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On Strictly Generalized PP Rings

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Abstract. A ring R is called *strictly generalized right PP* if for any nonzero element $x \in R$, there exists a positive integer n such that $x^n \neq 0$ and the right ideal $x^n R$ is projective. The class of strictly generalized PP rings is a new class of generalized PP rings and contains PP rings. In this paper, many properties of these rings are studied and some characterizations of von Neumann regular rings and PP rings are extended.

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1. Introduction

Throughout this paper, R is an associative ring with identity $1 \neq 0$ and all modules are unitary modules. We write M_R (resp. ${}_RM$) to indicate that M is a right (resp. left) R-module. The notation $A \leq M$ stands for the fact that A is a submodule of M. The right (resp. left) annihilator of a subset S of a ring R are denoted by $r(S)$ (resp. $l(S)$). If $S = \{x\}$, we usually abbreviate it to $r(x)$ (resp. $l(x)$). When $r(x) = l(x)$ for some $x \in R$, we write it $\text{ann}_R(x)$. J, Z_r, Z_l will stand for the Jacobson radical, the right singular ideal and the left singular ideal of R, respectively.

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A ring R is called *right principally injective* (or *right P-injective*) if every homomorphism $f: aR \to R,\ a \in R$, extends to R, equivalently if $lr(a) = Ra$ for all $a \in R$. A ring is called *right C2* if any right ideal of R isomorphic to a direct summand of R_R is itself a direct summand. *Left P-injective rings* and *left C2 rings* are defined symmetrically. A ring R is called a *Baer ring* if every right (left) annihilator in R is generated by an idempotent; and it is called a *right* (resp. *left*) *PP ring* if all principal *right* (resp. *left*) ideals are projective. If a ring is both left and right PP, then it is simply called a PP ring (see [17, 2, 11]). It is well-known that a ring R is a right PP ring if and only if for each element $a \in R$, the homomorphism $\varphi : R \to aR$ defined by $\varphi(r) = ar$ splits (i.e., Ker φ is a direct summand of R), if and only if the right annihilator of each element of R is generated by an idempotent (see Wisbauer [17]). Clearly, a Baer ring is left and right PP. For basic concepts and results that are not defined in this note we refer to the texts of Anderson and Fuller [1], Dung, Huynh, Smith and Wisbauer [4] , Faith [6], Lam [11], Wisbauer [17].

2. Strictly Generalized PP Rings

The concept of generalized PP rings is a generalization of PP rings and was studied by Hirano [7], Ôhori [14], Huh, Kim and Lee [8], etc. A ring R is called a *generalized right PP ring*, briefly *right GPP*, if for any $x \in R$, there exists a positive integer n (depending on x) such that the right ideal x^nR is projective, or equivalently, if for any $x \in R$ there exists a positive integer n (depending on x) such that the right annihilator of x^n is generated by an idempotent. Note that for a ring R to be right GPP, we only need to find a positive integer n such that x^nR is projective for each non-nilpotent element $x \in R$. We now consider the following class of rings.

Definition 2.1. *A ring R is called strictly generalized right PP, briefly right strictly GPP, if for any nonzero $x \in R$ there exists a positive integer n (depending on x) such that $x^n \neq 0$ and the right ideal x^nR is projective, or equivalently, if for any nonzero $x \in R$ the right annihilator of non-zero element x^n is generated by an idempotent for some positive integer n, depending on x. Strictly generalized left PP rings are defined similarly. A ring which is both left and right strictly GPP is called a strictly GPP ring.*

We have the following implications:

$$(right)\ PP \overset{(\star)}{\to} (right)\ strictly\ GPP \overset{(\star\star)}{\to} (right)\ GPP$$

The converse of $(\star\star)$ is not true, in general, as the following example shows.

Example 2.2. Let $\mathbb{Z}_2$ be the field of integers modulo 2, and

$$R = \{a_0 + a_1 i + a_2 j + a_3 k | a_i \in \mathbb{Z}_2,\ i = 0, 1, 2, 3\},$$

the Hamilton quaternions over $\mathbb{Z}_2$. Then R is a commutative ring isomorphic to the factor ring of the Hamilton quaternions over $\mathbb{Z}$ by the ideal $I = \{a_0 + a_1 i + a_2 j + a_3 k | a_i \in 2\mathbb{Z}, i = 0, 1, 2, 3\}$.

Since $(a_0+a_1i+a_2j+a_3)^2 = a_0^2-a_1^2-a_2^2-a_3^2 \in \mathbb{Z}_2$ with $a_0+a_1i+a_2j+a_3k \in R$, we have $(a_0 + a_1i + a_2j + a_3)^2 R$ is 0 or R, i.e., R is GPP. But the ring R is not a strictly GPP ring. In fact, all idempotents in R are 0 and 1. From $(1+i)^2 = 0$, we see that $1 + i \in r_R(1+i)$. Since $1 \notin r_R(1+i)$, $r_R(1+i)$ cannot be generated by an idempotent of R, and hence R is not a strictly GPP ring. Therefore, R is not a PP ring.

Question 2.3. Is a (right) strictly GPP ring is (right) PP?

We can not answer the above question but we will give some affirmative answers. Recall that a ring R is called *reduced* if it has no nonzero nilpotent elements. R is *abelian* (or *normal*) if all idempotents are in its center. It is called *semicommutative* if for every $a \in R$, $r(a)$ is an ideal of R. Clearly, reduced rings are semicommutative and semicommutative rings are abelian. If R is an abelian right PP ring, then R is PP from the fact that $r(x) = eR = Re = l(x)$ for every idempotent $e \in R$ and $x \in R$. Applying a result of Ôhori [14, Proposition 1] saying that a reduced ring R is PP if and only if R is GPP, we have immediately the following result.

Proposition 2.4. *For a reduced ring R the following conditions are equivalent:*

(i) *R is a (right) PP ring;*

(ii) *R is a (right) GPP ring;*

(iii) *R is a (right) strictly GPP ring.*

Clearly, the ring R in Example 2.2 above is abelian but not reduced. So, there exists an abelian GPP ring which is neither PP nor strictly GPP. However, we shall show that in the class of abelian rings, the notions of strictly GPP rings and PP rings coincide (Proposition 2.7). We first need the following useful lemma.

Lemma 2.5. *Let x be an element of a ring R and n a positive integer. If $r(x^n) = eR$ for some central idempotent e of R, then $r(x^n) = r(x^{n+1})$.*

Proof. See [14, Lemma 3]. ■

Using Lemma 2.5 with the same proof of Ôhori [14, Corollary 4], we have the following corollary.

Corollary 2.6. *Let R be an abelian strictly GPP ring. Then for every nonzero element $x \in R$, there exists a positive integer n such that $x^n \neq 0$ and the ideal $\mathrm{ann}_R(x^n)$ is generated by an idempotent.*

Proposition 2.7. *Let R be an abelian ring. Then R is PP if and only if R is strictly right GPP.*

Proof. It suffices to prove that if R is an abelian strictly right GPP ring then R is a PP ring. Let R be an abelian strictly right GPP ring and x a nonzero element of R. By the hypothesis, there exists a positive integer n such that $r(x^n) = eR$ for some idempotent $e \in R$. Since R is abelian, we have $(xe)^n = x^n e = 0$. We now show that $x^n e = (xe)^{n-1} = .. = xe = 0$. Indeed, assume on the contrary that $x^{n-1}e \neq 0$. Since R is strictly GPP, hence there exists an idempotent $f \neq 1$ of R such that $r((xe)^{n-1}) = fR$. But by Lemma 2.5, $r((xe)^{n-1}) = fR = r((xe)^n) = r(0) = R$, a contradiction. Thus $(xe)^{n-1}$ must be zero. By the same argument, xe also must be zero. From this fact we get $r(x) = eR$. In addition $r(x) = l(x)$ by the hypothesis. Therefore, R is a PP ring. ■

Example 2.8. Let S be an abelian ring and denote by R_n the ring of matrices of the form

$$\begin{bmatrix} a & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a & a_{23} & \dots & a_{2n} \\ 0 & 0 & a & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a \end{bmatrix},$$

with $a, a_{ij} \in S$ and n is a positive integer ≥ 2. The ring R_2 is called the trivial extension of S by S. Huh, Kim and Lee [8, Proposition 3] proved that $R_n, n \geq 2$ is right GPP if and only if S is right GPP. However, if S is right strictly GPP (even right PP), R_n is not right strictly GPP. For example, if S is a domain, then S is right PP and of course, R_2 is right GPP. But R_2 is not right PP from the fact that the element $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in R_2$ has its right nonzero annihilator $r(x) = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$ containing no nonzero idempotents. So, the abelian ring R_2 is also not strictly GPP.

Corollary 2.9. *Let R be a right strictly GPP ring. Then the following are equivalent:*

(i) *R is abelian;*

(ii) *R is semicommutative;*

(iii) *R is reduced;*

(iv) *$l(x) = r(x)$ for every $x \in R$.*

Proof. Since abelian right strictly GPP rings are right PP (Proposition 2.7) and abelian right PP rings are reduced ([8, Lemma 1]), it suffices to prove that, if R is strictly GPP such that $l(x) = r(x)$ for every $x \in R$, then R is abelian. Indeed, since R is right strictly GPP, if e is a nonzero idempotent and $x \in R$ then $r(e) = l(e) = R(1-e) = (1-e)R$. So, $(1-e)x = (1-e)x(1-e) = x(1-e)$ and hence, $ex = xe$. ■

The center of a Baer ring is also Baer by Kaplansky [10, Theorem 9]. In [8, Proposition 11], it was proved that, if a ring R is PP (resp. GPP, abelian right

GPP), then so is its center. By the same argument, it is easy to see that, the center of a strictly GPP ring is also strictly GPP. Since the center of a ring is always abelian, we obtain the following result by applying Proposition 2.7.

Corollary 2.10. *If R is a strictly GPP ring, then its center is a PP ring.*

It is well-known that every right PP ring is right nonsingular (see, e.g. [2, Lemma 8.3]). If R is a right GPP ring, then the right singular ideal Z_r of R is nil, not necessarily nonsingular ([14, Proposition 2]). However, a right strictly GPP ring is right nonsingular.

Lemma 2.11. *Every right strictly GPP is right nonsingular.*

Proof. Let R be a right strictly GPP ring. Assume on the contrary that $Z_r \neq 0$ and let $0 \neq x \in Z_r$. By the hypothesis there exists a positive integer n such that $x^n \neq 0$ and $r(x^n) = eR$ for some $e^2 = e \in R$. Since $x^n \neq 0$, $e \neq 1$, eR is not essential in R_R, which contradicts the fact that $x^n \in Z_r$. ∎

By [8, Proposition 4], there are no nonzero central nilpotent elements in a right PP ring. We have a similar result for right strictly GPP rings as follow.

Proposition 2.12. *Right strictly GPP rings have no nonzero central nilpotent elements.*

Proof. By the same argument as that given in the proof of [8, Proposition 4]. ∎

Chatter and Hajarnavis [2, Lemma 8.5] proved the following result for right PP rings and we now show that it is also true for right strictly GPP rings.

Proposition 2.13. *Let R be a right strictly GPP ring with a primitive idempotent e. If $x, y \in R$ with $xey = 0$, then $xe = 0$ or $ey = 0$.*

Proof. Assume that $xe \neq 0$. Since R is right strictly GPP, there exists a positive integer n such that $(xe)^n \neq 0$ and $(xe)^n R$ is projective. Define $\varphi : eR \to (xe)^n R$ by $\varphi(er) = (xe)^n r$ for all $r \in R$. It is easy to see that $\varphi \neq 0$ and φ is an epimorphism. We have $eR/Ker\varphi \cong (xe)^n R$, and hence $eR/Ker\varphi$ is projective. This implies that $Ker\varphi$ is a direct summand of eR. But eR is indecomposable and $\varphi \neq 0$, so $Ker\varphi = 0$. Since $ey \in Ker\varphi$, it follows that $ey = 0$, proving our proposition. ∎

Given a ring R and an idempotent e in R, if R is Baer (resp. right PP, right GPP) then so is eRe. These are the results of Kaplansky [10, Theorem 4], Colby and Rutter [3, Lemma 2.5] and Huh, Kim and Lee [8, Proposition 9]. Here, using the proof of [8, Proposition 9] (also of [10, Theorem 4]), we obtain the following result for right strictly GPP rings.

Proposition 2.14. *For any idempotent e of R, if R is a right strictly GPP ring, then so is eRe. In particular, if e is a primitive idempotent of R, then eRe*

(and also eR and Re) have no nonzero nilpotent elements, i.e., eRe is a right PP ring.

Proof. Let $0 \neq x = ere \in eRe$. Since R is right strictly GPP, there exists a positive n such that $x^n \neq 0$ and $r_R(x^n) = fR$ for some $f^2 = f \in R$. Then, $r_{eRe}(x^n) = fR \cap eRe$. Put $g = ef$. It can be checked that g is an idempotent of eRe and $r_{eRe}(x^n) = geRe$, as desired (see the proof of [8, Proposition 9]).

Now, let e be a primitive idempotent. Then eRe is a ring with only nonzero idempotent is e. Consequently, eRe is abelian and hence, it is a reduced PP ring by Proposition 2.7. To complete the proof, let er be a nilpotent element of eR with index n. Then, $(ere)^n = (er)^n e = 0$ and hence, ere is also a nilpotent element of eRe. But eRe is reduced, ere must be zero and so is er by Proposition 2.13. Thus, eR has no nonzero nilpotent elements. By symmetry, Re has no nonzero nilpotent elements.

■

A ring R is said to have *enough idempotents* if the identity element can be written as the sum of finite number of orthogonal primitive idempotents. If a ring R has no infinite sets of orthogonal idempotents, then R has enough idempotents, but the converse is not true (see, e.g. [2]). Gordon and Small [2, Lemma 8.6] showed that, if K is a nil right ideal of a right (or left) PP ring with enough idempotents, then K is nilpotent. But we now show that for right (or left) strictly GPP rings, any nil right ideal K must be zero.

Proposition 2.15. *If R is a right strictly GPP ring with enough idempotents then R has no nonzero nil right (or left) ideals.*

Proof. Let $\{e_1, ..., e_m\}$ be the set of orthogonal primitive idempotents of R with $e_1 + ... + e_m = 1$ and I be a nil right ideal of R. If $x \in I$, then $xe_i \in I$ is also nilpotent. By the proof of Proposition 2.14, xe_i must be zero and hence, $Ie_i = 0$ for $i = 1, .., m$. It follows that $I = I(e_1 + ... + e_m) = 0$. By the same argument (on the left) for nil left ideals we complete the proof. ■

The next result extends a result of Small [16, Theorem 1] and also give more affirmative answer for Question 2.3 by using the following lemma.

Lemma 2.16. ([14, Proposition 4]) *Let R be a right GPP ring in which there is no infinite set of orthogonal idempotents. Then for any left annihilator L, there is an idempotent e such that $L = Re \oplus (L \cap R(1-e))$ and $L \cap R(1-e)$ is nil.* ■

Corollary 2.17. *Let R be a ring which has no infinite sets of orthogonal idempotents. Then the following conditions are equivalent:*

(1) *R is Baer.*

(2) *R is right PP.*

(3) *R is left PP.*

(4) *R is right strictly GPP.*

(5) *R is left strictly GPP.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (4) are obvious, and by the symmetry in (1), it suffices to show (4) $\Rightarrow$ (1). Let R be a right strictly GPP ring and $S \subseteq R$. By Lemma 2.16, there exists an idempotent $e \in R$ such that $l(S) = Re \oplus (l(S) \cap R(1-e))$ and $l(S) \cap R(1-e)$ is nil. But R has no nonzero nil left (or right) ideals by Proposition 2.15, hence $l(S) = Re$, as desired. ∎

3. Characterizations of von Neumann Regular Rings

Recall that an element a in a ring R is called *von Neumann regular* if there exists an element $b \in R$ such that $a = aba$; and it is called *π-regular* if there exists a positive integer n (depending on x) and an element $b \in R$ such that $a^n = a^n b a^n$. A ring R is called *von Neumann regular* (resp. *π-regular*) if for any $a \in R$, a is von Neumann regular (resp. π-regular). A ring R is called *semiregular* if R/J is von Neumann regular and idempotents can be lifted modulo J, or equivalently if for each $a \in R$ there exists $e^2 = e \in aR$ such that $(1-e)a \in J$. In [18], an element a of a ring R is called *right generalized semiregular* if there exist two left ideals P, L of R such that $lr(a) = P \oplus L$, where $P \leq Ra$ and $Ra \cap L$ is small in R. A ring R is called right generalized semiregular if each of its elements is right generalized semiregular. A ring R is von Neumann regular if and only if R is right (resp. left) PP and right (resp. left) P-injective by Satyanarayana ([15, Theorem 1]), if and only if it is a right strictly GPP, right generalized semiregular with $J \leq Z_r$ by Xiao and Tong ([18, Theorem 3.12]). Note that in [18], the strictly GPP rings are only mentioned in terms of a condition that they called $(*)$. It is well-known that right P-injective rings are right C2 and by [18, Theorem 2.3(1)], a right generalized semiregular with $J \leq Z_r$ is also right C2. Now, we have a generalization of this result but firstly, we need the following lemma by a direct calculation.

Lemma 3.1. *Let a, b are elements of a ring R. If $a - aba$ is von Neumann regular, then so is a.*

Theorem 3.2. *Let R be a ring. Then the following are equivalent:*

(i) *R is von Neumann regular;*

(ii) *R is right strictly GPP and right C2;*

(iii) *R is left strictly GPP and left C2.*

Proof. By the symmetry, it suffices to prove (ii) $\Rightarrow$ (i).

Let $0 \neq x \in R$. Since R is right strictly GPP, there exists a positive integer n such that $x^n \neq 0$ and $r(x^n) = eR$ for some $e^2 = e \in R$. So $x^n R \cong R/r(x^n) \cong (1-e)R$. Since R satisfies C2, $x^n R$ is a direct summand of R and hence, x^n is von Neumann regular. If $n = 1$, then we are done. Otherwise, we show that x^{n-1} is also von Neumann regular, and by the induction, so is $x^1 = x$.

Indeed, let $c \in R$ be such that $x^n = x^n c x^n$. Put $y = x^{n-1} - x^{n-1}(cx)x^{n-1} =$

$x^{n-1} - x^{n-1}cx^n$. It is easy to check that $y^2 = 0$. Consider the following two cases

Case 1: $y = 0$. Then $x^{n-1} = x^{n-1}cx^n$, which implies that x^{n-1} is von Neumann regular.

Case 2: $y \neq 0$. By the similar argument of the proof above and from $y^2 = 0$, we see that y is von Neumann regular. Thus, x^{n-1} is also von Neumann regular by Lemma 3.1. ∎

Theorem 3.2 gives new characterizations of von Neumann regular rings. Moreover, the following is an immediate consequence.

Corollary 3.3. *Let R be a ring. Then the following are equivalent:*

(1) *R is von Neumann regular;*

(2) *R is right PP and right C2;*

(3) *R is right PP and right P-injective;*

(4) *R is right strictly GPP and right P-injective;*

(5) *R is left PP and left P-injective;*

(6) *R is left PP and left C2;*

(7) *R is left strictly GPP and left P-injective.*

The following result also extends a result of Xiao and Tong [18, Corollary 3.15]. It is obtained immediately from the proof of Theorem 3.2.

Corollary 3.4. *If R is a right (resp. left) GPP and right (resp. left) C2 ring, then R is π-regular.*

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