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Invariant Aproximations for Subcompatible Mappings

Hemant Kumar Nashine

Department of Mathematics, Disha Institute of Management and Technology, Satya Vihar, Vidhansabha-Chandrakhuri Marg (Baloda Bazar Road), Mandir Hasaud, Raipur-492101 (Chhattisgarh), India

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Abstract. Common fixed point results for generalized $\mathcal{I}$-nonexpansive subcompatible maps have been obtained in the present work. Some useful invariant approximation results have also been determined by its application. These results extend and generalize various existing known results with the aid of more general class of noncommuting mappings.

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1. Introduction

Fixed point theorems have been applied in the field of invariant approximation theory since the last four decades and several interesting and valuable results have been obtained.

Meinardus [7] was the first to employ a fixed-point theorem of Schauder to establish the existence of an invariant approximation. Further, Brosowski [2] obtained an interesting result and generalized the Meinardus's result. Later, several results [4, 11, 16] have been proved in the direction of Brosowski's result [2]. In the year 1988, Sahab et al [8] extended the result of Hicks and Humpheries [4] and Singh [11] by considering one linear and the other nonexpansive mappings. Al-Thagafi [1] generalized result of Sahab et al [8] and proved some results on invariant approximations for commuting mappings. The introduction of noncommuting maps to this area by Shahzad [9, 10] further extended Al-Thagafi's results and obtained a number of results regarding invariant approximation. Recently, using compatible maps Jungck and Hussain [6] unified, and generalized the results said above.

Some attempts have been made to find existence results on common fixed point theorem and to generalize $\mathcal{I}$-nonexpansive subcompatible maps which is further applied to prove some useful invariant approximation results. In this way, results of Jungck and Hussain [6] are unified, and generalized with the aid of more general class of noncommuting mappings instead of compatible mappings. Some known results of Al-Thagafi [1], Brosowski [2], Meinardus [7], Sahab et al [8] and Singh [11, 12] are also extended by considering Ciric's contraction type condition and more general class of noncommuting mappings in normed spaces. In this way, an approach has been made to give a new direction to the line of investigation initiated in [2].

2. Preliminaries

We need the following definitions.

Definition 2.1. [13] *Let $\mathcal{M}$ be a subset of a metric space $\mathcal{X}$. Let $x_0 \in \mathcal{X}$. An element $y \in \mathcal{M}$ is called a best approximant to $x_0 \in \mathcal{X}$, if*

$$d(x_0, y) = \text{dist}(x_0, \mathcal{M}) = \inf\{d(x_0, z) : z \in \mathcal{M}\}.$$

Let $\mathcal{P}_{\mathcal{M}}(x_0)$ be the set of best $\mathcal{M}-$approximants to x_0 and so

$$\mathcal{P}_{\mathcal{M}}(x_0) = \{z \in \mathcal{M} : d(x_0, z) = \text{dist}(x_0, \mathcal{M})\}.$$

Definition 2.2. [13] *Let $\mathcal{X}$ be a metric linear space. Then a nonempty subset $\mathcal{M}$ in $\mathcal{X}$ is said to be convex, if $\lambda x + (1 - \lambda)y \in \mathcal{M}$, whenever $x, y \in \mathcal{M}$ and $0 \leq \lambda \leq 1$.*

A subset $\mathcal{M}$ in $\mathcal{X}$ is said to be starshaped, if there exists at least one point $p \in \mathcal{M}$ such that the line segment $[x, p]$ joining x to p is contained in $\mathcal{M}$ for all $x \in \mathcal{M}$ (that is $\lambda x + (1 - \lambda)p \in \mathcal{M}$, for all $x \in \mathcal{M}$ and $0 < \lambda < 1$). In this case p is called the starcenter of $\mathcal{M}$.

Each convex set is starshaped with respect to each of its points, but not conversely.

Definition 2.3. [5] *A pair $(\mathcal{T}, \mathcal{I})$ of self-mappings of a metric space $\mathcal{X}$ is said to be compatible, if $d(\mathcal{T}\mathcal{I}x_n, \mathcal{I}\mathcal{T}x_n) \to 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{T}x_n, \mathcal{I}x_n \to t \in \mathcal{X}$.*

Every commuting pair of mappings is compatible but the converse is not true in general. Jungck introduced the concept of weakly compatible maps as follows.

Definition 2.4. [6] *A pair* $(\mathcal{I},\mathcal{T})$ *of self-mappings of a metric space* $\mathcal{X}$ *is said to be weakly compatible, if they commute at their coincidence points, i.e., if* $\mathcal{T}u=\mathcal{I}u$ *for some* $u\in\mathcal{X}$*, then* $\mathcal{T}\mathcal{I}u=\mathcal{I}\mathcal{T}u$.

It is easy to see that compatible maps are weakly compatible.

Definition 2.5. *Suppose that* $\mathcal{M}$ *is* p*-starshaped with* $p\in\mathcal{F}(\mathcal{I})$ *(set of fixed point) and is both* $\mathcal{T}$*- and* $\mathcal{I}$*-invariant. Then* $\mathcal{T}$ *and* $\mathcal{I}$ *are called* $\mathcal{R}$*-subcommuting on* $\mathcal{M}$*, if for all* $x\in\mathcal{M}$ *there exists a real number* $\mathcal{R}>0$ *such that* $d(\mathcal{I}\mathcal{T}x,\mathcal{T}\mathcal{I}x)\leq(\frac{\mathcal{R}}{k})d(((1-k)p+k\mathcal{T}x),\mathcal{I}x)$ *for each* $k\in(0,1]$*. If* $\mathcal{R}=1$*, then the maps are called* 1*-subcommuting. The* $\mathcal{I}$ *and* $\mathcal{T}$ *are called* $\mathcal{R}$*-subweakly commuting on* $\mathcal{M}$*, if for all* $x\in\mathcal{M}$ *there exists a real number* $\mathcal{R}>0$ *such that* $d(\mathcal{I}\mathcal{T}x,\mathcal{T}\mathcal{I}x)\leq\mathcal{R}d(\mathcal{I}x,[p,\mathcal{T}x])$*, where* $[p,x]=\{(1-k)p+kx:0\leq k\leq 1\}$.

Definition 2.6. *Suppose that* $\mathcal{M}$ *is* p*-starshaped with* $p\in\mathcal{F}(\mathcal{I})$*, define* $\bigwedge_p(\mathcal{I},\mathcal{T})=\{\bigwedge(\mathcal{I},\mathcal{T}_k):0\leq k\leq 1\}$ *where* $\mathcal{T}_kx=(1-k)q+k\mathcal{T}x$ *and* $\bigwedge(\mathcal{I},\mathcal{T}_k)=\Big\{\{x_n\}\subset\mathcal{M}:\lim_n\mathcal{I}x_n=\lim_n\mathcal{T}_kx_n=t\in\mathcal{M}\Rightarrow\lim_n d(\mathcal{I}\mathcal{T}_kx_n,\mathcal{T}_k\mathcal{I}x_n)=0\Big\}$*. Then* $\mathcal{I}$ *and* $\mathcal{T}$ *are called subcompatible* [14, 15] *if*

$$\lim_n d(\mathcal{I}\mathcal{T}x_n,\mathcal{T}\mathcal{I}x_n)=0$$

for all sequences $x_n\in\bigwedge_p(\mathcal{I},\mathcal{T})$.

Obviously, subcompatible maps are compatible but the converse does not hold, in general, as the following example shows.

Example 2.7. Let $\mathcal{X}=\mathbb{R}$ with usual norm and $\mathcal{M}=[1,\infty)$. Let $\mathcal{I}(x)=2x-1$ and $\mathcal{T}(x)=x^2$, for all $x\in\mathcal{M}$. Let $p=1$. Then $\mathcal{M}$ is p-starshaped with $\mathcal{I}p=p$. Note that $\mathcal{I}$ and $\mathcal{T}$ are compatible. For any sequence $\{x_n\}$ in $\mathcal{M}$ with $\lim_n x_n=2$, we have, $\lim_n\mathcal{I}x_n=\lim_n\mathcal{T}_{\frac{2}{3}}x_n=3\in\mathcal{M}$ hence $\lim_n\|\mathcal{I}\mathcal{T}_{\frac{2}{3}}x_n-\mathcal{T}_{\frac{2}{3}}\mathcal{I}x_n\|=0$. However, $\lim_n\|\mathcal{I}\mathcal{T}x_n-\mathcal{T}\mathcal{I}x_n\|=0$. Thus $\mathcal{I}$ and $\mathcal{T}$ are not subcompatible maps.

Note that $\mathcal{R}$-subweakly commuting and $\mathcal{R}$-subcommuting maps are subcompatible. The following simple example reveals that the converse is not true, in general.

Example 2.8. Let $\mathcal{X}=\mathbb{R}$ with usual norm and $\mathcal{M}=[0,\infty)$. Let $\mathcal{I}(x)=\frac{x}{2}$ if $0\leq x<1$ and $\mathcal{I}x=x$ if $x=1$, and $\mathcal{T}(x)=\frac{1}{2}$ if $0\leq x<1$ and $\mathcal{T}x=x^2$ if $x=1$. Then $\mathcal{M}$ is 1-starshaped with $\mathcal{I}1=1$ and $\bigwedge_p(\mathcal{I},\mathcal{T})=\{\{x_n\}:1\leq x_n<\infty\}$. Note that $\mathcal{I}$ and $\mathcal{T}$ are subcompatible but not $\mathcal{R}$-weakly commuting for all $\mathcal{R}>0$. Thus $\mathcal{I}$ and $\mathcal{T}$ are neither $\mathcal{R}$-subweakly commuting nor $\mathcal{R}$-subcommuting maps.

The weak commutativity of a pair of selfmaps on a metric space depends on the choice of the metric. This is true for compatibility, $\mathcal{R}$-weak commutativity and other variants of commutativity of maps as well.

Example 2.9. Let $\mathcal{X}=\mathbb{R}$ with usual norm and $\mathcal{M}=[0,\infty)$. Let $\mathcal{I}(x)=1+x$ and $\mathcal{T}(x)=2+x^2$. Then $|\mathcal{I}\mathcal{T}x-\mathcal{T}\mathcal{I}x|=2x$ and $|\mathcal{I}x-\mathcal{T}x|=|x^2-x+1|$. Thus the pair $(\mathcal{I}, \mathcal{T})$ is not weakly commuting on $\mathcal{M}$ with respect to the usual metric. But if $\mathcal{X}$ is endowed with the discrete metric d, then $d(\mathcal{I}\mathcal{T}x, \mathcal{T}\mathcal{I}x) = 1 = d(\mathcal{I}x, \mathcal{T}x)$ for $x > 1$. Thus the pair $(\mathcal{I}, \mathcal{T})$ is weakly commuting on $\mathcal{M}$ with respect to the discrete metric.

Throughout this paper $\mathcal{F}(\mathcal{T})$ (resp. $\mathcal{F}(\mathcal{I})$) denotes the set of fixed points of mapping $\mathcal{T}$ (resp. $\mathcal{I}$).

The following result will also be used in the sequel.

Theorem 2.10. [6, Theorem 2.1]. *Let $\mathcal{M}$ be a subset of a metric space $(\mathcal{X}, d)$, and let $\mathcal{T}$ and $\mathcal{I}$ be weakly compatible self mappings of $\mathcal{M}$. Assume that* $\mathrm{cl}\mathcal{T}(\mathcal{M}) \subset \mathcal{I}(\mathcal{M})$, $\mathrm{cl}\mathcal{T}(\mathcal{M})$ *is complete, and $\mathcal{T}$ and $\mathcal{I}$ satisfy*

$$\begin{aligned} &d(\mathcal{T}x, \mathcal{T}y) \\ &\leq h \max\left\{d(\mathcal{I}x, \mathcal{I}y), d(\mathcal{T}x, \mathcal{I}x), d(\mathcal{T}y, \mathcal{I}y), d(\mathcal{T}y, \mathcal{I}x), d(\mathcal{T}x, \mathcal{I}y)\right\}. \end{aligned} \tag{2.1}$$

for all $x, y \in \mathcal{M}$ and $0 \leq h < 1$, then $\mathcal{M} \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$ is a singleton.

3. Main Results

First, a general result in common fixed point theory for a more general class of noncommuting mappings is presented below.

Theorem 3.1. *Let $\mathcal{M}$ be a nonempty p-starshaped subset of a normed space $\mathcal{X}$. Suppose that $(\mathcal{T}, \mathcal{I})$ are subcompatible self-mappings of $\mathcal{M}$ such that* $\mathrm{cl}\mathcal{T}(\mathcal{M}) \subset \mathcal{I}(\mathcal{M})$, *and $\mathcal{I}$ is affine with $p \in \mathcal{F}(\mathcal{I})$. If $\mathcal{T}$ is continuous and $\mathcal{T}$ and $\mathcal{I}$ satisfy*

$$\begin{aligned} \|\mathcal{T}x - \mathcal{T}y\| \leq \max\Big\{&\|\mathcal{I}x - \mathcal{I}y\|,\ \mathrm{dist}([\mathcal{T}x, p], \mathcal{I}x), \mathrm{dist}([\mathcal{T}y, p], \mathcal{I}y), \\ &\mathrm{dist}([\mathcal{T}y, p], \mathcal{I}x), \mathrm{dist}([\mathcal{T}x, p], \mathcal{I}y)\Big\}, \end{aligned} \tag{3.1}$$

for all $x, y \in \mathcal{M}$, then $\mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$, provided one of the following conditions holds:

(1) $\mathrm{cl}\mathcal{T}(\mathcal{M})$ *is compact and $\mathcal{I}$ is continuous;*

(2) *$\mathcal{M}$ is complete, $\mathcal{F}(\mathcal{I})$ is bounded and $\mathcal{T}$ is a compact map;*

(3) *$\mathcal{M}$ is bounded and complete, $\mathcal{T}$ is hemicompact and $\mathcal{I}$ is continuous;*

(4) *$\mathcal{X}$ is complete, $\mathcal{M}$ is weakly compact, $\mathcal{I}$ is weakly continuous and $\mathcal{I} - \mathcal{T}$ is demiclosed at 0;*

(5) *$\mathcal{X}$ is complete, $\mathcal{M}$ is weakly compact, $\mathcal{T}$ is completely continuous and $\mathcal{I}$ is continuous.*

Proof. Choose a sequence $\{k_n\} \subset (0, 1)$ with $k_n \to 1$ as $n \to \infty$. Define for each $n \geq 1$ and for all $x \in \mathcal{M}$, a mapping $\mathcal{T}_n$ by

$$\mathcal{T}_n x = k_n \mathcal{T}x + (1 - k_n)p.$$

Then each $\mathcal{T}_n$ is a self-mapping of $\mathcal{M}$ and for each n, $\mathrm{cl}\mathcal{T}_n(\mathcal{M}) \subset \mathcal{I}(\mathcal{M})$ since $\mathcal{I}$ is affine, $p \in \mathcal{F}(\mathcal{I})$ and $\mathrm{cl}\mathcal{T}(\mathcal{M}) \subset \mathcal{I}(\mathcal{M})$. The subcompatibility of the pair $(\mathcal{I}, \mathcal{T})$ implies that

$$\begin{aligned}
0 &\leq \lim_n \|\mathcal{T}_n \mathcal{I} x_m - \mathcal{I}\mathcal{T}_n x_m\| \\
&\leq \lim_m k_n \|\mathcal{T}\mathcal{I} x_m - \mathcal{I}\mathcal{T} x_m\| + \lim_m (1-k_n)\|p - \mathcal{I}p\| \\
&= 0,
\end{aligned}$$

for any $\{x_m\} \subset \mathcal{M}$ with $\lim_m \mathcal{T}_n x_m = \lim_m \mathcal{I} x_m = t \in \mathcal{M}$.

Thus $(\mathcal{T}_n, \mathcal{I})$ are compatible and hence weakly compatible on $\mathcal{M}$ for each n. Also

$$\begin{aligned}
&\|\mathcal{T}_n x - \mathcal{T}_n y\| \\
&= k_n \|\mathcal{T} x - \mathcal{T} y\| \\
&\leq k_n \max\{\|\mathcal{I}x - \mathcal{I}y\|, \mathrm{dist}([\mathcal{T}x, p], \mathcal{I}x), \mathrm{dist}([\mathcal{T}y, p], \mathcal{I}y), \\
&\qquad \mathrm{dist}([\mathcal{T}y, p], \mathcal{I}x), \mathrm{dist}([\mathcal{T}x, p], \mathcal{I}y)\} \\
&\leq k_n \max\{\|\mathcal{I}x - \mathcal{I}y\|, \|\mathcal{T}_n x - \mathcal{I}x\|, \|\mathcal{T}_n y - \mathcal{I}y\|, \|\mathcal{T}_n y - \mathcal{I}x\|, \|\mathcal{T}_n x - \mathcal{I}y\|\} \\
&\leq k_n \max\{\|\mathcal{I}x - \mathcal{I}y\|, \|\mathcal{T}_n x - \mathcal{I}x\|, \|\mathcal{T}_n y - \mathcal{I}y\|, \|\mathcal{T}_n y - \mathcal{I}x\|, \|\mathcal{T}_n x - \mathcal{I}y\|\}
\end{aligned}$$

for all $x, y \in \mathcal{M}$.

(1) Since $\mathrm{cl}\mathcal{T}(\mathcal{M})$ is compact, $\mathrm{cl}\mathcal{T}_n(\mathcal{M})$ is also compact. By Theorem 2.10, for each $n \geq 1$, there exists $y_n \in \mathcal{M}$ such that $y_n = \mathcal{I}y_n = \mathcal{T}_n y_n$. The compactness of $\mathrm{cl}\mathcal{T}(\mathcal{M})$ implies that there exists a subsequence $\{\mathcal{T}y_m\}$ of $\{\mathcal{T}y_n\}$ such that $\mathcal{T}y_m \to y$ as $m \to \infty$. Then the definition of $\mathcal{T}_m y_m$ implies $y_m \to y$, so by the continuity of $\mathcal{T}$ and $\mathcal{I}$ we have $y \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$. Thus $\mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.

(2) As in (1), there is a unique $y_n \in \mathcal{M}$ such that $y_n = \mathcal{T}_n y_n = \mathcal{I}y_n$. As $\mathcal{T}$ is compact and $\{y_n\}$ being in $\mathcal{F}(\mathcal{I})$ is bounded so $\{\mathcal{T}y_n\}$ has a subsequence $\{\mathcal{T}y_m\}$ such that $\{\mathcal{T}y_m\} \to y$ as $m \to \infty$. Then the definition of $\mathcal{T}_m y_m$ implies $y_m \to y$, so by the continuity of $\mathcal{T}$ and $\mathcal{I}$ we have $y \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$. Thus $\mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.

(3) As in (1) there exists $y_n \in \mathcal{M}$ such that $y_n = \mathcal{I}y_n = \mathcal{T}_n y_n$. And $\mathcal{M}$ is bounded, so $y_n \to \mathcal{T}y_n = (1 - (k_n)^{-1})(y_n - p) \to 0$ as $n \to \infty$ and hence $d(y_n, \mathcal{T}y_n) \to 0$ as $n \to \infty$. The hemicompactness of $\mathcal{T}$ implies that $\{y_n\}$ has a subsequence $\{y_j\}$ which converges to some $z \in \mathcal{M}$. By the continuity of $\mathcal{T}$ and $\mathcal{I}$ we have $z \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$. Thus $\mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.

(4) As in (1) there exists $y_n \in \mathcal{M}$ such that $y_n = \mathcal{I}y_n = \mathcal{T}_n y_n$. Since $\mathcal{M}$ is weakly compact, we can find a subsequence $\{y_m\}$ of $\{y_n\}$ in $\mathcal{M}$ converging weakly to $y \in \mathcal{M}$ as $m \to \infty$ and as $\mathcal{I}$ is weakly continuous so $\mathcal{I}y = y$. By (3) $\mathcal{I}y_m - \mathcal{T}y_m \to 0$ as $m \to \infty$. The demiclosedness of $\mathcal{I} - \mathcal{T}$ at 0 implies that $\mathcal{I}y = \mathcal{T}y$. Thus $\mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.

(5) As in (4), we can find a subsequence $\{y_m\}$ of $\{y_n\}$ in $\mathcal{M}$ converging weakly to $y \to \mathcal{M}$ as $m \to \infty$. Since $\mathcal{T}$ is completely continuous, $\mathcal{T}y_m \to \mathcal{T}y$ as $m \to \infty$. Since $k_n \to 1$, $y_m = \mathcal{T}_m y_m = k_m \mathcal{T} y_m + (1 - k_m)p \to \mathcal{T}y$ as $m \to \infty$. Thus $\mathcal{T}y_m \to \mathcal{T}^2 y$ as $m \to \infty$ and consequently $\mathcal{T}^2 y = \mathcal{T}y$ implies that $\mathcal{T}w = w$, where $w = \mathcal{T}y$. Also, since $\mathcal{I}y_m = y_m \to \mathcal{T}y = w$, using the continuity of $\mathcal{I}$ and the uniqueness of the limit, we have $\mathcal{I}w = w$. Hence $\mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$. ■

An immediate consequence from Theorem 3.1 is the following

Corollary 3.2. *Let $\mathcal{M}$ be a p-starshaped subset of a normed space $\mathcal{X}$, and let $\mathcal{T}$ and $\mathcal{I}$ be continuous self-maps of $\mathcal{M}$. Suppose that $\mathcal{I}$ is affine with $p \in \mathcal{F}(\mathcal{I})$,* $\mathrm{cl}\mathcal{T}(\mathcal{M}) \subset \mathcal{I}(\mathcal{M})$ *and that* $\mathrm{cl}\mathcal{T}(\mathcal{M})$ *is compact. If the pair $(\mathcal{T}, \mathcal{I})$ is $\mathcal{R}$-subweakly commuting and satisfies (3.1) for all $x, y \in \mathcal{M}$, then $\mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.*

As an application of Theorem 3.1, we obtain the following more general results in invariant approximations theory with the aid of a more general class of noncommuting, namely, subcompatible mappings.

Theorem 3.3. *Let $\mathcal{X}$ be a normed space and $\mathcal{T}, \mathcal{I} : \mathcal{X} \to \mathcal{X}$. Let $\mathcal{M}$ be a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$. Suppose $\mathcal{I}$ is affine on $\mathcal{P}_{\mathcal{M}}(x_0)$, $p \in \mathcal{F}(\mathcal{I})$, $\mathcal{P}_{\mathcal{M}}(x_0)$ is closed and p-starshaped, $\mathcal{I}(\mathcal{P}_{\mathcal{M}}(x_0)) = \mathcal{P}_{\mathcal{M}}(x_0)$, and* $\mathrm{cl}\mathcal{T}(\mathcal{P}_{\mathcal{M}}(x_0))$ *is compact. If the pair $(\mathcal{T}, \mathcal{I})$ is continuous, subcompatible and satisfies*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \begin{cases} \|\mathcal{I}x - \mathcal{I}x_0\| \ \ \textit{if } y = x_0, \\ \max\Big\{\|\mathcal{I}x - \mathcal{I}y\|, \mathrm{dist}([\mathcal{T}x, p], \mathcal{I}x), \mathrm{dist}([\mathcal{T}y, p], \mathcal{I}y), \\ \mathrm{dist}([\mathcal{T}y, p], \mathcal{I}x), \mathrm{dist}([\mathcal{T}x, p], \mathcal{I}y)\Big\}, \textit{if } y \in \mathcal{P}_{\mathcal{M}}(x_0), \end{cases} \tag{3.2}$$

for all $x \in \mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$, then $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.

Proof. Let $y \in \mathcal{P}_{\mathcal{M}}(x_0)$. Then $y \in \partial\mathcal{M}$ and so $\mathcal{T}y \in \mathcal{M}$, because $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$. Now since $\mathcal{T}x_0 = x_0 = \mathcal{I}x_0$, we have

$$\|\mathcal{T}y - x_0\| = \|\mathcal{T}y - \mathcal{T}x_0\| \leq \|\mathcal{I}y - \mathcal{I}x_0\| = \|\mathcal{I}y - x_0\| = \mathrm{dist}(x_0, \mathcal{M}).$$

This shows that $\mathcal{T}y \in \mathcal{P}_{\mathcal{M}}(x_0)$. Consequently, $\mathcal{T}(\mathcal{P}_{\mathcal{M}}\ (x_0)) \subseteq \mathcal{P}_{\mathcal{M}}(x_0) = \mathcal{I}(\mathcal{P}_{\mathcal{M}}(x_0))$. Now Theorem 3.1 guarantees that

$$\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset.$$

■

Define $\mathcal{C}^{\mathcal{I}}_{\mathcal{M}}(x_0) = \{x \in \mathcal{M} : \mathcal{I}x \in \mathcal{P}_{\mathcal{M}}(x_0)\}$ and $\mathcal{D}^{\mathcal{I}_{\mathcal{M}}}(x_0) = \mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{C}^{\mathcal{I}_{\mathcal{M}}}(x_0)$.

Theorem 3.4. *Let $\mathcal{X}$ be a normed space and $\mathcal{T}, \mathcal{I} : \mathcal{X} \to \mathcal{X}$. Let $\mathcal{M}$ be a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$. Suppose $\mathcal{I}$ is affine on*

$\mathcal{D}^* = \mathcal{D}^{\mathcal{I}}_{\mathcal{M}}(x_0)$, $p \in \mathcal{F}(\mathcal{I})$, $\mathcal{D}^*$ is compact and p-starshaped, $\mathcal{I}(\mathcal{D}^*) = \mathcal{D}^*$, $\mathcal{I}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$ and $\mathrm{cl}\mathcal{T}(\mathcal{D}^*)$ is compact. If the pair $(\mathcal{T}, \mathcal{I})$ is continuous, subcompatible on $\mathcal{D}^*$ and if $\mathcal{T}$ and $\mathcal{I}$ satisfy

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \begin{cases} \|\mathcal{I}x - \mathcal{I}x_0\| \quad \text{if } y = x_0, \\ \max\Big\{\|\mathcal{I}x - \mathcal{I}y\|, [\mathcal{T}x, p], \mathcal{I}x), \mathrm{dist}([\mathcal{T}y, p], \mathcal{I}y), \\ \mathrm{dist}([\mathcal{T}y, p], \mathcal{I}x), \mathrm{dist}([\mathcal{T}x, p], \mathcal{I}y)\Big\} \text{ if } y \in \mathcal{D}^*, \end{cases} \tag{3.3}$$

for all $x \in \mathcal{D}^* \cup \{x_0\}$, then $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.

Proof. First, we show that $\mathcal{T}$ is a self map on $\mathcal{D}^*$, i.e., $\mathcal{T} : \mathcal{D}^* \to \mathcal{D}^*$. Let $y \in \mathcal{D}^*$, then $\mathcal{I}y \in \mathcal{D}^*$, since $\mathcal{I}(\mathcal{D}^*) = \mathcal{D}^*$. By the definition of $\mathcal{D}^*$, $y \in \partial\mathcal{M}$. Also $\mathcal{T}y \in \mathcal{M}$, since $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$. Now since $\mathcal{T}x_0 = x_0 = \mathcal{I}x_0$,

$$\|\mathcal{T}y - x_0\| = \|\mathcal{T}y - \mathcal{T}x_0\| \leq \|\mathcal{I}y - \mathcal{I}x_0\|.$$

As $\mathcal{I}x_0 = x_0$,

$$\|\mathcal{T}y - \mathcal{T}x_0\| \leq \|\mathcal{I}y - x_0\| = \mathrm{dist}(x_0, \mathcal{M}),$$

since $\mathcal{I}y \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}y$ is also closest to x_0, so $\mathcal{T}y \in \mathcal{P}_{\mathcal{M}}(x_0)$. As $\mathcal{I}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$,

$$\begin{aligned} \|\mathcal{I}\mathcal{T}y - x_0\| = \|\mathcal{I}\mathcal{T}y - \mathcal{I}x_0\| &\leq \|\mathcal{T}y - x_0\| = \|\mathcal{T}y - \mathcal{T}x_0\| \\ &\leq \|\mathcal{I}y - \mathcal{I}x_0\| = \|\mathcal{I}y - x_0\|. \end{aligned}$$

Thus, $\mathcal{I}\mathcal{T}y \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}y \in \mathcal{C}^{\mathcal{I}_{\mathcal{M}}}(x_0)$ and $\mathcal{T}y \in \mathcal{D}^*$. So $\mathcal{T}$ and $\mathcal{I}$ are selfmaps on $\mathcal{D}^*$. Hence, all the conditions of Theorem 3.1 are satisfied. Thus, there exists $z \in \mathcal{P}_{\mathcal{M}}(x_0)$ such that $z = \mathcal{I}z = \mathcal{T}z$. ■

Theorem 3.5. *Let $\mathcal{X}$ be a normed space and $\mathcal{T}, \mathcal{I} : \mathcal{X} \to \mathcal{X}$. Let $\mathcal{M}$ be a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{M} \cap \mathcal{M}) \subseteq \mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$. Suppose $\mathcal{I}$ is linear on $\mathcal{D}^* = \mathcal{D}^{\mathcal{I}}_{\mathcal{M}}(x_0)$, $p \in \mathcal{F}(\mathcal{I})$, $\mathcal{D}^*$ is compact and p-starshaped, $\mathcal{I}(\mathcal{D}^*) = \mathcal{D}^*$, $\mathcal{I}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$, and $\mathrm{cl}\mathcal{T}(\mathcal{D}^*)$ is compact. If the pair $(\mathcal{T}, \mathcal{I})$ is continuous, subcompatible on $\mathcal{D}^*$ and $\mathcal{T}$ and $\mathcal{I}$ satisfy* (3.3) *for all $x \in \mathcal{D}^* \cup \{x_0\}$, then $\mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) \neq \emptyset$.*

Proof. Let $x \in \mathcal{D}^*$. Then, $x \in \mathcal{P}_{\mathcal{M}}(x_0)$ and hence $\|x - x_0\| = \mathrm{dist}(x_0, \mathcal{M})$. Note that for any $k \in (0, 1)$,

$$\|kx_0 + (1-k)x - x_0\| = (1-k)\|x - x_0\| < \mathrm{dist}(x_0, \mathcal{M}).$$

It follows that the line segment $\{kx_0 + (1-k)x : 0 < k < 1\}$ and the set $\mathcal{M}$ are disjoint. Thus x is not in the interior of $\mathcal{M}$ and so $x \in \partial\mathcal{M} \cap \mathcal{M}$. Since $\mathcal{T}(\partial\mathcal{M} \cap \mathcal{M}) \subset \mathcal{M}$, $\mathcal{T}x$ must be in $\mathcal{M}$. Along with the lines of the proof of Theorem 3.4, we have the result. ■

Remark 3.6. It is observed that $\mathcal{I}(\mathcal{P}_{\mathcal{M}}(x_0)) \subset \mathcal{P}_{\mathcal{M}}(x_0)$ implies $\mathcal{P}_{\mathcal{M}}(x_0) \subset \mathcal{D}^*$ and hence $\mathcal{D}^* = \mathcal{P}_{\mathcal{M}}(x_0)$. Consequently, Theorems 3.4, 3.5 remain valid when $\mathcal{D}^* = \mathcal{P}_{\mathcal{M}}(x_0)$.

Remark 3.7. Theorem 3.1 - Theorem 3.5 generalize the results of Jungck and Hussain [6, Theorem 2.3 - Theorem 2.5] in the sense that the more general noncommuting mappings, namely, subcompatible mappings, have been used in place of compatible mappings.

Remark 3.8. Similarly, all other results of Jungck and Hussain [6, Theorem 2.9 - Theorem 2.12] hold by using subcompatible mappings instead of compatible mappings.

Remark 3.9. Theorem 3.1 contains [1, Theorem 2.2], [3, Theorem 1] and [10, Theorem 2.2].

Remark 3.10. Theorem 3.3 - Theorem 3.5 contain Theorem 3.2 of Al - Thagafi [1], Theorem 3 of Sahab et al [8] and Singh [11, 12] in the sense that more general noncommuting mappings (subcommuting mappings) and relatively nonexpansive maps have been used in place of relatively nonexpansive commuting maps.

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