))[x] \to ((H^j_I(R))[x])_x \to H^{j+1}_{(I,x)}(R[x]) \to 0.$$

3. *For any $j \geq 0$, $H^j_I(R) \neq 0$ if and only if $H^{j+1}_{(I,x)}(R[x]) \neq 0$.*

In particular if x_1 is transcendental over R, x_{i+1} is transcendental over $R[x_1, ..., x_i]$, for all i, and $I \subset R$ is an ideal then for any $j \geq 0$, $H^j_I(R) \neq 0$ if and only if $H^{j+n}_{(I,x_1,...,x_n)}(R[x_1, ..., x_n]) \neq 0$.

Proof. The first item is clear since $R[x]$ is a free $R-$module. The second item follows from the long exact sequence associated to local cohomology

$$... \to H^j_{I,x}(R[x]) \to (H^j_I(R))[x] \to ((H^j_I(R))[x])_x \to H^{j+1}_{(I,x)}(R[x]) \to ...,$$

and the following observation: let M be a non-zero R-module. Then $M[x]$ is an $R[x]$-module and the natural map $M[x] \to (M[x])_x$ is injective. So $H^0_{(I,x)}(R)[x] = 0$ and the above long exact sequence splits into short exact sequences

$$0 \to (H^j_I(R))[x] \to ((H^j_I(R))[x])_x \to H^{j+1}_{(I,x)}(R[x]) \to 0,$$

for any $j \geq 0$.

The third item follows from the second item, since for any M non zero $R-$module the natural map $M[x] \to (M[x])_x$ is never surjective. The last statement is an immediate consequence of the third item. ∎

$\mathbb{Z}^n$**-graduation.** Let consider a polynomial ring S in n variables, $S = K[x_1, ..., x_n]$. For any monomial in S, $\underline{x}^\alpha := x_1^{\alpha_1}...x_t^{\alpha_n}$ we set $deg_{\mathbb{Z}^n}(\underline{x}^\alpha) = \alpha$. S has a $\mathbb{Z}^n$-graduation

$$S = \oplus_{\alpha \in \mathbb{N}^n} K\underline{x}^\alpha.$$

It follows that ideals generated by monomials are the only ideals with $\mathbb{Z}^n$-graduation. It is well known that if I is a monomial ideal then the local cohomology modules $H_I^i(S)$ are $\mathbb{Z}^n$-graded $S-$modules. For any S-module M with a $\mathbb{Z}^n$-graduation, M_α will denote the $K-$ vector space of elements in M of degree α. The following proposition is well known.

Proposition 2.
1. *Let* $\mathbf{m} = (x_1, ..., x_n)$ *be the maximal ideal in* S *then* $H^j_{\mathbf{m}}(S) \neq 0$ *if and only if* $j = n$. *Moreover*

$$H^n_{\mathbf{m}}(S) \simeq \bigoplus_{\alpha \in (1,...,1)+\mathbb{N}^n} KX^{-\alpha} \simeq (K[\mathbf{m}^{-1}])[\alpha(\mathbf{m})]$$

as $\mathbb{Z}^n$*-graded* S*-modules, where* $\alpha(m) = (1, ..., 1) \in \mathbb{Z}^n$.

2. *Let* $V = \{x_1, ..., x_n\}$*, and* $Q \subset V$ *be a subset of variables, with* $\operatorname{card} Q = q$*, then* $H^j_{(Q)}(S) \neq 0$ *if and only if* $j = q$. *Moreover*

$$H^q_{(Q)}(S) \simeq (H^q_{(Q)}(K[Q])[V \setminus Q] \simeq (K[Q^{-1}][V \setminus Q])[\alpha(Q)]$$

as $\mathbb{Z}^n$*-graded* S*-modules, where for a subset* Q *of variables we set*

$$\alpha(Q)_k = \begin{cases} 1 & \textit{if } x_k \in Q \\ 0 & \textit{else} \end{cases} \quad \textit{and } \alpha(Q) = (\alpha(Q)_1, ..., \alpha(Q)_n).$$

For $\alpha \in \mathbb{Z}^n$ *we can write* $\alpha = \alpha_+ - \alpha_-$*, where* α_+, α_- *have positive entries. It follows that*

$$H^q_{(Q)}(S) \simeq \bigoplus_{\alpha \in \mathbb{Z}^n, \operatorname{supp}(\alpha_-) = \operatorname{supp}(Q)} K\underline{x}^{-\alpha_-}\underline{x}^{\alpha_+}.$$

In particular

$$(H^q_{(Q)}(S))_\alpha \neq 0 \Longleftrightarrow \operatorname{supp}(\alpha_-) = \operatorname{supp}(Q),$$

where $\operatorname{supp}(\alpha_-) = \{i \ / \ \alpha_i < 0\}$ *and* $\operatorname{supp}(Q) = \{i \ / \ x_i \in Q\}$.

The structure of $H^q_{(Q)}(S)$ *as a* $S-$*module is given by: For any monomials* $\underline{x}^\beta \in S$ *and* $\underline{x}^\alpha \in H^q_{(Q)}(S)$*, we have*

$$\underline{x}^\beta \underline{x}^\alpha = \begin{cases} \underline{x}^{\beta+\alpha} & \text{if } \operatorname{supp}((\beta+\alpha)_-) = \operatorname{supp}(Q) \\ 0 & \text{else.} \end{cases}$$

3. 2-regular Monomial Ideals

Let $\mathcal{Q} \subset S := K[V]$ be a square free monomial ideal.

Theorem 2. [16] *Let $S := K[V]$, where V is a finite set of variables. The following are equivalent:*

1. A reduced monomial ideal $\mathcal{Q} \subset S$ is $2-$regular.

2. There exists an ordered sequence of linear ideals $\mathcal{Q}_1, \ \dots \ , \mathcal{Q}_l \ \subset \ S$, such that $\mathcal{Q} \ = \ \mathcal{Q}_1 \ \cap \ \dots \ \cap \ \mathcal{Q}_l$ is the minimal prime decomposition of $\mathcal{Q}$, satisfying the following properties:

a) $\exists \ D_i, \ P_i \subset V$, *with* $D_l = \emptyset, \ P_1 = \emptyset$,

b) *for all* $i = 1, ..., l$, $\mathcal{Q}_i$ *is generated by* $D_i \ \cup \ P_i$,

c) $D_1 \ \supset \ D_2 \ \supset \ \dots \ \supset \ D_l$ (*Strictly decreasing*),

d) *for all* $i = 2, ..., l$, $D_{i-1} \ \cap P_i \ = \ \emptyset$,

e) $\bigcap_{j=1}^{k-1} \mathcal{Q}_j \ \subseteq \ (P_k, D_{k-1}) \ \forall k = 2, ..., l.$

Proof. The proof is contained in [16], but in order to be clear we sketch it: Any monomial ideal $\mathcal{Q} \subset S$ satisfying the property 2 is a 2-regular ideal by [16, Theorem 3]. On the other hand by [10, Proposition 3.4] any 2-regular ideal is "linearly joined", so [16, Corollary 3] implies that the ideal $\mathcal{Q} \subset S$ has a decomposition as stated in 2.

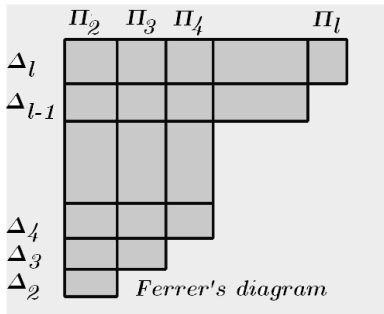
Let Δ_k be the complement of D_k in D_{k-1}. It follows from [16] that

$$\bigcap_{k=1}^{i} \mathcal{Q}_k = (D_i, \bigcup_{k=2,\dots,i} \Delta_k \times P_k),$$

$$\mathcal{Q} = \bigcap_{k=1}^{l} \mathcal{Q}_k = (\bigcup_{k=2,\dots,l} \Delta_k \times P_k),$$

where for two sets A, B of variables $A \times B = \{ab \mid a \in A, b \in B\}$. ∎

Ferrer's diagrams and Ferrer's ideals. A Ferrer's diagram is a way to represent partitions of a natural number N. Let N, m be natural numbers. A partition of N is a sum of natural numbers: $N = \lambda_1 + \lambda_2 + \dots + \lambda_m$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$. A partition is described by a Young diagram which consists of m rows, with the first row containing λ_1 boxes, the second row containing λ_2 boxes, etc. Each row is left-justified. Let $\lambda_{m+1} = 0, \delta_0 = 0$, and δ_1 be the highest integer such that $\lambda_1 = \dots = \lambda_{\delta_1}$, and by induction we define δ_{i+1} as the highest integer such that $\lambda_{\delta_i+1} = \dots = \lambda_{\delta_{i+1}}$, and take l such that $\delta_{l-1} = m$. Let $n = \lambda_1$, we consider two disjoint sets of variables : $\{x_1, x_2, ..., x_m\}, \{y_1, y_2, ..., y_n\}$. Let $V\{x_1, x_2, ..., x_m\} \cup \{y_1, y_2, ..., y_n\}$and $S = K[V]$ the ring of polynomials in the variables V. From the following picture of a Ferrer's tableau



we define the following sets:

$$\Delta_{l-i} = \{x_{\delta_i+1}, \dots, x_{\delta_{i+1}}\},$$
$$\Pi_{l-i} = \{y_{\lambda_{\delta_i+1}}, \dots, y_{\lambda_{\delta_{i+1}+1}+1}\},$$

for $i = 0, \dots, l-2$.

Proposition 3. [16] *With the above notations, we set for all* $i = 1, ..., l$

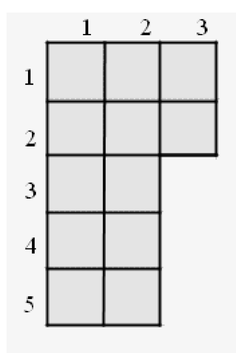
$$D_i = \Delta_{i+1} \cup ... \cup \Delta_l \qquad P_i = \Pi_2 \cup ... \cup \Pi_i, \qquad V = D_1 \cup P_l, \qquad S := K[V],$$

and the linear ideals $\mathcal{Q}_i = (P_i \cup D_i)$*. The Ferrer's ideal associated to the above Ferrer's diagram is given by*

$$\mathcal{Q} = \bigcap_{k=1}^{l} \mathcal{Q}_k = (\bigcup_{i=2,\dots,l} \Delta_i \times P_i).$$

Note that Ferrer's ideals satisfy the conditions a)-e) of Theorem 3. So Ferrer's ideals are 2-regular monomial ideals.

Example 1. Consider the following partition: $3 \geq 3 \geq 2 \geq 2 \geq 2$, and its Ferrer's diagram:



The Ferrer's ideal $\mathcal{I}$ has a minimal primary decomposition

$$\mathcal{I} = (x_1, ..., x_5) \cap (x_1, x_2, y_1, y_2) \cap (y_1, y_2, y_3),$$

and is generated by

$$(x_3, x_4, x_5) \times (y_1, y_2), (x_1, x_2) \times (y_1, y_2, y_3).$$

4. Local Cohomology Modules of 2-regular Ideals

Let $\mathcal{Q} := \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_l \subset K[V]$ be a 2-regular monomial ideal. There are two important invariants for the local cohomology modules: grade($\mathcal{I}$) is the smallest natural number a such that $H^a_{\mathcal{I}}(S) \neq 0$ and the cohomological dimension cd($\mathcal{I}$) is the highest integer q such that $H^q_{\mathcal{I}}(S) \neq 0$. In this section we will give the set of integers q for which $H^q_{\mathcal{I}}(S) \neq 0$.

We recall that for any two ideals $\mathcal{J}_1, \mathcal{J}_2 \subset S$ we have the following exact sequence

$$0 \to S/\mathcal{J}_1 \cap \mathcal{J}_2 \to S/\mathcal{J}_1 \oplus S/\mathcal{J}_2 \to S/(\mathcal{J}_1 + \mathcal{J}_2) \to 0,$$

which gives rise to the Mayer-Vietoris long exact sequence

$$\to H^{h-1}_{\mathcal{J}_1 \cap \mathcal{J}_2}(S) \to H^h_{\mathcal{J}_1 + \mathcal{J}_2}(S) \to H^h_{\mathcal{J}_1}(S) \oplus H^h_{\mathcal{J}_2}(S) \to H^h_{\mathcal{J}_1 \cap \mathcal{J}_2}(S) \to .$$

Theorem 4. *Let $\mathcal{Q} := \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_l \subset K[V]$ be a 2-regular monomial ideal, that is $\mathcal{Q}_i$ is generated by a set of variables $Q_i \subset V$ in the polynomial ring $S := K[V]$. We have a decomposition $Q_i = D_i \cup P_i$ satisfying the properties of Theorem 3. Let $\mathcal{Q}^{(i)} := (Q_1) \cap ... \cap (Q_i)$ then*

1. *For any $i = 2, ..., l$ and a natural number j we have the following short exact sequence*

$$0 \to H^j_{\mathcal{Q}^{(i-1)}}(S) \oplus H^j_{\mathcal{Q}_i}(S) \to H^j_{\mathcal{Q}^{(i)}}(S) \to H^{j+1}_{\mathcal{Q}^{(i-1)} + \mathcal{Q}_i}(S) \to 0.$$

2. $H^j_{\mathcal{Q}^{(i)}}(S) \neq 0$ *if and only if either*

a) $j =$ card (Q_k) *for some $k = 1, ..., i$, and in this case* $\dim_K(H^j_{\mathcal{Q}^{(i)}}(S))_{-\alpha(Q_k)} = 1$.

b) $j+1 =$ card $(D_{k-1} \cup P_k)$ *for some $k = 2, ..., i$, and in this case* $\dim_K(H^j_{\mathcal{Q}^{(i)}}(S))_{-\alpha(D_{k-1} \cup P_k)}$ *is the number of occurrences of the set $D_{k-1} \cup P_k$ among the sets $D_1 \cup P_2, \ldots, D_{i-1} \cup P_i$.*

c) *The graded part $(H^j_{\mathcal{Q}^{(i)}}(S))_\alpha \neq 0$ if and only if* supp $(\alpha_-) =$ supp (Q_k) *for some $k \leq i$ such that* card $(Q_k) = j$ *or* supp $(\alpha_-) =$ supp $(D_{k-1} \cup P_k)$ *for some $k \leq i$ such that* card $(D_{k-1} \cup P_k) = j - 1$.

3. $H^j_{\mathcal{Q}}(S) \neq 0$ *if and only if either*

a) $j =$ card (Q_k) *for some $k = 1, ..., l$, and in this case* $\dim_K(H^j_{\mathcal{Q}}(S))_{-\alpha(Q_k)} = 1$.

b) $j+1 = \operatorname{card}(D_{k-1} \cup P_k)$ *for some* $k = 2, ..., l$, *and in this case* $\dim_K(H^j_{\mathcal{Q}}(S))_{-\alpha(D_{k-1}\cup P_k)}$ *is the number of occurrences of the set* $D_{k-1} \cup P_k$ *among the sets* $D_1 \cup P_2, ..., D_{l-1} \cup P_l$.

c) *The graded part* $(H^j_{\mathcal{Q}}(S))_\alpha \neq 0$ *if and only if* $\operatorname{supp}(\alpha_-) = \operatorname{supp}(Q_k)$ *for some* $k \leq l$ *such that* $\operatorname{card}(Q_k) = j$ *or* $\operatorname{supp}(\alpha_-) = \operatorname{supp}(D_{k-1} \cup P_k)$ *for some* $k \leq l$ *such that* $\operatorname{card}(D_{k-1} \cup P_k) = j - 1$.

Proof. Let $Q \subset V$ be a set of q variables, and (Q) be the ideal generated by Q. We recall that from Proposition 2 we have that $H^j_{(Q)}(S) \neq 0$ if and only if $j = q$. Moreover

$$(H^q_{(Q)}(S))_\alpha \neq 0 \Longleftrightarrow \operatorname{supp}(\alpha_-) = \operatorname{supp}(Q).$$

Remark that Claim 3 is Claim 2 for $i = l$. The proof of the theorem is by induction on both Claims 1 and 2.

For $i = 2$, we have the Mayer-Vietoris long exact sequence

$$\to H^{j-1}_{(Q_1)\cap(Q_2)}(S) \to H^j_{(Q_1)+(Q_2)}(S) \to H^j_{(Q_1)}(S) \oplus H^j_{(Q_2)}(S) \to H^j_{(Q_1)\cap(Q_2)}(S) \to ...$$

and we know from the preliminaries that

- $H^j_{(Q_1)}(S) \neq 0$ if and only if $j = \operatorname{card}(D_1)$,
- $H^j_{(Q_2)}(S) \neq 0$ if and only if $j = \operatorname{card}(D_2 \cup P_2)$,
- $H^j_{(Q_1)+(Q_2)}(S) \neq 0$ if and only if $j = \operatorname{card}(D_1 \cup P_2)$, since $(Q_1) + (Q_2)$ is a linear ideal generated by $D_1 \cup P_2$.

It is enough to prove that the following map is the zero map

$$H^j_{(Q_1)+(Q_2)}(S) \to H^j_{(Q_1)}(S) \oplus H^j_{(Q_2)}(S).$$

This is certainly the case if $H^j_{(Q_1)+(Q_2)}(S) = 0$. So we only need to consider the case where $j = \operatorname{card}(D_1 \cup P_2)$. Let $\alpha \in \mathbb{Z}^t$ such that $(H^j_{(Q_1)+(Q_2)}(S))_\alpha \neq 0$. Then $\operatorname{supp}(\alpha_-) = \operatorname{supp}(D_1 \cup P_2)$. Suppose that the map $(H^j_{(Q_1)+(Q_2)}(S))_\alpha \to (H^j_{(Q_1)}(S))_\alpha \oplus (H^j_{(Q_2)}(S))_\alpha$ is nonzero. Proposition 2 implies that either $\operatorname{supp}(\alpha_-) = \operatorname{supp}(Q_1)$ or $\operatorname{supp}(\alpha_-) = \operatorname{supp}(Q_2)$. Both cases are excluded since the sets $Q_1, Q_2, D_1 \cup P_2$ are all distinct. So Claim 1 is true for $i = 1$. Claim 2 follows immediately from the short exact sequence

$$0 \to H^j_{(Q_1)}(S) \oplus H^j_{(Q_2)}(S) \to H^j_{\mathcal{Q}^{(2)}}(S) \to H^{j+1}_{(Q_1)+(Q_2)}(S) \to 0.$$

Now suppose that $i \geq 3$. By Theorem 3, we have for $k = 2, ..., l$ that

$$(Q_k) + \mathcal{Q}^{(i)} = (Q_k) + \bigcap_{j=1}^{k-1} (Q_j) = (P_k \cup D_{k-1}).$$

By the induction hypothesis we have short exact sequences

$$0 \to H^j_{\mathcal{Q}^{(k-1)}}(S) \oplus H^j_{\mathcal{Q}_k}(S) \to H^j_{\mathcal{Q}^{(k)}}(S) \to H^{j+1}_{\mathcal{Q}^{(k-1)}+\mathcal{Q}_k}(S) \to 0,$$

for all $k = 2, ..., i-1$ and also that the graded part $(H^j_{\mathcal{Q}^{(i-1)}}(S))_\alpha \neq 0$ if and only if $\text{supp}\,(\alpha_-) = \text{supp}\,(Q_k)$ for some $k \leq i-1$ such that $\text{card}\,(Q_k) = j$ or $\text{supp}\,(\alpha_-) = \text{supp}\,(D_{k-1} \cup P_k)$ for some $k \leq i-1$ such that $\text{card}\,(D_{k-1} \cup P_k) = j-1$.

Let consider the long exact sequence

$$... \to H^{j-1}_{\mathcal{Q}_i \cap \mathcal{Q}^{(i-1)}}(S) \to H^j_{\mathcal{Q}_i + \mathcal{Q}^{(i-1)}}(S) \to H^j_{\mathcal{Q}_i}(S) \oplus H^j_{\mathcal{Q}^{(i-1)}}(S) \to H^j_{\mathcal{Q}_i \cap \mathcal{Q}^{(i-1)}}(S) \to ...$$

We know that $\mathcal{Q}_i \cap \mathcal{Q}^{(i-1)} = \mathcal{Q}^{(i)}$, and $\mathcal{Q}_i + \mathcal{Q}^{(i-1)}$ is a linear ideal generated by $D_{i-1} \cup P_i$. In order to prove our theorem it is enough to show that the following map is the zero map:

$$H^j_{\mathcal{Q}_i + \mathcal{Q}^{(i-1)}}(S) \to H^j_{\mathcal{Q}_i}(S) \oplus H^j_{\mathcal{Q}^{(i-1)}}(S).$$

This is certainly the case if $j \neq \text{card}\,(D_{i-1} \cup P_i)$, since in this case $\mathcal{Q}_i + \mathcal{Q}^{(i-1)} = (D_{i-1} \cup P_i)$, so $H^j_{\mathcal{Q}_i + \mathcal{Q}^{(i-1)}}(S) = 0$. It follows that we only need to consider the case where $j = \text{card}\,(D_{i-1} \cup P_i)$. Let $\alpha \in \mathbb{Z}^t$ such that $(H^j_{\mathcal{Q}_i + \mathcal{Q}^{(i-1)}}(S))_\alpha \neq 0$. Then $\text{supp}\,(\alpha_-) = \text{supp}\,(D_{i-1} \cup P_i)$. Suppose that the map $(H^j_{\mathcal{Q}_i + \mathcal{Q}^{(i-1)}}(S))_\alpha \to (H^j_{\mathcal{Q}_i}(S))_\alpha \oplus (H^j_{\mathcal{Q}^{(i-1)}}(S))_\alpha$ is nonzero. The induction hypothesis and Proposition 2 imply that either $\text{supp}\,(\alpha_-) = \text{supp}\,(Q_i)$ or $\text{supp}\,(\alpha_-) = \text{supp}\,(Q_k)$ for some $k \leq i-1$ such that $\text{card}\,(Q_k) = j$ or $\text{supp}\,(\alpha_-) = \text{supp}\,(D_{k-1} \cup P_k)$ for some $k \leq i-1$ such that $\text{card}\,(D_{k-1} \cup P_k) = j-1$. Now remark first that $D_{i-1} \cup P_i \neq Q_i$ and if $D_{i-1} \cup P_i = Q_k$ for some $k \leq i-1$ then $Q_k = D_{i-1} \cup P_i \supset Q_i$ and so the prime decomposition of $\mathcal{Q}$ will be redundant. This is a contradiction. On the other hand we cannot have $D_{i-1} \cup P_i = D_{k-1} \cup P_k$ for some $k \leq i-1$ such that $\text{card}\,(D_{k-1} \cup P_k) = j-1$, since $D_{i-1} \cup P_i, D_{k-1} \cup P_k$ have different cardinals. This shows that our Claim 1 is true for i. Our Claim 2 follows immediately from the short exact sequence

$$0 \to H^j_{\mathcal{Q}^{(i-1)}}(S) \oplus H^j_{\mathcal{Q}_i}(S) \to H^j_{\mathcal{Q}^{(i)}}(S) \to H^{j+1}_{\mathcal{Q}^{(i-1)}+\mathcal{Q}_i}(S) \to 0.$$

This completes the induction and the proof of our theorem. ■

Corollary 1. *Let $\mathcal{Q} := \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_l \subset K[V]$ be a 2-regular monomial ideal. With the above notations we have*

1. $\text{grade}(\mathcal{Q}) = \text{ht}(\mathcal{Q}) = \min_{i=1,...,l}\{\text{ht}(\mathcal{Q}_i)\} = \min_{i=1,...,l}\{\text{card}\,(P_i \cup D_i)\}$.
2. $\text{cd}(\mathcal{Q}) = \max_{i=2,...,l}\{\text{card}\,(P_i \cup D_{i-1}) - 1\}$.
3. $\text{cd}(\mathcal{Q}) = \text{projdim}\,(S/\mathcal{Q})$.

The first two assertions are immediate from the theorem while the last assertion is Lyubeznik's Theorem [15].

Corollary 2. *The multigraded Hilbert function of the local cohomology modules are*

$$H(H^j_{\mathcal{Q}}(S),\underline{x}) = \sum_{Q_i\ /\ \mathrm{card}\,(Q_i)=j} \prod_{x_k\in Q_i} \frac{x_k^{-1}}{1-x_k^{-1}} \prod_{x_k\notin Q_i} \frac{1}{1-x_k}$$
$$+ \sum_{D_{i-1}\cup P_i\ /\ \mathrm{card}\,(D_{i-1}\cup P_i)=j+1} \prod_{x_k\in D_{i-1}\cup P_i} \frac{x_k^{-1}}{1-x_k^{-1}} \prod_{x_k\notin D_{i-1}\cup P_i} \frac{1}{1-x_k}.$$

Proof. From Theorem 4 we have exact sequences

$$0 \to H^j_{\mathcal{Q}^{(i-1)}}(S) \oplus H^j_{\mathcal{Q}_i}(S) \to H^j_{\mathcal{Q}^{(i)}}(S) \to H^{j+1}_{\mathcal{Q}^{(i-1)}+\mathcal{Q}_i}(S) \to 0.$$

Remark that $\mathcal{Q}^{(i-1)} + \mathcal{Q}_i = (D_{i-1}, P_i)$. The proof follows from these exact sequences. Since the multigraded Hilbert function is additive for exact sequences, for any subset $Q \subset V$, we have that

$$H^q_{(Q)}(S) \simeq \bigoplus_{\alpha\in\mathbb{Z}^n,\mathrm{supp}\,(\alpha_-)=\mathrm{supp}\,(Q)} K\underline{x}^{-\alpha_-}\underline{x}^{\alpha_+},$$

and

$$H(H^j_{(Q)}(S),\underline{x}) = \prod_{x_k\in Q} \frac{x_k^{-1}}{1-x_k^{-1}} \prod_{x_k\notin Q} \frac{1}{1-x_k}.$$

■

Remark 1. The following result is a straight consequence of [3] and our theorem. Let $\mathcal{Q} \subset S$ be a 2-regular square free monomial ideal. The ideal $\mathcal{Q}$ and the local cohomology $S-$modules $H^i_{\mathcal{Q}}(S)$ have $\mathbb{Z}^n$-graded resolutions. We have the following results:

1. If K is a field of characteristic zero, $S = K[x_1, ..., x_n]$, and

$$A_n(K) = K[x_1, ..., x_n, \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}]$$

   is the Weyl algebra, it is well known that $H^j_{\mathcal{Q}}(S)$ is a holonomic $A_n(K)-$module. Then the characteristic cycle of $H^j_{\mathcal{Q}}(S)$ is

$$CC(H^j_{\mathcal{Q}}(S)) = \sum_{Q_i/\ \mathrm{card}\,(Q_i)=j} T^*_{X_{Q_i}} A^n_K + \sum_{D_{i-1}\cup P_i/\ \mathrm{card}\,(D_{i-1}\cup P_i)=j+1} T^*_{X_{D_{i-1}\cup P_i}} A^n_K.$$

2. Let $K = \mathbb{R}$, and $X \subset A^n_{\mathbb{R}}$ be the hyperplane arrangement defined by the ideal $\mathcal{Q}$. Then

$$\dim_{\mathbb{Q}} \tilde{H}_{i-1}(A^n_{\mathbb{R}} - X) = \mathrm{card}\,\{Q_k/\ \mathrm{card}\,(Q_k) = i\} + \mathrm{card}\,\{D_{k-1}\cup P_k/\ \mathrm{card}\,(D_{k-1}\cup P_k) = i+1\}.$$

In particular we have that

$$\min\{i/\tilde{H}_{i-1}(A^n_{\mathbb{R}} - X, \mathbb{Q}) \neq 0\} = \min_k\{\text{ card}\,(Q_k)\} = \text{ ht}(\mathcal{Q}),$$

$$\max\{i/\tilde{H}_{i-1}(A^n_{\mathbb{R}} - X, \mathbb{Q}) \neq 0\} = \max_k\{\text{ card}\,(D_{k-1} \cup P_k) - 1\} = \text{projdim}\,(S/\mathcal{Q}).$$

5. Alexander Duality, Syzygies of codim 2 Cohen-Macaulay Monomial Ideals

Let $\mathcal{Q} \subset S := K[V]$ be a square free monomial ideal and $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap ... \cap \mathcal{Q}_l$ be its minimal primary decomposition, where $\mathcal{Q}_i$ is generated by a set of variables Q_i. The Alexander dual ideal $\mathcal{Q}^\vee$ is the square free monomial ideal generated by the monomials $\underline{x}^{Q_1}, ..., \underline{x}^{Q_l}$ where if $Q = \{x_{i_1}, ..., x_{i_s}\}$ we set $\underline{x}^{Q_1} = x_{i_1} ... x_{i_s}$. It is well known that $(\mathcal{Q}^\vee)^\vee = \mathcal{Q}$. Now we recall the following results:

Theorem 5. *Let $\mathcal{Q} \subset S := K[x_1, ..., x_n]$ be a square free monomial ideal.*

1. [11] *The ring $S/\mathcal{Q}$ is Cohen-Macaulay if and only if the Alexander dual ideal $Q^\vee$ has a linear resolution.*
2. [18] projdim $(S/\mathcal{Q}) =$ reg $(\mathcal{Q}^\vee)$.
3. [17] *The ideals $\mathcal{Q}, \mathcal{Q}^\vee$, and the local cohomology modules $H^i_{\mathcal{Q}}(S)$ have $\mathbb{Z}^n$-graded resolutions. Let $\Omega = \{-1, 0\}^n$, and $\alpha \in \Omega$, then*

$$\beta_{i,-\alpha}(\mathcal{Q}^\vee) = \dim{}_K(H^{|\alpha|-i}_{\mathcal{Q}}(S))_\alpha$$

and all other graded Betti numbers are zero.

Proposition 4. *Let $\mathcal{Q} := \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_l \subset K[V]$ be a 2-regular monomial ideal, $Q_i = D_i \cup P_i$ satisfying the properties of Theorem 3. Then the minimal free resolution of $\mathcal{Q}^\vee$ is*

$$0 \longrightarrow \bigoplus_{i=2}^{l} S[-\alpha(D_{i-1} \cup P_i)] \xrightarrow{M} \bigoplus_{i=1}^{l} S[-\alpha(Q_i)] \xrightarrow{N} \mathcal{Q}^\vee \longrightarrow 0.$$

It then follows that

1. *$S/\mathcal{Q}^\vee$ is a Cohen-Macaulay ring and has projective dimension equal 2.*
2. *The multigraded Betti numbers of this free resolution are*

$$\beta_{0,\alpha}(\mathcal{Q}^\vee) = 1 \Leftrightarrow \alpha = \alpha(Q_i), \quad i = 1, ..., l.$$

$\beta_{1,\alpha}(\mathcal{Q}^\vee)$ is the number of occurrences of α in the sequence $\alpha(D_1 \cup P_2), ..., \alpha(D_{l-1} \cup P_l)$.

Let Δ_{i+1} denote the complement of D_{i+1} in D_i. In the above free resolution $N = (\underline{x}^{Q_1}, ..., \underline{x}^{Q_l})$ and M is given by the following set of minimal relations between the generators of $\mathcal{Q}^\vee$:

a) *If $P_i \subset P_{i+1}$ we have the relation*

$$\underline{x}^{P_{i+1}\setminus P_i}\underline{x}^{Q_i} - \underline{x}^{\Delta_{i+1}}\underline{x}^{Q_{i+1}}.$$

(Remark that in this case we also have $(D_i, P_{i+1}) = (D_i, P_i) + (D_{i+1}, P_{i+1})$.)

b) *If $P_i \not\subset P_{i+1}$, let $j < i$ be the smallest number such that $P_j \subset P_{i+1}$ and $P_{j+1} \not\subset P_{i+1}$, it follows then that $\Delta_k \subset P_{i+1}$ for all $j+1 \leq k \leq i$, so $P_{i+1} \supset P_j \cup \cup_{j+1\leq k\leq i}\Delta_k$ and we have the relation*

$$\underline{x}^{P_{i+1}\setminus(P_j\cup\cup_{j+1\leq k\leq i}\Delta_k)}\underline{x}^{Q_j} - \underline{x}^{\Delta_{i+1}}\underline{x}^{Q_{i+1}}.$$

(Remark that in this case we also have $(D_i, P_{i+1}) = (D_j, P_j) + (D_{i+1}, P_{i+1})$, since $D_j = D_i \cup \cup_{j+1\leq k\leq i}\Delta_k$).

3. *When $\mathcal{Q}$ is a Ferrer's ideal, the minimal free resolution of $(\mathcal{Q}^\vee)$ is*

$$0 \longrightarrow \oplus_{i=2}^{l} S[-(\alpha(D_{i-1}\cup P_i))] \xrightarrow{M} \oplus_{i=1}^{l} S[-(\alpha(Q_i))] \xrightarrow{N} \mathcal{Q}^\vee \longrightarrow 0,$$

where $N = (\underline{x}^{Q_1}, ..., \underline{x}^{Q_l})$ and

$$M = \begin{pmatrix} \underline{x}^{\Pi_2} & 0 & 0 & ... & ... & 0 & 0 \\ -\underline{x}^{\Delta_2} & \underline{x}^{\Pi_3} & 0 & ... & ... & 0 & 0 \\ 0 & -\underline{x}^{\Delta_3} & 0 & 0 & ... & ... & 0 \\ 0 & 0 & ... & ... & ... & ... & 0 \\ 0 & 0 & ... & ... & ... & ... & \underline{x}^{\Pi_l} \\ 0 & 0 & ... & ... & ... & ... & -\underline{x}^{\Delta_l} \end{pmatrix}.$$

Proof. The first claim is an immediate consequence of the above theorem. The third claim is a consequence of the second one. In order to show the second claim, remark that the shape of this free resolution is an immediate consequence of our Theorem 4, and the results by Eagon-Reiner [11], Terai [18], and [17] in the above theorem. Also the relations between the generators of $\mathcal{Q}^\vee$ have the right degree by our Theorem 4, so they are the minimal ones. We can also give a direct proof by using the Buchsbaum-Eisenbud theorem to check that the complex

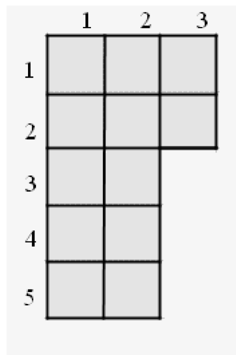
$$0 \longrightarrow \oplus_{i=2}^{l} S[-(\alpha(D_{i-1}\cup P_i))] \xrightarrow{M} \oplus_{i=1}^{l} S[-(\alpha(Q_i))] \xrightarrow{N} I^\vee \longrightarrow 0$$

is a minimal free resolution, but it is quite long and not interesting. ∎

Remark 2. The above proposition gives a characterization of perfect codimension two reduced monomial ideals $\mathcal{I}$. By Alexander duality, Theorem 3 provides an effective method to find the prime decomposition $\mathcal{I} = \mathcal{I}_1 \cap ... \cap \mathcal{I}_e$, where each ideal $\mathcal{I}_i$ is a linear monomial ideal generated by two elements and $e = deg(S/\mathcal{I}) = \mu(\mathcal{I}^\vee)$ is the number of generators of $\mathcal{I}^\vee$.

6. Examples

1. Consider the following partition: $3 \geq 3 \geq 2 \geq 2 \geq 2$, and its Ferrer's diagram:



the corresponding ideal $\mathcal{I}$ has minimal primary decomposition

$$\mathcal{I} = (x_1, ..., x_5) \cap (x_1, x_2, y_1, y_2) \cap (y_1, y_2, y_3),$$

the local cohomology modules are given by:

$$H^3_{\mathcal{I}}(S) \simeq H^3_{(y_1,y_2,y_3)}(S),$$
$$H^5_{\mathcal{I}}(S) \simeq H^5_{(x_1,\dots,x_5)}(S),$$
$$H^6_{\mathcal{I}}(S) \simeq H^7_{(x_1,\dots,x_5,y_1,y_2)}(S),$$

and the exact sequence

$$0 \to H^4_{(x_1,x_2,y_1,y_2)}(S) \to H^4_{\mathcal{I}}(S) \to H^5_{(x_1,x_2,y_1,y_2,y_3)}(S) \to 0.$$

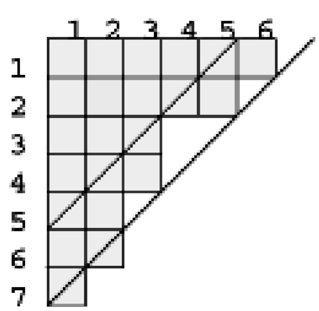
2. The second example corresponds to the partition: $6 \geq 5 \geq 3 \geq 3 \geq 2 \geq 2 \geq 1$, the corresponding ideal $\mathcal{I}$ is generated by

$$\mathcal{I} = (x_1y_1, x_1y_2, ..., x_1y_6, x_2y_1, ..., x_2y_5, x_3y_1, ..., x_3y_3, x_4y_1, ..., x_4y_3, x_5y_1, x_5y_2, x_6y_1, x_6y_2, x_7y_1),$$

and its minimal primary decomposition is

$$\mathcal{I} = (x_1, ..., x_7) \cap (x_1, ..., x_6, y_1) \cap (x_1, ..., x_4, y_1, y_2) \cap (x_1, x_2, y_1, y_2, y_3) \cap (x_1, y_1, ..., y_5) \cap (y_1, ..., y_6).$$

By applying 1 we have grade$(\mathcal{I}) =$ ht$(\mathcal{I}) = 5$, cd$(\mathcal{I}) = 7 =$ projdim $(S/(\mathcal{I}))$.



3. The third example corresponds to the partition: $m \geq m \geq ... \geq m$, where m is repeated n times, and we suppose that $m > n$. The corresponding ideal $\mathcal{I}$ has minimal primary decomposition

$$\mathcal{I} = (x_1, ..., x_n) \cap (y_1, ..., y_m).$$

Let $\mathcal{Q}_1 = (x_1, ..., x_n)$, $\mathcal{Q}_2 = (y_1, ..., y_m)$. We have that

$$\mathcal{Q}_1 + \mathcal{Q}_2 = (x_1, ..., x_n, y_1, ..., y_m).$$

It then follows that $H^n_{\mathcal{Q}_1 \cap \mathcal{Q}_2}(S) \simeq H^n_{\mathcal{Q}_1}(S)$ and

- if $n = 1$ we have the exact sequence

$$0 \to H^m_{\mathcal{Q}_2}(S) \to H^m_{\mathcal{Q}_1 \cap \mathcal{Q}_2}(S) \to H^{m+1}_{\mathcal{Q}_1 + \mathcal{Q}_2}(S) \to 0,$$

- if $n \geq 2$ then

$$H^m_{\mathcal{Q}_1 \cap \mathcal{Q}_2}(S) \simeq H^m_{\mathcal{Q}_1}(S), H^{m+n-1}_{\mathcal{Q}_1 \cap \mathcal{Q}_2}(S) \simeq H^{m+n}_{\mathcal{Q}_1 + \mathcal{Q}_2}(S).$$

All others local cohomology modules are zero. Hence $\operatorname{grade}(\mathcal{I}) = \operatorname{ht}(\mathcal{I}) = n$, $\operatorname{cd}(\mathcal{I}) = m + n - 1 = \operatorname{projdim}(S/(\mathcal{I}))$.

7. Associated Primes of $H^i_{\mathcal{Q}}(S)$

In this section we will prove the following theorem which describes the associated primes of $H^i_{\mathcal{Q}}(S)$, for $\mathcal{Q}$ a 2-regular monomial ideal.

Theorem 6. *Let $\mathcal{Q}$ be a 2-regular square free monomial ideal. We set*

$$\mathcal{A}_{i,\mathcal{Q}} = \{(Q_j)/ \operatorname{card} Q_j = i, \},$$
$$\mathcal{B}_{i,\mathcal{Q}} = \{(D_{k-1}, P_k), \operatorname{card} D_{k-1} + \operatorname{card} P_k = i + 1\}.$$

We say that an element $(D_{k-1}, P_k) \in \mathcal{B}_{i,\mathcal{Q}}$ belongs to the set $\widetilde{\mathcal{B}_{i,\mathcal{Q}}}$ if $(D_{s-1}, P_s) = (Q_j) + (Q_k)$, for some j, k such that $\operatorname{card} Q_j$, $\operatorname{card} Q_k < i$. With these notations, we have

$$Ass(H^i_{\mathcal{Q}}(S)) = \mathcal{A}_{i,\mathcal{Q}} \cup \widetilde{\mathcal{B}_{i,\mathcal{Q}}}.$$

Before going to the proof we need some definitions and lemmas. Let $\mathcal{Q}$ be any square free monomial ideal and $\mathcal{Q} = \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_l$ its prime decomposition. Let $\Omega = \{0, 1\}^l$. For any $\alpha \in \Omega$ we set $\mathcal{Q}_\alpha = \alpha_1 \mathcal{Q}_1 + ... + \alpha_l \mathcal{Q}_l$, where $\alpha_i \mathcal{Q}_i = 0$ if $\alpha_i = 0$ and $\alpha_i \mathcal{Q}_i = \mathcal{Q}_i$ if $\alpha_i = 1$. Finally we set $\mathcal{I}_{\Omega,0} := \{\mathcal{Q}_\alpha \mid \alpha \in \Omega\}$.

We apply the following algorithm:

Input: The set $\mathcal{I}_{\Omega,0}$.

For $i = 1$ to l

Do Step i: Prune the ideal I_α and $I_{\alpha+\varepsilon_i}$, for all $I_\alpha \in \mathcal{I}_{\Omega,i-1}$ such that $\alpha_i = 0$, where ε_i is the $i-$th unit vector and prune means remove both ideals if they are equal.

Next

Output: The set $\widetilde{\mathcal{I}}$.

Let denote by $\widetilde{\mathcal{I}}_t = \{\mathcal{Q}_\alpha \in \widetilde{\mathcal{I}} \ \mid \mid \alpha \mid = t\}$. We illustrate this algorithm with a Ferrer's ideal.

Example 2. Let $\mathcal{Q}$ be a Ferrer's ideal, that is $\mathcal{Q} = \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_s$ is its prime decomposition, where $\mathcal{Q}_k = (D_k, P_k)$ with

$$D_1 \supset ... \supset D_l = \emptyset, \emptyset = P_1 \subset ... \subset P_l, D_1 \cap P_l = \emptyset.$$

Remark that for any sequence of natural numbers $i_1 < i_2 < ... < i_k$ we have $\mathcal{Q}_{i_1} + \mathcal{Q}_{i_2} + ... + \mathcal{Q}_{i_k} = \mathcal{Q}_{i_1} + \mathcal{Q}_{i_k}$. Hence applying the above algorithm, we get

$$\widetilde{\mathcal{I}}_1 := \{\mathcal{Q}_1, ..., \mathcal{Q}_l\},$$

$$\widetilde{\mathcal{I}}_2 := \{\mathcal{Q}_k + \mathcal{Q}_{k+1} = (D_k, P_{k+1}) \mid \forall 1 \leq k < l\},$$

and $\widetilde{\mathcal{I}} = \widetilde{\mathcal{I}}_1 \cup \widetilde{\mathcal{I}}_2$.

Remark 3. As it was remarked in [1, page 102], and in [2, 3.2] the above algorithm is the same as the algorithm to pass from the Taylor resolution of the Alexander dual monomial ideal $\mathcal{Q}^\vee$ to a *minimal* free resolution of $\mathcal{Q}^\vee$. So if $\mathcal{Q}$ is a 2-regular square free monomial ideal, by Proposition 4 we get

1. $\widetilde{\mathcal{I}}_1 = \{(Q_j)/ \text{ card}\, Q_j = i\}$,
2. $\widetilde{\mathcal{I}}_2 = \{(D_{k-1}, P_k), \text{ card}\, D_{k-1} + \text{ card}\, P_k = i+1\}$.

Following [1] we say that two sums $\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_{t+1}}$ and $\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_t}$ are almost paired if $\text{ht}(\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_{t+1}}) = \text{ht}(\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_t}) + 1$.

The reader should take care that $\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_t}$ and $\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_{t+1}}$ can be almost paired, but it is possible that $\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_{t+1}}$ and $\mathcal{Q}_{i_1} + ... + \widehat{\mathcal{Q}_{i_\tau}} + ... + \mathcal{Q}_{i_{t+1}}$ are not almost paired, where $\widehat{}$ means to delete this ideal. In this last case we will say that $\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_{t+1}}$ is non almost paired.

For any ideal $\mathcal{P}$ generated by some variables (linear ideal) and j a natural integer, let $\mathcal{Q}_{\mathcal{P},j,\mathcal{P}}$ be the set of all ideals of the form $\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_j} \in \widetilde{\mathcal{I}}_j$ such that $\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_{j+1}}$ is non almost paired, and $\mathcal{P} + (\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_j}) = \mathcal{P}$.

Example 3. Let

$$S = K[a, b, c, d, e, f, g],$$
$$\mathcal{Q} = (cg, dg, cf, df, ce, de, bc, bd, bg, bf, be, ae),$$
$$\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \mathcal{Q}_3 \cap \mathcal{Q}_4,$$

where $\mathcal{Q}_1 = (a, b, c, d)$, $\mathcal{Q}_2 = (b, c, d, e)$, $\mathcal{Q}_3 = (c, d, e, f, g)$, $\mathcal{Q}_4 = (b, e, f, g)$.

We have that $(D_1, P_2) = (a, b, c, d, e), (D_2, P_3) = (b, c, d, e, f, g) = (D_3, P_4)$. So $(b, c, d, e, f, g) = \mathcal{Q}_2 + \mathcal{Q}_3 = \mathcal{Q}_2 + \mathcal{Q}_4 = \mathcal{Q}_3 + \mathcal{Q}_4$. Let $\mathcal{P} := (b, c, d, e, f, g)$, then

$$\mathcal{Q}_{\mathcal{P},1,\mathcal{P}} = \{\mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4\},$$

$$\mathcal{Q}_{\mathcal{P},2,\mathcal{P}} = \{\mathcal{Q}_2 + \mathcal{Q}_4\}, \mathcal{Q}_{\mathcal{P},3,\mathcal{P}} = \emptyset.$$

We can now state Proposition 4.3.14 of [1]:

Proposition 5. *Let $\mathcal{Q}$ be any square free monomial ideal and $\mathcal{Q} = \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_s$ its prime decomposition. Let $\mathcal{P}$ be a linear ideal. The following are equivalent:*

1. $\mathcal{P} \in Ass(H^i_{\mathcal{Q}}(S))$.
2. *There exists* $\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_j} \in \mathcal{Q}_{\mathcal{P},j,\mathcal{P}}$ *such that* $\mathrm{ht}(\mathcal{Q}_{i_1} + ... + \mathcal{Q}_{i_j}) = i + (j-1) = \mathrm{ht}\mathcal{P}$.

Now we are ready to prove Theorem 6.

Let $\mathcal{Q}$ be a square free 2-regular monomial ideal. By Theorem linearly-joined. there exists an ordered sequence of linear ideals $\mathcal{Q}_1, ..., \mathcal{Q}_l \subset S := K[V]$, such that $\mathcal{Q} = \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_l$ and

1. $\exists D_i, P_i \subset V$, with $D_l = \emptyset, P_1 = \emptyset$, and for all $i = 1, ..., l$, $\mathcal{Q}_i = (D_i, P_i)$,
2. $D_1 \supset D_2 \supset ... \supset D_l$. (Strictly decreasing) and for all $i = 2, ..., l, D_{i-1} \cap P_i = \emptyset$,
3. Let $\mathcal{Q}^{(k)} := (Q_1) \cap ... \cap (Q_k)$ then $\mathcal{Q}^{(k-1)} + \mathcal{Q}_k = (D_{k-1}, P_k)$.

Applying Theorem 4, for any $k = 2, ..., l$ and a natural number j we have the following short exact sequence:

$$0 \to H^i_{\mathcal{Q}^{(k-1)}}(S) \oplus H^i_{\mathcal{Q}_k}(S) \to H^i_{\mathcal{Q}^{(k)}}(S) \to H^{i+1}_{\mathcal{Q}^{(k-1)} + \mathcal{Q}_k}(S) \to 0.$$

Since $\mathcal{Q}^{(k-1)} + \mathcal{Q}_k = (D_{k-1}, P_k)$, this exact sequence implies that

$$Ass(H^i_{\mathcal{Q}^{(k-1)}}(S)) \cup \{\mathcal{Q}_k, if \ \ \mathrm{ht}\mathcal{Q}_k = i\} \subset Ass(H^i_{\mathcal{Q}^{(k)}}(S))$$

and

$$\begin{aligned} \mathrm{Ass}\,(H^i_{\mathcal{Q}^{(k)}}(S)) \subset Ass(H^i_{\mathcal{Q}^{(k-1)}}(S)) &\cup \{\mathcal{Q}_k, if \ \ \mathrm{ht}\mathcal{Q}_k = i\} \\ &\cup \{(D_{k-1}, P_k) if \ \ \mathrm{ht}(D_{k-1}, P_k) = i + 1\}. \end{aligned}$$

By using induction on the number of prime components of $\mathcal{Q}$ we get that

$$\{\mathcal{Q}_k, | \ \ \mathrm{ht}\mathcal{Q}_k = i\} \subset Ass(H^i_{\mathcal{Q}}(S)) \subset \{\mathcal{P} \mid \exists k, \mathcal{P} = (D_{k-1}, P_k), \ \mathrm{ht}\mathcal{P} = i + 1\}.$$

Now remark that $(D_{k-1}, P_k) = \mathcal{Q}_j + \mathcal{Q}_k$ for some $j < k$. Let Δ_j be the complement of D_j in D_{j-1}.

1. If $P_{k-1} \subset P_k$ then $(D_{k-1}, P_k) = \mathcal{Q}_{k-1} + \mathcal{Q}_k$.

2. If $P_{k-1} \not\subset P_k$ let $j < k$ be the smallest number such that $P_j \subset P_k$ and $P_{j+1} \not\subset P_k$, it follows then that $\Delta_r \subset P_k$ for all $j+1 \leq r \leq k-1$, then $(D_{k-1}, P_k) = \mathcal{Q}_j + \mathcal{Q}_k$.

By Proposition 5, $\mathcal{P} \in Ass(H^i_{\mathcal{Q}}(S))$ if and only if there exists some non almost paired sum $Q_{i_1} + ... + Q_{i_j}$ such that

$$\mathcal{P} + Q_{i_1} + ... + Q_{i_j} = \mathcal{P},\ \mathrm{ht} Q_{i_1} + ... + Q_{i_j} = i + (j-1) = \mathrm{ht}\mathcal{P}.$$

So by applying Remark 3 we have two cases

1. $\mathrm{ht}\mathcal{P} = i$ and in this case $j = 1, \mathcal{P} = Q_{i_1}$, or
2. $\mathrm{ht}\mathcal{P} = i+1$, in this case $\mathcal{P} = (D_{k-1}, P_k)$ for some k, and $j = 2$, $\mathrm{ht} Q_{i_1} + Q_{i_2} = i+1 = \mathrm{ht}\mathcal{P}$ so $Q_{i_1} + Q_{i_2} = \mathcal{P}$ with the condition $\mathrm{ht} Q_{i_1} < i$ and $\mathrm{ht} Q_{i_2} < i$, so $\mathcal{P} \in \widetilde{\mathcal{B}}_{i,\mathcal{Q}}$.

For Ferrer's ideals the situation is much simpler.

Theorem 7. *Let $\mathcal{Q}$ be a Ferrer's ideal, $\mathcal{Q} = \mathcal{Q}_1 \cap ... \cap \mathcal{Q}_l$ its minimal prime decomposition as introduced before, where $\mathcal{Q}_i$ is generated by a set of variables Q_i such that $Q_k = D_k \cup P_k$ with*

$$D_1 \supset ... \supset D_l = \emptyset,\ \emptyset = P_1 \subset ... \subset P_l,\ D_1 \cap P_l = \emptyset.$$

We set $\mathcal{A}_{i,\mathcal{Q}} = \{(Q_j)/ \operatorname{card} Q_j = i,\}$, $\mathcal{B}_{i,\mathcal{Q}} = \{\mathcal{P} := (D_{k-1}, P_k),\ \operatorname{card} D_{k-1} + \operatorname{card} P_k = i+1\}$. Then $\operatorname{Ass}(H^i_{\mathcal{Q}}(S))$ has no embedded primes, more precisely

$$Ass(H^i_{\mathcal{Q}}(S)) = \mathcal{A}_{i,\mathcal{Q}} \cup \{\mathcal{P} \in \mathcal{B}_{i,\mathcal{Q}} \ \mid \nexists\ j,\ \mathcal{P} \supset (Q_j) and\ \operatorname{card} Q_j = i\}.$$

Proof.

1. If $\mathcal{P} \in \mathcal{B}_{i,\mathcal{Q}}$ and there is no j, such that $\mathcal{P} \supset (Q_j)$ with $\operatorname{card} Q_j = i$, then $\mathcal{P}$ is non almost paired since $\mathcal{P} = \mathcal{Q}_{k-1} + \mathcal{Q}_k$, and $\mathrm{ht}\mathcal{Q}_{k-1} < i$, $\mathrm{ht}\mathcal{Q}_k < i$. Hence by Proposition of Alvarez, $\mathcal{P} \in Ass(H^i_{\mathcal{Q}}(S))$.
2. Let $\mathcal{P} \in Ass(H^i_{\mathcal{Q}}(S))$ such that $\mathrm{ht}\mathcal{P} = i+1$. Then $\mathcal{P} = \mathcal{Q}_j + \mathcal{Q}_k$, with $j < k$ and $\mathcal{Q}_j + \mathcal{Q}_k$ is non almost paired, that is $\mathrm{ht}\mathcal{Q}_j < i$ and $\mathrm{ht}\mathcal{Q}_k < i$. So $\mathcal{P} = \mathcal{Q}_j + \mathcal{Q}_k = (D_j, P_k)$. Suppose that there exist some $\mathcal{Q}_t$, $t \neq j, k$, $\mathrm{ht}\mathcal{Q}_t = i$, with $\mathcal{P} \supset \mathcal{Q}_t$, that is $\mathcal{P} = (D_j, P_k) \supset (D_t, P_t)$. Since $D_1 \cap P_l = \emptyset$, this implies that $j < t < k$ and D_j must properly contain D_t, and P_k must properly contain P_t, so $\mathrm{ht}\mathcal{P} \geq \mathrm{ht}\mathcal{Q}_t + 2$. This contradicts the assumption that $\mathrm{ht}\mathcal{P} = \mathrm{ht}\mathcal{Q}_t + 1$. ■

Example 4. Let

$$S = K[a, b, c, d, e, f, g],$$

and

$$\begin{aligned}\mathcal{Q} &= (cg, dg, cf, df, ce, de, bc, bd, bg, bf, be, ae)\\ &= (a, b, c, d) \cap (b, c, d, e) \cap (c, d, e, f, g) \cap (b, e, f, g).\end{aligned}$$

Set $\mathcal{Q}_1 = (a, b, c, d), \mathcal{Q}_2 = (b, c, d, e), \mathcal{Q}_3 = (c, d, e, f, g), \mathcal{Q}_4 = (b, e, f, g)$. We have $(D_1, P_2) = (a, b, c, d, e), (D_2, P_3) = (b, c, d, e, f, g) = (D_3, P_4)$. So $(b, c, d, e, f, g) = \mathcal{Q}_2 + \mathcal{Q}_3 = \mathcal{Q}_2 + \mathcal{Q}_4 = \mathcal{Q}_3 + \mathcal{Q}_4$ and $\mathcal{Q}_2 + \mathcal{Q}_4$ is a non almost paired sum. We get

$$Ass(H^4_{\mathcal{Q}}(S)) = \{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_4\}, Ass(H^5_{\mathcal{Q}}(S)) = \{\mathcal{Q}_3, \mathcal{Q}_2 + \mathcal{Q}_3\}.$$

In particular $Ass(H^5_{\mathcal{Q}}(S))$ has an embedded associated prime.

We give some easy corollaries of Proposition 5.

Let x be a new variable. In general it is not easy to relate the associated primes of $H^i_{\mathcal{Q}}(S[x])$ and $H^i_{\mathcal{Q}+(x)}(S[x])$, but for a square free monomial ideal this will be given by the following corollary:

Corollary 3. *Let $\mathcal{Q} \subset S$ be any square free monomial ideal in some set of variables Γ and Δ be a set of variables disjoint from Γ. Let $\mathcal{P}$ be a linear ideal. The following are equivalent:*

1. $\mathcal{P} \in Ass(H^i_{\mathcal{Q}}(S))$.
2. $(\mathcal{P}, \Delta) \in Ass(H^i_{(\mathcal{Q},\Delta)}(S))$.

The proof is immediate from the above proposition by using the definition of the sets $\mathcal{Q}_{\mathcal{P},j,\mathcal{P}}$.

Corollary 4. *Suppose that $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2$ has only two prime components, with $\mathcal{Q}_1$ generated by $D_1 := \Delta_2 \cup D_2$, and $\mathcal{Q}_2$ generated by $D_2 \cup P_2$ such that $D_1 \cap P_2 = \emptyset$. Then $\mathrm{Ass}\,(H^i_{(\mathcal{Q})}(S))$ is given by the (possible empty) set*

$$\{\mathcal{Q}_1, if \ \ \mathrm{ht}\mathcal{Q}_1 = i\} \cup \{\mathcal{Q}_2, if \ \ \mathrm{ht}\mathcal{Q}_2 = i\} \\ \cup \{(D_1, P_2) if \ \ \mathrm{ht}(D_1, P_2) = i + 1, \ \mathrm{ht}\mathcal{Q}_1, \ \mathrm{ht}\mathcal{Q}_2 < i\}.$$

The claim follows from Proposition 5, since in this case, the only non possible empty sets $\mathcal{Q}_{\mathcal{P},j,\mathcal{P}}$ are non empty for $j = 1$ or $j = 2$. For $j = 1$ we have only the sets $\mathcal{Q}_{\mathcal{Q}_k,1,\mathcal{Q}_k}$ for $k = 1, 2$, and $\mathcal{Q}_{\mathcal{Q}_k,1,\mathcal{Q}_k} = \{\mathcal{Q}_k,\} \ if \ \ \mathrm{ht}\mathcal{Q}_1 = i$ or empty. For $j = 2$ we have that $\mathcal{Q}_1 + \mathcal{Q}_2 = (D_1, P_2)$ and if $\mathrm{ht}(D_1, P_2) = i + 1$, the set $\mathcal{Q}_{(D_1,P_2),2,(D_1,P_2)}$ will be non empty and equal to $\{(D_1, P_2)\}$ if $\mathcal{Q}_1 + \mathcal{Q}_2 = (D_1, P_2)$ is non almost paired, that is if we have that $\mathrm{ht}\mathcal{Q}_1, \ \mathrm{ht}\mathcal{Q}_2 < i$.

Remark 4. Using the results in this section we can deduce informations on the Lyubeznik's numbers and the Bass numbers. Since they are straight consequences of [2], we refer the reader to this paper.

Acknowledgement. The second author thanks Uli Walther and Josep Alvarez Montaner for useful discussions on local cohomology modules.

This work was done during the stay of the first author in the Institut Fourier, Université de Grenoble I, and she wants to thank the Institut Fourier for financial support. The research work of the first author is also partially covered by the National Council for research in Vietnam.

The authors thank the referee for a careful reading of the paper and pertinent remarks.

References

1. J. Alvarez Montaner, *Local cohomology modules supported on monomial ideals.* Thesis's work, Universitat de Barcelona, 2002.
2. J. Alvarez Montaner, Characteristic cycles of local cohomology modules of monomial ideals II, *JPAA* **192** (2004), 1 – 20.
3. J. Alvarez Montaner, R. Garcia Lopez, and S. Zarzuela Armengou, Local cohomology, arrangements of subspaces and monomial ideals, *Adv. Math.* **174**, (2003), 35 – 56.
4. M. Barile and M. Morales, On the equations defining minimal varieties, *Comm. Alg.* **28** (2000), 1223 – 1239.
5. M. Barile and M. Morales, On unions of scrolls along linear spaces, *Rend. Sem. Mat. Univ. Padova* **111** (2004), 161 – 178.
6. E. Bertini, *Introduzione alla geometria projettiva degli iperspazi con appendice sulle curve algebriche e loro singolarita,* Pisa: E. Spoerri, 1907.
7. M. Brodmann and R. Y. Sharp, *Local cohomology*, Cambridge studies in Advanced Math. **60** (1998).
8. P. Del Pezzo, Sulle superficie di ordine n immerse nello spazio di $n+1$ dimensioni, *Nap. rend.* **XXIV** (1885), 212 – 216.
9. D. Eisenbud and S. Goto, Linear free resolutions and minimal multiplicity, *J. Algebra* **88** (1984), 89 – 133.
10. D. Eisenbud, M. Green, K. Hulek, and S. Popescu, Restricting linear syzygies: algebra and geometry, *Compos. Math.* **141** (2005), 1460 – 1478.
11. J. A. Eagon and V. Reiner, Resolutions of Stanley-Reisner rings and Alexander duality, *J. Pure Appl. Algebra* **130** (1998), 265 – 275.
12. R. Fröberg, On Stanley – Reisner rings, *Banach Center Publ.* **26** (1990), 57 – 70.
13. J. Harris, *Algebraic Geometry, A First Course.* Springer-Verlag, New York, 1992.
14. C. Huneke and A. Taylor, Lectures on local cohomology, Avramov, Luchezar et al. (ed.). Interactions between homotopy theory and algebra. Summer school, University of Chicago, IL, USA, July 26–August 6, 2004. Providence, RI: American Mathematical Society (AMS). *Contemporary Mathematics* **436** (2007), 51 – 99.
15. G. Lyubeznik, On the local cohomology modules $H^i_{\mathcal{I}}(R)$ for ideals $\mathcal{I}$ generated by monomials in an R-sequence, *Lect. Notes Math.* **1092** (1984), 214 – 220.
16. M. Morales, *Simplicial ideals, 2-linear ideals and arithmetical rank.* math. AC/0702668.
17. M. Mustata, Local cohomology at monomial ideals, *J. Symb. Comput.* **29** (2000), 709 – 720.
18. N. Terai, Alexander duality theorem and Stanley-Reisner rings, RIMS Kokyuroku **1078** (1999), 174 – 184.
19. U. Walther, Algorithmic computation of local cohomology modules and the local cohomological dimension of algebraic varieties, *J. Pure Appl. Algebra* **139** (1999),

303 —321.

20. S. Xambo, On projective varieties of minimal degree, *Collect. Math.* **32**(1981), 149–163.