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Homomorphisms, Amenability and Weak Amenability of Banach Algebras

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Abstract. In this paper we find some necessary and sufficient conditions for a Banach algebra to be amenable or weakly amenable, by applying the homomorphisms on Banach algebras.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. Then X^* is a Banach $\mathcal{A}$-bimodule if for each $a \in \mathcal{A}$, $x \in X$ and $x^* \in X^*$ we define

$$\langle x, ax^* \rangle = \langle xa, x^* \rangle, \qquad \langle x, x^*a \rangle = \langle ax, x^* \rangle.$$

Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Banach algebra homomorphism, then $\mathcal{B}$ is an $\mathcal{A}$-bimodule by the following module actions

$$a.b = \varphi(a)b, \qquad b.a = b\varphi(a) \qquad (a \in \mathcal{A}, b \in \mathcal{B}).$$

We denote by $\mathcal{B}_\varphi$ the above $\mathcal{A}$-bimodule. For a Banach algebra $\mathcal{A}$, $\mathcal{A}^{**}$ with the first Arens product is a Banach algebra. Let X be a Banach $\mathcal{A}-$module, we can extend the actions of $\mathcal{A}$ on X to actions of $\mathcal{A}^{**}$ on X^{**} via

$$a''.x'' = w^*\text{-}\lim_\alpha \lim_\beta a_\alpha x_\beta$$

and

$$x''.a'' = w^*\text{-}\lim_\beta \lim_\alpha x_\beta a_\alpha,$$

where $a'' = w^*\text{-}\lim_\alpha a_\alpha$, $x'' = w^*\text{-}\lim_\beta x_\beta$.

If X is a Banach $\mathcal{A}$-bimodule then a derivation from $\mathcal{A}$ into X is a continuous linear map D, such that for every $a, b \in \mathcal{A}$, $D(ab) = D(a).b + a.D(b)$. If $x \in X$, and we define $\delta_x : \mathcal{A} \longrightarrow X$ by $\delta_x(a) = a.x - x.a \qquad (a \in \mathcal{A})$, then δ_x is a derivation, derivations of this form are called inner derivations. A Banach algebra $\mathcal{A}$ is amenable if $H^1(\mathcal{A}, X^*) = \{0\}$ for every $\mathcal{A}$-bimodule X, where $H^1(\mathcal{A}, X^*)$ is the first cohomology group from $\mathcal{A}$ with coefficients in X^*. This definition was introduced by Johnson in [4]. $\mathcal{A}$ is weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. Bade, Curtis and Dales have introduced the concept of weak amenability for commutative Banach algebras [1]. In this paper we show that for amenability of Banach algebra $\mathcal{A}$, it is enough to show that for every Banach algebra $\mathcal{B}$ and every injective homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, $H^1(\mathcal{A}, \mathcal{B}_\varphi{}^*) = \{0\}$. So we introduce two new notations in amenability of Banach algebras and we related this notations to weak amenability.

2. Amenability

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule, then $X \oplus_1 \mathcal{A}$ is a Banach space, with the following norm

$$\|(x, a)\| = \|x\| + \|a\| \qquad (a \in \mathcal{A}\ ,\ x \in X).$$

So $X \oplus_1 \mathcal{A}$ is a Banach algebra with the product

$$(x_1, a_1)(x_2, a_2) = (x_1 \cdot a_2 + a_1 \cdot x_2, a_1 a_2).$$

$X \oplus_1 \mathcal{A}$ is called a module extension Banach algebra. It is easy to show that $(X \oplus_1 \mathcal{A})^* = X^* \oplus \mathcal{A}^*$, where the sum is $\mathcal{A}$-bimodule $l_\infty-$sum. In this section we use module extension Banach algebras to finding an easy equivalent condition for amenability of a Banach algebra.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra. Then the following assertions are equivalent:*

a) *$\mathcal{A}$ is amenable.*

b) *For every Banach algebra $\mathcal{B}$ and every homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, $H^1(\mathcal{A}, \mathcal{B}_\varphi{}^*) = \{0\}$.*

c) *For every Banach algebra $\mathcal{B}$ and every injective homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, $H^1(\mathcal{A}, \mathcal{B}_\varphi{}^*) = \{0\}$.*

d) *For every Banach algebra $\mathcal{B}$ and every injective homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, if $d_\varphi : \mathcal{A} \longrightarrow \mathcal{B}_\varphi{}^*$ is a (bounded) derivation satisfying*

$$\langle d_\varphi(a), \varphi(b)\rangle + \langle d_\varphi(b), \varphi(a)\rangle = 0 \qquad (a, b \in \mathcal{A}),$$

then d_φ is an inner derivation.

e) *For every Banach algebra $\mathcal{B}$ and every injective homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, $H^1(\mathcal{A}, \mathcal{B}_\varphi{}^{**}) = \{0\}$.*

Proof. The proofs of $a) \Longrightarrow b)$, $a) \Longrightarrow e)$, $b) \Longrightarrow c)$ and $c) \Longrightarrow d)$ are immediate. We prove $d) \Longrightarrow a)$ and $e) \Longrightarrow a)$. Suppose that d) holds, X a Banach $\mathcal{A}$-bimodule and that $D : \mathcal{A} \longrightarrow X^*$ is a derivation. As above, we know that $X \oplus_1 \mathcal{A}$ is a Banach algebra and obviously the map

$$\varphi : a \mapsto (0, a), \qquad \mathcal{A} \longrightarrow X \oplus_1 \mathcal{A}$$

is an injective Banach algebras homomorphism. Then $H^1(\mathcal{A}, ((X \oplus_1 \mathcal{A})_\varphi)^*) = \{0\}$. We define $D_1 : \mathcal{A} \longrightarrow (X \oplus_1 \mathcal{A})^*$ by $D_1(a) = (D(a), 0)$. For $a, b \in \mathcal{A}$ we have

$$\begin{aligned} D_1(ab) &= (D(ab), 0) = (D(a)b + aD(b), 0) \\ &= (D(a), 0)(0, b) + (0, a)(D(b), 0) \\ &= D_1(a)\varphi(b) + \varphi(a)D_1(b). \end{aligned}$$

Thus D_1 is a derivation from $\mathcal{A}$ into $((X \oplus_1 \mathcal{A})_\varphi)^*$. Also for every $a, b \in \mathcal{A}$, we have

$$\langle D_1(a), \varphi(b)\rangle + \langle D_1(b), \varphi(a)\rangle = \langle (D(a), 0), (0, b)\rangle + \langle (D(b), 0), (0, a)\rangle = 0.$$

Then D_1 is an inner derivation. In other words there exist $a' \in \mathcal{A}^*, x' \in X^*$ such that $D_1 = \delta_{(x', a')}$. For every $a \in \mathcal{A}$ we have

$$\begin{aligned} (D(a), 0) = D_1(a) &= \delta_{(x', a')}(a) \\ &= \varphi(a)(x', a') - (x', a')\varphi(a) \\ &= (0, a)(x', a') - (x', a')(0, a) \\ &= (ax' - x'a, aa' - a'a). \end{aligned}$$

Thus $D = \delta_{x'}$. So $\mathcal{A}$ is amenable. To prove $e) \Longrightarrow a)$, let X be a Banach $\mathcal{A}$-bimodule and let $D : \mathcal{A} \longrightarrow X^{**}$ be a derivation. If $\varphi : \mathcal{A} \longrightarrow X \oplus_1 \mathcal{A}$ is the above injective Banach algebras homomorphism, then it is easy to show that $\varphi^{**} : \mathcal{A}^{**} \longrightarrow (X \oplus_1 \mathcal{A})^{**}$ the second transpose of φ is a Banachs algebra homomorphism and that $((X \oplus_1 \mathcal{A})_\varphi)^{**} \simeq (X^{**} \oplus_1 \mathcal{A}^{**})_{\varphi^{**}}$ as $\mathcal{A}^{**}$-bimodules. Then

$$H^1(\mathcal{A}, (X^{**} \oplus_1 \mathcal{A}^{**})_{\varphi^{**}}) = H^1(\mathcal{A}, ((X \oplus_1 \mathcal{A})_\varphi)^{**}) = \{0\}. \tag{2.1}$$

Now we define $D_1 : \mathcal{A} \longrightarrow X^{**} \oplus_1 \mathcal{A}^{**}$ by $D_1(a) = (D(a), 0)$. For $a, b \in \mathcal{A}$ we have

$$D_1(ab) = D_1(a)\varphi^{**}(\hat{b}) + \varphi^{**}(\hat{a})D_1(b).$$

Thus D_1 is a derivation from $\mathcal{A}$ into $(X^{**} \oplus_1 \mathcal{A}^{**})_{\varphi^{**}}$. By (2.1), D_1 is inner. Therefore there exist $a'' \in \mathcal{A}^{**}, x'' \in X^{**}$ such that $D_1 = \delta_{(x'', a'')}$, and by a similar proof as above we can show that D is inner. Then we have $H^1(\mathcal{A}, X^{**}) = \{0\}$, and by Proposition 2.8.59 of [2], $\mathcal{A}$ is amenable. ∎

Let $\mathcal{A}$ has a bounded approximate identity, and let X be an essential Banach $\mathcal{A}$-bimodule, then it is easy to show that $(X \oplus_1 \mathcal{A})_\varphi$ is an essential Banach $\mathcal{A}$-bimodule when $\varphi : \mathcal{A} \longrightarrow X \oplus_1 \mathcal{A}$ is defined by $\varphi(a) = (0, a)$. By the same technique as above and by using Corollary 2.9.28 of [2], we have the following theorem.

Theorem 2.2. *Let A be a Banach algebra with a bounded approximate identity. Then $\mathcal{A}$ is amenable if and only if $H^1(\mathcal{A}, {\mathcal{B}_\varphi}^*) = \{0\}$, for every Banach algebra $\mathcal{B}$ and every injective homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ in which $\mathcal{B}_\varphi$ is essential.*

3. Weak Amenability

In this section we find the relationship between weak amenability and homomorphisms of Banach algebras. First we introduce two new notations of amenability of Banach algebras.

Definition 3.1. *Let $\mathcal{A}$ be a Banach algebra. Then*

a) *$\mathcal{A}$ is supper weakly amenable if for every Banach algebra $\mathcal{B}$ and every continuous homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, if d_φ is a (bounded) derivation from $\mathcal{A}$ into ${\mathcal{B}_\varphi}^*$, then the following condition holds*

$$\langle d_\varphi(a), \varphi(b)\rangle + \langle d_\varphi(b), \varphi(a)\rangle = 0 \qquad (a, b \in \mathcal{A}) \tag{3.1}$$

b) *$\mathcal{A}$ is semiweakly amenable if every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$, with the following property*

$$\langle D(a), b\rangle + \langle D(b), a\rangle = 0 \qquad (a, b \in \mathcal{A}), \tag{3.2}$$

is an inner derivation.

Example 1. Let $\mathbb{T}$ be the unit circle. We write $(\hat{f}(n) : n \in \mathbb{Z})$ for the sequence of Fourier coefficients of a function $f \in L^1(\mathbb{T})$. For $\alpha \in (\frac{1}{2}, 1)$, let $\mathcal{A} = \mathrm{lip}_\alpha(\mathbb{T})$, we define $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ by

$$\langle D(f), g\rangle = \sum n\hat{g}(n)\hat{f}(-n), \qquad (f, g \in \mathcal{A}).$$

D is a non-inner derivation (see [1]) and we have

$$\langle D(f), g\rangle + \langle D(g), f\rangle = 0 \qquad (f, g \in \mathcal{A}).$$

Thus $\mathcal{A}$ is not semiweakly amenable.

Theorem 3.2. *Let $\mathcal{A}$ be a supper weakly amenable Banach algebra. Then*

a) *$\mathcal{A}$ is essential.*

b) *There are no no-zero continuous point derivations on $\mathcal{A}$.*

Proof. a) Let $a_0 \in \mathcal{A} - \bar{\mathcal{A}}^2$, then by Hahn-Banach theorem there exists $f \in \mathcal{A}^*$ such that $\langle f, a_0\rangle = 1$ and $f(\bar{\mathcal{A}}^2) = \{0\}$. The mapping $D : a \longmapsto f(a)f$, $\mathcal{A} \longrightarrow \mathcal{A}^*$ is a derivation and we have $\langle D(a_0), a_0\rangle + \langle D(a_0), a_0\rangle = 2 \neq 0$. Thus $\mathcal{A}$ is not supper weakly amenable.

b) Let $\varphi \in \Omega_{\mathcal{A}}$. If $\varphi = 0$, then by a), every derivation from $\mathcal{A}$ into ${\mathbb{C}_\varphi}^*$ is zero. If $\varphi \neq 0$, and $d_\varphi : \mathcal{A} \longrightarrow \mathbb{C}_\varphi$ is a point derivation at φ, then by Definition

3.1, for every $a \in \mathcal{A}$ we have $\langle d_\varphi(a), \varphi(a)\rangle = d_\varphi(a)\varphi(a) = 0$. Therefore we have $d_\varphi|(\mathcal{A} \setminus M_\varphi) = 0$. Thus $d_\varphi = 0$. ■

Example 2. Let $\mathcal{A} = \mathbb{C}$ by the product $ab = 0, (a, b \in \mathbb{C})$. Then by Theorem 3.2 a), $\mathcal{A}$ is not supper weakly amenable. But it is easy to check that $\mathcal{A}$ is semiweakly amenable.

Example 3. Let S be a discrete semigroup in which $S^2 \neq S$, then by Theorem 3.2 a), $l^1(S)$ is not supper weakly amenable. Let $S = \{t, 0\}$ by products $t0 = 0t = t^2 = 0^2 = 0$, then $l^1(S)$ is not supper weakly amenable but for every derivation $D : l^1(S) \longrightarrow l^1(S)^*$ if $\langle D(\delta_t), \delta_0\rangle + \langle D(\delta_0), \delta_t\rangle = 0$, then we have $D = 0$. Thus $l^1(S)$ is semiweakly amenable.

Now we find an equivalent condition for weak amenability of Banach algebras.

Theorem 3.3. *Let $\mathcal{A}$ be a Banach algebra. Then*

a) *$\mathcal{A}$ is weakly amenable if and only if $\mathcal{A}$ is supper weakly amenable and semiweakly amenable.*

b) *Let $\mathcal{A}$ be a unital Banach algebra then $\mathcal{A}$ is supper weakly amenable if and only if for every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$, and for every $a \in \mathcal{A}$, we have $\langle D(a), 1\rangle = 0$.*

c) *Let $\mathcal{A}$ be a unital supper weakly amenable Banach algebra. Then for every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ and $\varphi \in \Omega_\mathcal{A} - \{0\}$, there exists $F \in \mathcal{A}^{**}$ such that* $\mathrm{Im}(D) \subseteq \mathrm{Ker}(F)$ *and* $\langle F, \varphi\rangle = 1$.

Proof. a) Let $\mathcal{A}$ be weakly amenable. Obviously $\mathcal{A}$ is semiweakly amenable. For Banach algebra $\mathcal{B}$ and for (continuous) homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, let $d_\varphi : \mathcal{A} \longrightarrow {\mathcal{B}_\varphi}^*$ be a derivation. We define $D = d_\varphi \otimes \varphi : \mathcal{A} \longrightarrow \mathcal{A}^*$ as follows

$$\langle D(a), b\rangle = \langle d_\varphi(a), \varphi(b)\rangle \qquad (a, b \in \mathcal{A}). \tag{3.3}$$

For every $a, b, c \in \mathcal{A}$, we have

$$\begin{aligned}\langle D(ab), c\rangle &= \langle d_\varphi(ab), \varphi(c)\rangle \\ &= \langle d_\varphi(a)\varphi(b), \varphi(c)\rangle + \langle \varphi(a)d_\varphi(b), \varphi(c)\rangle \\ &= \langle d_\varphi(a), \varphi(b)\varphi(c)\rangle + \langle d_\varphi(b), \varphi(c)\varphi(a)\rangle \\ &= \langle d_\varphi(a), \varphi(bc)\rangle + \langle d_\varphi(b), \varphi(ca)\rangle \\ &= \langle D(a), bc\rangle + \langle D(b), ca\rangle \\ &= \langle D(a)b + aD(b), c\rangle.\end{aligned}$$

Therefore D is a derivation. Then there exists $f \in \mathcal{A}^*$ such that $D = \delta_f : \mathcal{A} \longrightarrow \mathcal{A}^*$. Thus for every $a, b \in \mathcal{A}$, we have

$$\begin{aligned}\langle D(a), b\rangle + \langle D(b), a\rangle &= \langle \delta_f(a), b\rangle + \langle \delta_f(b), a\rangle \\ &= \langle af - fa, b\rangle + \langle bf - fb, a\rangle \\ &= 0.\end{aligned}$$

So $\mathcal{A}$ is supper weakly amenable. The converse is trivially since $id : \mathcal{A} \longrightarrow \mathcal{A}$ is a homomorphism in which $\mathcal{A}^* = {\mathcal{A}_{id}}^*$.

b) Let for every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$, and for every $a \in \mathcal{A}$, the equality $\langle D(a), 1\rangle = 0$ hold, and let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Banach algebra homomorphism. If $d_\varphi : \mathcal{A} \longrightarrow {\mathcal{B}_\varphi}^*$ is a derivation, then $D = d_\varphi \otimes \varphi : \mathcal{A} \longrightarrow \mathcal{A}^*$ defined by (3.3), is a derivation and for every $a, b \in \mathcal{A}$, we have

$$\langle d_\varphi(a), \varphi(b)\rangle + \langle d_\varphi(b), \varphi(a)\rangle = \langle D(a), b\rangle + \langle D(b), a\rangle = \langle D(ab), 1\rangle = 0.$$

The converse is trivial.

c) Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a derivation and let $a_0 \in \mathcal{A}$. If $D(a_0) \in \Omega_\mathcal{A}$, then by (b) for every $a \in \mathcal{A}$ we have

$$\langle D(a_0), a\rangle = \langle D(a_0), a.1\rangle = \langle D(a_0), a\rangle\langle D(a_0), 1\rangle = 0.$$

Thus $D(a_0) = 0$. Let now $\varphi \in \Omega_\mathcal{A} - \{0\}$. Then φ is not in $Im(D)$, so by Hahn-Banach theorem there exists $F \in \mathcal{A}^{**}$ such that $\mathrm{Im}(D) \subseteq \mathrm{Ker}(F)$ and $\langle F, \varphi\rangle = 1$. ∎

Corollary 3.4. (Theorem 2.8.63 of [2]) *Let $\mathcal{A}$ be a weakly amenable Banach algebra, then $\mathcal{A}$ is essential and there are no non-zero, (continuous) point derivations on $\mathcal{A}$.*

Corollary 3.5. *Let G be a locally compact topological group. Then G is discrete if and only if $M(G)$ is supper weakly amenable.*

Proof. Dales, Ghahramani and Helmeskii [3] showed that G is discrete if and only if there are no nonzero point derivations on $M(G)$. By applying Theorems 3.2 b) and 3.3 a), we conclude that G is discrete if and only if $M(G)$ is supper weakly amenable. ∎

By the following theorem we can show that the supper weak amenability is different from the weak amenability and semiweak amenability.

Theorem 3.6. *Let $\mathcal{A}$ be a supper weakly amenable Banach algebra, and let $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous Banach algebra homomorphism with dense range. Then $\mathcal{B}$ is supper weakly amenable.*

Proof. Let $\varphi : \mathcal{B} \longrightarrow \mathcal{C}$ be a Banach algebra homomorphism and let $d_\varphi : \mathcal{B} \longrightarrow {\mathcal{C}_\varphi}^*$ be a derivation. Then for every $a, b \in \mathcal{A}$, we have

$$d_\varphi o\theta(ab) = d_\varphi o\theta(a)\varphi o\theta(b) + \varphi o\theta(a) d_\varphi o\theta(b).$$

Therefore $d_\varphi o\theta$ is a derivation from $\mathcal{A}$ into $(\mathcal{C}_{\varphi o\theta})^*$. Since $\mathcal{A}$ is supper weakly amenable, then for every $a, b \in \mathcal{A}$, we have

$$\langle d_\varphi o\theta(a), \varphi o\theta(b)\rangle + \langle d_\varphi o\theta(b), \varphi o\theta(a)\rangle = 0.$$

Since $\theta(\mathcal{A})$ is dense in $\mathcal{B}$, then for every $a', b' \in \mathcal{B}$,

$$\langle d_\varphi(a'), \varphi(b')\rangle + \langle d_\varphi(b'), \varphi(a')\rangle = 0.$$

Thus $\mathcal{B}$ is supper weakly amenable. ∎

Corollary 3.7. *There exists a supper weakly amenable, non-semiweakly amenable Banach algebra.*

Proof. Let E be a Banach space without approximation property and take $\mathcal{A}$ to be the nuclear algebra $E\hat{\otimes}E^*$ (see Definition 2.5.4 of [2]). The identification of $E \otimes E^*$ with $\mathcal{F}(E)$ extends to an epimorphism $R : E\hat{\otimes}E^* \longrightarrow \mathcal{N}(E)$ (see Theorem 2.5.3 of [2]). Set $K = kerR$, then by Corollary 2.8.43 of [2], $\mathcal{A}$ is biprojective and hence weakly amenable. If $dimK \geq 2$ then K does not have trace extension property. So by Proposition 2.8.65 c) of [2], $\mathcal{N}(E) = \frac{\mathcal{A}}{K}$ is not weakly amenable. On the other hand by a) of Theorem 3.3, above, $\mathcal{A}$ is supper weakly amenable and by Theorem 3.6, $\mathcal{N}(E) = \frac{\mathcal{A}}{K}$ is supper weakly amenable. Thus by Theorem 3.3, $\mathcal{N}(E)$ is a supper weakly amenable, non-semiweakly amenable Banach algebra. ∎

We finish this section with a theorem about semiweak amenability of unitization of Banach algebras, and its application to finding an example of non-supper weakly amenable Banach algebra whose unitization is supper weakly amenable.

Theorem 3.8. *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}^\sharp$ (the unitization of $\mathcal{A}$) is semiweakly amenable, then $\mathcal{A}$ is semiweakly amenable.*

Proof. Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a derivation in which (3.2) holds. We define $D^\sharp : \mathcal{A}^\sharp \longrightarrow \mathcal{A}^{\sharp *}$ as follows

$$\langle D^\sharp(a, c), (b, c')\rangle = \langle D(a), b\rangle \qquad (a, b \in \mathcal{A}, c, c' \in \mathbb{C}).$$

Then for every $a, b, d \in \mathcal{A}$ and $c, c', c'' \in \mathbb{C}$, we have

$$\begin{aligned}\langle D^\sharp((a, c)(b, c')), (d, c'')\rangle &= \langle D^\sharp(ab + cb + c'a, cc'), (d, c'')\rangle = \langle D(ab + cb + c'a), d\rangle\\ &= \langle D(a)b + aD(b) + cD(b) + c'D(a), d\rangle\\ &= \langle D(a), bd + c''b + c'd\rangle + \langle D(b), da + cd + c''a\rangle\\ &= \langle D^\sharp(a, c), (bd + c''b + c'd, c'c'')\rangle\\ &\quad + \langle D^\sharp(b, c'), (da + cd + c''a, cc'')\rangle\\ &= \langle (D^\sharp(a, c))(b, c'), (d, c'')\rangle + \langle (a, c)(D^\sharp(b, c')), (d, c'')\rangle.\end{aligned}$$

Thus $D^\sharp$ is a derivation. So we have

$$\langle D^\sharp(a, c), (b, c')\rangle = \langle D(a), b\rangle = \langle D(b), a\rangle = \langle D^\sharp(b, c'), (a, c)\rangle$$

where $a, b \in \mathcal{A}, c, c' \in \mathbb{C}$.

$\mathcal{A}^\sharp$ is semiweakly amenable, then there is $u' \in \mathcal{A}^{\sharp^*}$ such that $D^\sharp = \delta_{u'}$. So we have $D = \delta_{(u'|_{\mathcal{A}})}$. Thus $\mathcal{A}$ is semiweakly amenable. ■

Let $\mathcal{A}$ be the augmentation ideal of $L^1(PS(2,\mathbb{R}))$, then we know that $\mathcal{A}^\sharp$ is weakly amenable and that $\mathcal{A}$ is not weakly amenable (see [5]). By the above theorem, $\mathcal{A}$ is semiweakly amenable. So by Theorem 3.3, $\mathcal{A}^\sharp$ is supper weakly amenable and $\mathcal{A}$ is not supper weakly amenable. Thus we have the following

Corollary 3.9. *There exists a semiweakly amenable Banach algebra $\mathcal{A}$ such that $\mathcal{A}^\sharp$ is supper weakly amenable, and $\mathcal{A}$ is not supper weakly amenable.* ■

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