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Sensitivity Analysis of an Upper-linear-return Consciously Co-operative Economy of a Firm

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Abstract. We consider a firm, in which the activities can be partitioned into the activities for private goods and the activities for public goods. An economy of the firm is to indicate the value of the activities for private goods and the value of the activities for public goods that maximize the total value of the activities and assign a given value for the firm utility. In this article we study the conscious co-operativeness and the property of upper-linear return of the economy. The further analysis would reveal other behaviours of the economy.

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1. Introduction

The problem that we are concerned with consists of a firm in which the activities can be partitioned into the activities for private goods and the activities for public goods. The firm activities consume a given set of resources. An economy of the firm is to indicate the value of the activities for private goods and the value of the activities for public goods that maximize the total value of the activities and assign a given value for the firm utility. The different equilibrium characteristics of private goods and public goods were studied in various economic models (cf. [6, 1, 2]). In this article the analysis of the economy concerning private goods and public goods is motivated by the study of the firm behaviours that show the ability of return and the relationship between the activities for the private goods and the public goods. For the conscious co-operativeness of the economy there are several interpretations. In the first alternative, the economy can be described as a problem of maximizing the value of the activities for public goods with the lower bound constraint of the firm utility. In the second alternative, the economy can be described as a problem of maximizing the sum of the firm utility and the weighted value of the activities for public goods. The study of the co-operativeness in the economic theory, especially in games, could be appreciated in more general contexts (cf. [4, 3, 2]). Using the formulation of the economy that exposes the conscious co-operativeness we can analyse the balance and the imbalance of the set of consumed resources of the economy. The consequence of the imbalance of the set of resources ranges from the infeasibility to the inefficiency of the economy. Some recent studies of the production functions with the balanced set of resources in the framework of nonconvexity can be found in [7]. With the compatible increase of the firm utility we can prove that the economy yields the property of upper-linear return. The activities for private goods remain bounded even when the set of resources approaches the unboundedness. So, the value of the activities for private goods can be negligeable with respect to (briefly, w.r.t.) the value of the activities for public goods with great enough resources.

Section 2 formulates the economy of a firm. Section 3 presents the conscious co-operativeness of the economy. Section 4 is devoted to the study on the balance of the set of resources. Section 5 shows the property of upper-linear return of the economy. Finally, Sec. 6 draws several concluding remarks.

2. The Economy of a Firm

The activities of the firm under our consideration are indicated by an n-dimensional vector $x = (x_1, x_2, \dots, x_n)$ where x_i ($x_i \geq 0$) indicates the value of the i-th activity for any $i \in \{1, 2, \dots, n\}$. The firm activities consume m types of resources. For any $j \in \{1, 2, \dots, m\}$ the j-th resource consumed by the activity vector x is $h_j(x)$, where h_j is an increasing convex homogeneous function defined on R^n_+. Let $d_1, d_2, \dots, d_m$ be positive values of the first resource, the second resource,..., the m^{th} resource, respectively, and denote by $d = (d_1, d_2, \dots, d_m)$ a resource vector. An activity vector x is called feasible to d if for any $j \in \{1, 2, \dots, m\}$ the j-th resource consumed by x is not greater than d_j:

$$h_j(x) \ \leq \ d_j \quad j = 1, 2, \dots, m.$$

Denote by $X(d)$ the set of the activity vectors that maximize the sum of the values of the activities over the set of feasible activity vectors :

$$X(d) \ = \ \left\{ \overline{x} \geq 0 : \ \sum_{i=1}^{n} \overline{x}_i = \theta(d), \ h_j(\overline{x}) \leq d_j \quad j = 1, 2, \dots, m \right\},$$

where

$$\theta(d) \ = \ \max \left\{ \sum_{i=1}^{n} x_i : \ h_j(x) \leq d_j \quad j = 1, 2, \dots, m, \ x \geq 0 \right\}.$$

It is obvious that $\theta(d)$ is positive : $\theta(d) > 0$. Let t be a positive number such that $t < \theta(d)$.

For any $i \in \{1, 2, \ldots, n\}$ the i-th activity x_i can be partitioned into the i-th activity x_i^1 for private goods and the i-th activity x_i^2 for public goods:

$$x_i = x_i^1 + x_i^2 \quad i = 1, 2, \ldots, n.$$

The private goods are the goods for the use of the firm only. The examples of private goods that can be taken into account are, for instance, foods, beverages, clothes... The public goods are the goods such that the firm use of these goods does not preclude the public use of them. The examples of public goods are, for instance, broadcasting, military defence, road networks... Denote

$$\begin{aligned} x^1 &= (x_1^1, x_2^1, \ldots, x_n^1), \\ \theta_1 &= \sum_{i=1}^n x_i^1, \\ x^2 &= (x_1^2, x_2^2, \ldots, x_n^2), \\ \theta_2 &= \sum_{i=1}^n x_i^2. \end{aligned}$$

The firm utility is a function of the value θ_1 of the activities for private goods and the value θ_2 of the activities for public goods defined as follows

$$\theta_1 + p(\theta_2),$$

where p is an increasing differentiable concave function defined on R_+ such that

$$\begin{aligned} &p(0) = 0, \\ &p(\eta) < t \quad \forall \eta \geq 0, \\ &p'(\eta) < 1 \quad \forall \eta > 0, \end{aligned}$$

where $p'(\eta)$ denotes the derivative of p at η. The property of the utility function tells that in order to increase the firm utility it is natural to increase θ_1 rather than to increase θ_2. Furthermore for fixed θ_2 the values of the firm utility are unbounded with the unbounded values of θ_1, but for fixed θ_1 the values of the firm utility are bounded even with the unbounded values of θ_2.

The economy of the firm is represented as a problem of finding the activity vector x^1 for private goods and the activity vector x^2 for public goods such that $x^1 + x^2 \in X(d)$ and $\theta_1 + p(\theta_2) = t$:

$$\begin{aligned} &x^1 + x^2 \in X(d), \ x^1 \geq 0, \ x^2 \geq 0, \\ &\theta_1 = \sum_{i=1}^n x_i^1, \ \theta_2 = \sum_{i=1}^n x_i^2, \theta_1 + p(\theta_2) = t. \end{aligned} \qquad (1)$$

Theorem 2.1. *The problem* (1) *is solvable, and the values of* θ_1 *and* θ_2 *for any solution of it are unique.*

Proof. Let $x \in X(d)$ and for any $\rho \in [0, 1]$ we define

$$x^{1,\rho} = \rho x,$$
$$x^{2,\rho} = (1-\rho)x.$$

Then, $x^{1,\ \rho} + x^{2,\ \rho} = x$ for any $\rho \in [0, 1]$. By setting

$$\theta_{1,\ \rho} = \sum_{i=1}^{n} x_i^{1,\ \rho},$$
$$\theta_{2,\ \rho} = \sum_{i=1}^{n} x_i^{2,\ \rho},$$

we have

$$\theta_{1,\ 0} + p(\theta_{2,\ 0}) = p(\theta_{2,\ 0}) < t,$$
$$\theta_{1,\ 1} + p(\theta_{2,\ 1}) = \theta_{1,\ 1} = \theta(d) > t.$$

Since $\theta_{1,\ \rho}$ and $\theta_{2,\ \rho}$ are continuous in ρ, this implies that there is $\overline{\rho} \in [0, 1]$ such that

$$\theta_{1,\ \overline{\rho}} + p(\theta_{2,\ \overline{\rho}}) \ = \ t.$$

Thus, we obtain a solution $(x^{1,\ \overline{\rho}}, x^{2,\ \overline{\rho}}, \theta_{1,\ \overline{\rho}}, \theta_{2,\ \overline{\rho}})$ of the problem (1). So, the problem (1) is solvable. Since

$$\theta_1 \ = \ \theta(d) - \theta_2, \tag{2}$$

if θ_2 is a solution of the problem (1) then it is a root of the following equation

$$p(\theta_2) - \theta_2 \ = \ t - \theta(d). \tag{3}$$

Since the derivative of the function $p(\theta_2) - \theta_2$ at θ_2 is $p'(\theta_2) - 1$ that is less than 0, the function $p(\theta_2) - \theta_2$ is concave, decreasing in $\theta_2 \in R_+$. So, the equation (3) yields the unique root $\overline{\theta}_2$, hence the solution $\overline{\theta}_2$ of the problem (1) is unique. This together with (2) implies that the solution $\overline{\theta}_1$ of the problem (1) is also unique. ■

3. The Conscious Co-operativeness of the Economy

As discussed in the last section the economy of the firm yields the unique value $\overline{\theta}_2$ of the activities for public goods that assigns the value t to the firm utility. The condition, in which the firm utility is assigned to the value t, reflexes the consciousness of the economy that ensures the utility value t for any solution of the economy. For any feasible activity vector $x \notin X(d)$ and $x^1 \geq 0$, $x^2 \geq 0$ such that $x^1 + x^2 = x$ we have

$$\theta_1 + \theta_2 \; = \; \theta \; < \; \theta(d),$$

hence the equation

$$p(\theta_2) - \theta_2 \; = \; t - \theta$$

yields the root θ_2' that is less than $\overline{\theta}_2$: $\theta_2' < \overline{\theta}_2$. Thus, the condition $x^1 + x^2 \in X(d)$ of the economy ensures the maximum value of the activities for public goods. This shows the co-operativeness of the economy. Now we consider the following alternative formulation of the economy

$$\begin{aligned} &\max \theta_2, \\ &\text{s.t. } \theta_1 + p(\theta_2) \; \geq \; t, \\ &\quad \theta_1 \; = \; \sum_{i=1}^{n} x_i^1, \; \theta_2 \; = \; \sum_{i=1}^{n} x_i^2, \\ &\quad x \; = \; x^1 + x^2, \; x^1 \; \geq \; 0, \; x^2 \; \geq \; 0, \\ &\quad h_j(x) \; \leq \; d_j \quad j = 1, 2, \ldots, m. \end{aligned} \tag{4}$$

This is a convex program.

Theorem 3.1. *The problems* (1) *and* (4) *are equivalent. More concretely,* $(\overline{x}^1, \overline{x}^2, \overline{\theta}_1, \overline{\theta}_2)$ *is an optimal solution of the problem* (4) *if and only if it is a solution of the problem* (1).

Proof. Let $(\overline{x}^1, \overline{x}^2, \overline{\theta}_1, \overline{\theta}_2)$ be a feasible solution of (4). Set $\theta = \overline{\theta}_1 + \overline{\theta}_2$. Then

$$\theta \; \leq \; \theta(d). \tag{5}$$

Set $t' = \overline{\theta}_1 + p(\overline{\theta}_2)$. Then

$$t' \; \geq \; t. \tag{6}$$

Since $\overline{\theta}_2$ is the unique root of the following equation of θ_2

$$p(\theta_2) - \theta_2 \; = \; t' - \theta,$$

from (5) and (6) it follows that $\overline{\theta}_2$ is maximum if and only if $\theta = \theta(d)$ and $t' = t$. Thus, $\overline{\theta}_2$ is maximum if and only if $(\overline{x}^1, \overline{x}^2, \overline{\theta}_1, \overline{\theta}_2)$ is a solution of the problem (1). ■

Regarding to the interpretation (4) the concious co-operativeness of the economy is interpreted in the objective that maximizes the value of the activities for public goods and in the constraint where the firm utility is not less than the value t. In the sequel we present another interpretation of the economy where the concious co-operativeness can be interpreted in only the objective that maximizes the sum of the firm utility and the weighted value of the activities for public goods.

Theorem 3.2. *The problem* (1) *is equivalent to the following problem*

$$\begin{aligned} \max\ & \mu_d\theta_2 + \theta_1 + p(\theta_2), \\ \text{s.t.}\ \theta_1 &= \sum_{i=1}^n x_i^1,\ \theta_2 = \sum_{i=1}^n x_i^2, \\ x &= x^1 + x^2,\ x^1 \ge 0,\ x^2 \ge 0, \\ h_j(x) &\le d_j \quad j = 1, 2, \ldots, m, \end{aligned} \tag{7}$$

where $\mu_d = 1 - p'(\overline{\theta}_2)$ *and* $\overline{\theta}_2$ *is the solution of the problem (1). More concretely,* $(\overline{x}^1, \overline{x}^2, \overline{\theta}_1, \overline{\theta}_2)$ *is optimal solution of the problem* (7) *if and only if it is a solution of the problem* (1).

Proof. Let $(\overline{x}^1, \overline{x}^2, \overline{\theta}_1, \overline{\theta}_2)$ be a solution of the problem (1). By Theorem 3.1 the problem (1) is equivalent to the problem (4). So, there is $\eta \ge 0$ that is a Lagrange multiplier of the constraint $\theta_1 + p(\theta_2) \ge t$ such that an optimal solution of (4) is also optimal to the following problem

$$\begin{aligned} \max\ & \theta_2 + \eta(\theta_1 + p(\theta_2)), \\ \text{s.t.}\ \theta_1 &= \sum_{i=1}^n x_i^1,\ \theta_2 = \sum_{i=1}^n x_i^2, \\ x &= x^1 + x^2,\ x^1 \ge 0,\ x^2 \ge 0, \\ h_j(x) &\le d_j \quad j = 1, 2, \ldots, m, \end{aligned} \tag{8}$$

If $\eta = 0$ then $\overline{\theta}_1$ equals 0 and $\overline{\theta}_2$ equals $\theta(d)$. This is contradictory, because

$$t = \overline{\theta}_1 + p(\overline{\theta}_2) = p(\overline{\theta}_2).$$

So, $\eta > 0$. Since p is concave and $\overline{\theta}_1, \overline{\theta}_2$ are solutions of (8), they are also optimal solutions of the following problem

$$\begin{aligned} \max\ & \theta_2 + \eta\theta_1 + \eta p'(\overline{\theta}_2)\theta_2, \\ \text{s.t.}\ \theta_1 &= \sum_{i=1}^n x_i^1,\ \theta_2 = \sum_{i=1}^n x_i^2, \\ x &= x^1 + x^2,\ x^1 \ge 0,\ x^2 \ge 0, \\ h_j(x) &\le d_j \quad j = 1, 2, \ldots, m. \end{aligned} \tag{9}$$

If

$$\eta p'(\overline{\theta}_2) + 1 > \eta,$$

then the optimal values of θ_1 and θ_2 of (9) must be 0 and $\theta(d)$, respectively. This conflicts with the fact that $\overline{\theta}_1$ and $\overline{\theta}_2$ are solutions of (1). If $\eta p'(\overline{\theta}_2) + 1 < \eta$, then the optimal values of θ_1 and θ_2 of (9) must be $\theta(d)$ and 0, respectively. This also conflicts with the fact that $\overline{\theta}_1$ and $\overline{\theta}_2$ are optimal solutions of (1). So, $\eta p'(\overline{\theta}_2) + 1 = \eta$, or equivalently $\eta = \frac{1}{1 - p'(\overline{\theta}_2)} = \frac{1}{\mu_d}$.

This together with the fact that $(\overline{x}^1, \overline{x}^2, \overline{\theta}_1, \overline{\theta}_2)$ is optimal to (8) implies that it is also optimal to (7). Thus, we have proved that any solution of the problem (1) is optimal to the problem (7). Since $\overline{\theta}_1 = t - p(\overline{\theta}_2)$, the optimal value of (7) is $\mu_d\overline{\theta}_2 + t - p(\overline{\theta}_2) + p(\overline{\theta}_2) = \mu_d\overline{\theta}_2 + t$.

Now suppose that $(\tilde{x}^1, \tilde{x}^2, \tilde{\theta}_1, \tilde{\theta}_2)$ is an optimal solution of the problem (7). Then

$$\mu_d\tilde{\theta}_2 + \tilde{\theta}_1 + p(\tilde{\theta}_2) = \mu_d\overline{\theta}_2 + t. \tag{10}$$

Since $(\tilde{x}^1, \tilde{x}^2, \tilde{\theta}_1, \tilde{\theta}_2)$ is optimal to (8) with $\eta = 1/\mu_d$, it is also optimal to (9) with $\eta = 1/\mu_d$. However, for $\eta = 1/\mu_d$ the problem (9) is as follows

$$\begin{aligned}
&\max \eta(\theta_1 + \theta_2),\\
&\text{s.t. } \theta_1 = \sum_{i=1}^{n} x_i^1,\ \theta_2 = \sum_{i=1}^{n} x_i^2,\\
&\qquad x = x^1 + x^2,\ x^1 \geq 0,\ x^2 \geq 0,\\
&\qquad h_j(x) \leq d_j \quad j = 1, 2, \dots, m.
\end{aligned}$$

Thus, $\tilde{x}^1 + \tilde{x}^2 \in X(d)$, i.e., $\tilde{\theta}_1 + \tilde{\theta}_2 = \theta(d)$. So, $\tilde{\theta}_1 = \theta(d) - \tilde{\theta}_2,\ 0 \leq \tilde{\theta}_2 \leq \theta(d)$.

Replacing $\tilde{\theta}_1$ by $\theta(d) - \tilde{\theta}_2$ in (10) we obtain

$$\mu_d\tilde{\theta}_2 + \theta(d) - \tilde{\theta}_2 + p(\tilde{\theta}_2) = \mu_d\overline{\theta}_2 + t,$$

or equivalently,

$$\mu_d\tilde{\theta}_2 - \tilde{\theta}_2 + p(\tilde{\theta}_2) = \mu_d\overline{\theta}_2 + t - \theta(d).$$

From $\mu_d = 1 - p'(\overline{\theta}_2)$, $t = \overline{\theta}_1 + p(\overline{\theta}_2)$, and $\theta(d) = \overline{\theta}_1 + \overline{\theta}_2$ it follows that

$$\begin{aligned}
\mu_d\tilde{\theta}_2 - \tilde{\theta}_2 + p(\tilde{\theta}_2) &= (1 - p'(\overline{\theta}_2))\overline{\theta}_2 + \overline{\theta}_1 + p(\overline{\theta}_2) - \overline{\theta}_1 - \overline{\theta}_2\\
&= (1 - p'(\overline{\theta}_2))\overline{\theta}_2 - \overline{\theta}_2 + p(\overline{\theta}_2)\\
&= p(\overline{\theta}_2) - p'(\overline{\theta}_2)\overline{\theta}_2\\
&= p(\overline{\theta}_2) + p'(\overline{\theta}_2)(\tilde{\theta}_2 - \overline{\theta}_2) - p'(\overline{\theta}_2)\tilde{\theta}_2.
\end{aligned}$$

Since p is strictly concave, this implies that

$$\mu_d\tilde{\theta}_2 - \tilde{\theta}_2 + p(\tilde{\theta}_2) \geq p(\tilde{\theta}_2) - p'(\overline{\theta}_2)\tilde{\theta}_2,$$

and the equality holds only if $\tilde{\theta}_2 = \overline{\theta}_2$. However,

$$\begin{aligned}
\mu_d\tilde{\theta}_2 - \tilde{\theta}_2 + p(\tilde{\theta}_2) &= (1 - p'(\overline{\theta}))\tilde{\theta}_2 - \tilde{\theta}_2 + p(\tilde{\theta}_2)\\
&= p(\tilde{\theta}_2) - p'(\overline{\theta}_2)\tilde{\theta}_2.
\end{aligned}$$

Therefore, $\tilde{\theta}_2 = \overline{\theta}_2$, consequently, $\tilde{\theta}_1 = \overline{\theta}_1$. So, $(\tilde{x}^1, \tilde{x}^2, \tilde{\theta}_1, \tilde{\theta}_2)$ solves (1). ∎

In the above proof we can see that μ_d is the unique optimal Lagrange multiplier of the constraint in which the firm utility is not less than t.

4. The Balance of the Set of Resources

In order to obtain the value θ_2 of the activities for public goods the economy consumes the resources $d_1, d_2, \ldots, d_m$. Suppose that the j-th resource can be converted from a special resource that is called the scarce resource for any $j = 1, 2, \ldots, m$. Let α_j be the conversion coefficient of the j-th resource, and assume that $\alpha_j > 0$ for any $j = 1, 2, \ldots, m$. In order to produce the value d_j of the j-th resource we need the value $\alpha_j d_j$ of the scarce resource. Set

$$\gamma \ = \ \sum_{i=1}^{m} \alpha_j d_j.$$

So, γ is the value of the scarce resource consumed by the economy. A vector $d' = (d'_1, d'_2, \ldots, d'_m) \in R^m_+$ of resources is called feasible w.r.t. γ if the value of the scarce resource converted from $d'_1, d'_2, \ldots, d'_m$ is γ, $\gamma \ = \ \sum_{j=1}^{m} \alpha_j d'_j$.

A resource vector d' is called feasible w.r.t. the economy if the problem (1), where d is replaced by d', is solvable. The resource vector d' is called balanced w.r.t. γ (briefly, balanced) if it is feasible w.r.t. γ, feasible w.r.t. the economy, and any solution $(\overline{x}^1, \overline{x}^2)$ of the problem (1), where d is replaced by d', consumes all resources:

$$h_j(\overline{x}) \ = \ d'_j \quad j = 1, 2, \ldots, m,$$

where $\overline{x} = \overline{x}^1 + \overline{x}^2$.

Let us consider the following problem

$$\begin{aligned}
&\max \theta_2, \\
&\text{s.t. } \theta_1 + p(\theta_2) \ \geq \ t, \\
&\quad \theta_1 \ = \ \sum_{i=1}^{n} x_i^1, \ \theta_2 \ = \ \sum_{i=1}^{n} x_i^2, \\
&\quad x \ = \ x^1 + x^2, \ x^1 \ \geq \ 0, \ x^2 \ \geq \ 0, \\
&\quad h_j(x) \ \leq \ d'_j, \ d'_j \ \geq \ 0 \quad j = 1, 2, \ldots, m, \\
&\quad \sum_{j=1}^{m} \alpha_j d'_j \ = \ \gamma.
\end{aligned} \tag{11}$$

This problem is solvable because of the boundedness of its feasible domain which contains d.

Theorem 4.1. *The resource vector $d^* = (d_1^*, d_2^*, \ldots, d_m^*)$ is balanced w.r.t. γ if and only if there are vectors $\overline{x}^1$ and $\overline{x}^2$ such that $\overline{x}^1$, $\overline{x}^2$ and d^* are optimal to the problem* (11).

Before proving this theorem we consider the following Lemma.

Lemma 4.1. *A resource vector d^* feasible w.r.t. γ is a balanced resource vector if and only if there is a vector $\overline{x}$ such that $\overline{x}$ and d^* are optimal to the following problem*

$$\begin{aligned} &\max \sum_{i=1}^{n} x_i, \\ &\text{s.t. } h_j(x) \ \leq \ d'_j, \ d'_j \ \geq 0 \quad j = 1, 2, \ldots, m, \\ &\qquad \sum_{j=1}^{m} \alpha_j d'_j \ = \ \gamma, \ x \geq 0. \end{aligned} \tag{12}$$

Proof. Let $\theta(d')$ be the optimal value of the following problem

$$\max \sum_{i=1}^{n} x_i, \text{ s.t. } h_j(x) \leq d'_j \ \ j = 1, 2, \ldots, m, \ x \geq 0. \tag{13}$$

Since this is a concave maximization problem depending on the right hand side, $\theta(\cdot)$ is concave in $d' \in R^m_+$ (cf. [5]). The problem (12) can be rewritten as

$$\max \ \theta(d'), \text{ s.t. } \sum_{j=1}^{m} \alpha_j d'_j = \gamma, \ d'_j \geq 0 \ \ j = 1, 2, \ldots, m. \tag{14}$$

Suppose that $\overline{x}$ and d^* are optimal to (12). Then, d^* is optimal to (14). Hence, $\theta(d^*) \geq \theta(d) > t$. Therefore, by Theorem 2.1, the problem (1), where d is replaced by d^*, is solvable. So, d^* is feasible w.r.t. the economy. Assume that d^* is not balanced. Let $\overline{x}^1$ and $\overline{x}^2$ be a solution of (1), where d is replaced by d^*, such that $h_k(\overline{x}) < d^*_k$ for some $k \in \{1, 2, \ldots, m\}$ and $\overline{x} = \overline{x}^1 + \overline{x}^2$. Let $\tilde{d}$ be a resource vector defined as follows

$$\begin{aligned} \tilde{d}_k &= \frac{1}{2} \ (d^*_k + h_k(\overline{x})), \\ \tilde{d}_j &= \frac{1}{2(m-1)} \ (d^*_k - h_k(\overline{x})) + d^*_j \quad j \neq k, \ j = 1, 2, \ldots, m. \end{aligned}$$

Then $\tilde{d}$ is feasible w.r.t. γ, because

$$\sum_{j=1}^{m} \tilde{d}_j \ = \ \sum_{j=1}^{m} d^*_j \ = \ \gamma.$$

However,

$$h_j(\overline{x}) \ < \ \tilde{d}_j \quad j = 1, 2, \ldots, m.$$

Therefore, there is $\xi > 1$ such that

$$h_j(\xi \overline{x}) \ < \ \tilde{d}_j \quad j = 1, 2, \ldots, m.$$

Hence,

$$\theta(\tilde{d}) \geq \sum_{i=1}^{n} \xi \overline{x}_i = \xi \sum_{i=1}^{n} \overline{x}_i > \sum_{i=1}^{n} \overline{x}_i = \theta(d^*).$$

This conflicts with the fact that d^* solves (14).

Conversely suppose that d^* is a balanced resource vector. Then any optimal solution x^* of (13), in which d' is replaced by d_j^*, satisfies

$$h_j(x^*) \ = \ d_j^* \quad j = 1, 2, \ldots, m.$$

Consequently, $d^* > 0$. Furthermore,

$$\begin{aligned}
&\sum_{i=1}^{n} x_i^* + \sum_{j=1}^{m} \alpha_j (d_j^* - h_j(x^*)) \\
&= \sum_{i=1}^{n} x_i^* \\
&= \max \left\{ \sum_{i=1}^{n} x_i : \ h_j(x) \leq d_j^* \quad j = 1, 2, \ldots, m, \ x \geq 0 \right\} \\
&= \max_{x \geq 0} \min_{u \geq 0} \left\{ \sum_{i=1}^{n} x_i + \sum_{j=1}^{m} u_j (d_j^* - h_j(x)) \right\}.
\end{aligned}$$

Since the constraint qualification holds in the problem of maximizing $\sum_{i=1}^{n} x_i$ on the domain $\{x \geq 0 : \ h_j(x) \leq d_j^* \ j = 1, 2, \ldots, m\}$, the above maximin value equals the minimax value:

$$\begin{aligned}
&\max_{x \geq 0} \min_{u \geq 0} \left\{ \sum_{i=1}^{n} x_i + \sum_{j=1}^{m} u_j (d_j^* - h_j(x)) \right\} \\
&= \min_{u \geq 0} \max_{x \geq 0} \left\{ \sum_{i=1}^{n} x_i + \sum_{j=1}^{m} u_j (d_j^* - h_j(x)) \right\}
\end{aligned}$$

(cf. [5]). Thus, (x^*, α) is a saddle point of the saddle function

$$\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} u_j (d_j^* - h_j(x))$$

on $R_+^n \times R_+^m$. So, α is an optimal vector of Lagrange multipliers of the m constraints $h_j(x) \ \leq \ d_j^* \ \ j = 1, 2, \ldots, m$, of the problem (13), in which d' is replaced by d^*. Hence, α is a subgradient of θ at d^*. Therefore, d^* is optimal to (14), or equivalently to (12). ∎

The proof of Theorem 4.1. Suppose that d^* is balanced w.r.t. γ. We have by Lemma 4.1

$$\begin{aligned}
&\max\left\{\sum_{i=1}^{n} x_i : \; h_j(x) \le d_j^* \;\; j = 1, 2, \ldots, m, \; x \ge 0\right\} \\
&\ge \max\left\{\sum_{i=1}^{n} x_i : \; h_j(x) \le d_j \;\; j = 1, 2, \ldots, m, \; x \ge 0\right\} \\
&= \theta(d) > t.
\end{aligned}$$

By Theorem 3.2 the folowing problem

$$\begin{aligned}
&\max \theta_2, \\
&\text{s.t. } \theta_1 + p(\theta_2) \;\ge\; t, \\
&\qquad \theta_1 \;=\; \sum_{i=1}^{n} x_i^1, \; \theta_2 \;=\; \sum_{i=1}^{n} x_i^2, \\
&\qquad x \;=\; x^1 + x^2, \; x^1 \ge 0, \; x^2 \ge 0, \\
&\qquad h_j(x) \;\le\; d_j^* \quad j = 1, 2, \ldots, m,
\end{aligned} \tag{15}$$

is equivalent to the following problem

$$\begin{aligned}
&x \;\in\; X(d^*), \\
&\theta_1 \;=\; \sum_{i=1}^{n} x_i^1, \; \theta_2 \;=\; \sum_{i=1}^{n} x_i^2, \; \theta_1 + p(\theta_2) \;=\; t,
\end{aligned} \tag{16}$$

where

$$X(d^*) \;=\; \left\{\overline{x} \ge 0 : \; \sum_{i=1}^{n} \overline{x}_i = \theta(d^*), \; h_j(\overline{x}) \le d_j^* \;\; j = 1, 2, \ldots, m\right\}$$

and $\theta(d^*)$ is the optimal value of the problem (13) in which d' is replaced by d^*. Let $(x^1, x^2, \theta_1, \theta_2, d')$ be any feasible solution of the problem (11). Then, by setting $x = x^1 + x^2$, $\theta' = \sum_{i=1}^{n} x_i$ we have $\theta' \;\le\; \theta(d^*)$. Since $p(\theta_2) - \theta_2 \;\ge\; t - \theta'$, whereas the solution θ_2^* of the problem (18), or equivalently, of the problem (16) is the unique root of the following equation

$$p(\theta_2') - \theta_2' \;=\; t - \theta(d^*),$$

from $\theta(d^*) \;\ge\; \theta'$ it follows that $\theta_2 \;\le\; \theta_2^*$. Thus, θ_2^* is optimal to (11). So, d^* is optimal to the problem (11). Conversely suppose that d^* is optimal to the problem (11) Similar to the above arguments, this implies that $\theta(d^*)$ is maximized, i.e., d^* solves (12). Hence by Lemma 4.1, d^* is balanced. ■

Let $D(\gamma)$ be the set defined as follows

$$D(\gamma) \;=\; \left\{d' \ge 0 : \; \sum_{j=1}^{m} \alpha_j d_j' = \gamma, \; \theta(d') > t\right\}.$$

It is clear that d belongs to $D(\gamma)$, and any balanced resource vector d^* also belongs to $D(\gamma)$. By Theorem 3.1, the problem (1) and the problem (4), where d is replaced by $d' \in D(\gamma)$, are equivalent. A resource vector $d' \in D(\gamma)$ is called imbalanced if it is not balanced. As a consequence of Theorem 4.1 we immediately obtain the following corollary.

Corollary 4.1. *A resource vector $d' \in D(\gamma)$ is imbalanced if and only if the value of the activities for public goods w.r.t. d' is not maximized on $D(\gamma)$.*

We note that the value of the activities for public goods w.r.t. d' is the value of θ_2 in the following problem

$$\begin{aligned} x^1 + x^2 \;\in\; X(d'),\; x^1 \;\geq\; 0,\; x^2 \;\geq\; 0, \\ \theta_1 \;=\; \sum_{i=1}^{n} x_i^1,\; \theta_2 \;=\; \sum_{i=1}^{n} x_i^2,\; \theta_1 + p(\theta_2) \;=\; t, \end{aligned} \tag{17}$$

where

$$X(d') \;=\; \left\{ \overline{x} \geq 0 : \sum_{i=1}^{n} \overline{x}_i \;=\; \theta(d'),\; h_j(\overline{x}) \leq d_j' \;\; j = 1, 2, \ldots, m \right\}.$$

The objective function in the interpretation (7) of the economy is the sum of the firm utility $\theta_1 + p(\theta_2)$ and the weighted value $\mu_d\theta_2$ of the activities for public goods. The higher the weight μ_d is, the more co-operative the economy is.

Theorem 4.2. *A resource vector $d' \in D(\gamma)$ is imbalanced if and only if the weight $\mu_{d'}$ of the value of the activities for public goods in the problem* (7)*, where d is replaced by d', is not maximized on $D(\gamma)$.*

Proof. Denote by θ_2' the value of the activities for public goods w.r.t. d' and by θ_2^* the value of the activities for public goods w.r.t. d^* where d^* is a balanced resource vector. From Theorem 4.1 and Corollary 4.1 it follows that d' is imbalanced if and only if $\theta_2' < \theta_2^*$. Furthermore,

$$\begin{aligned} \theta_2' \;<\; \theta_2^* &\Leftrightarrow p'(\theta_2') \;>\; p'(\theta_2^*) \\ &\Leftrightarrow 1 - p'(\theta_2') \;<\; 1 - p'(\theta_2^*) \\ &\Leftrightarrow \mu_{d'} \;<\; \mu_{d^*}. \end{aligned}$$

This completes the proof. ∎

5. Upper-linear Return of the Economy

Suppose that the value of the activities for public goods can be converted into the scarce resource, namely, the value α_0 of the activities for public goods can be converted into one unit of the scarce resource. Denote by d^* a balanced resource vector w.r.t. γ and by θ_2^* the value of the activities for public goods w.r.t. d^*. From the economic point of view the economy is socially benificial only if the value $\alpha_0\theta_2^*$ substructed by γ remains positive:

$$\delta \ := \ \alpha_0\theta_2^* - \gamma \ > \ 0. \tag{18}$$

From now on we suppose that (18) is satisfied.

Let $\beta \geq 1$, and suppose that the value of the scarce resource consumed by the economy is $\beta\gamma$. By Lemma 4.1 we have

$$\begin{aligned}
&\max\left\{\sum_{i=1}^{n} x_i : \ h_j(x) \leq d_j^* \ \ j=1,2,\ldots,m, \ x \geq 0\right\} \\
&= \max\left\{\sum_{i=1}^{n} x_i : \ h_j(x) \leq d'_j, \ d'_j \geq 0 \ \ j=1,2,\ldots,m, \ \sum_{j=1}^{m} \alpha_j d'_j = \gamma, \ x \geq 0\right\}.
\end{aligned}$$

Since h_j $j=1,2,\ldots,m$ are homogeneous, from this it follows that

$$\begin{aligned}
&\max\left\{\sum_{i=1}^{n} x_i : \ h_j(x) \leq \beta d_j^* \ \ j=1,2,\ldots,m, \ x \geq 0\right\} \\
&= \max\left\{\sum_{i=1}^{n} x_i : \ h_j(x) \leq d'_j, \ d'_j \geq 0 \ \ j=1,2,\ldots,m, \ \sum_{j=1}^{m} \alpha_j d'_j = \beta\gamma, \ x \geq 0\right\}.
\end{aligned}$$

So, again by Lemma 4.1, βd^* is a balanced resource vector w.r.t. $\beta\gamma$. When the economy consumes the value $\beta\gamma$ of the scarce resource, the firm utility value would be $t(\beta)$, where

$$t(\beta) \ = \ \frac{p(\beta\theta_2^*)}{p(\theta_2^*)} \ t \quad \forall \beta \geq 1.$$

It is clear that $t(\beta)$ increases as β increases. The value $t(\beta)$ is called the firm utility compatible to the scarce resource value $\beta\gamma$ consumed by the economy.

For any $\beta \geq 1$ we consider the economy in the following representation

$$\begin{aligned}
&x^1 + x^2 \ \in \ \beta X(d^*), \\
&\theta_1 \ = \ \sum_{i=1}^{n} x_i^1, \ \theta_2 \ = \ \sum_{i=1}^{n} x_i^2, \ \theta_1 + p(\theta_2) \ = \ t(\beta).
\end{aligned} \tag{19}$$

We would mention that

$$\begin{aligned}
\beta X(d^*) &= \left\{\beta\overline{x} : \ \sum_{i=1}^{n} \overline{x}_i = \theta(d^*), \ h_j(\overline{x}) \leq d_j^* \ \ j=1,2,\ldots,m, \ \overline{x} \geq 0\right\} \\
&= \left\{\overline{x} : \ \sum_{i=1}^{n} \overline{x}_i = \theta(\beta d^*), \ h_j(\overline{x}) \leq \beta d_j^* \ \ j=1,2,\ldots,m, \ \overline{x} \geq 0\right\} \\
&= X(\beta d^*),
\end{aligned}$$

where $\theta(\beta d^*)$ is the optimal value in the problem (13) where d' is replaced by βd^*. Since $\beta p(\theta_2^*) \ \geq \ p(\beta\theta_2^*)$, we have $\beta t \geq t(\beta)$. Therefore, $\theta(\beta d^*) = \beta\theta(d^*) > \beta t \geq t(\beta)$.

Since $\beta X(d^*) = X(\beta d^*)$, by Theorem 3.1 the problem (19) is equivalent to the following problem

$$\begin{aligned} &\max \theta_2, \\ &\text{s.t. } \theta_1 + p(\theta_2) \ \geq \ t(\beta), \\ &\qquad \theta_1 \ = \ \sum_{i=1}^n x_i^1, \ \theta_2 \ = \ \sum_{i=1}^n x_i^2, \\ &\qquad x \ = \ x^1 + x^2, \ x^1 \geq \ 0, \ x^2 \ \geq \ 0, \\ &\qquad h_j(x) \ \leq \ \beta d_j^* \quad j = 1, 2, \dots, m. \end{aligned} \tag{20}$$

Denote by $\theta_2(\beta)$ the optimal of the problem (20) that is the value of the activities for public goods in return of the value $\beta\gamma$ of the consumed scarce resource. The return value $s(\beta)$ of the economy in the representation (20) would be the following

$$s(\beta) \ = \ \alpha_0\theta_2(\beta) - \beta\gamma \quad (\beta \geq 1).$$

Since $\theta_2(\beta)$ is the optimal value of the problem (20), which is a concave maximization in $(\beta, x^1, x^2, \theta_1, \theta_2)$, the function $\theta_2(\cdot)$ defined on $[1, +\infty)$ is concave in β. Therefore, the function $s(\cdot)$ is concave on $[1, +\infty)$.

Theorem 5.1. *The return function $s(\cdot)$ of the economy is increasing and upper-linear, namely*

$$s(\beta) \ \geq \ \beta\delta \quad \forall \beta \geq 1,$$

where δ is defined via (18).

Proof. Let $(x^{*1}, x^{*2}, \theta_1^*, \theta_2^*)$ be an optimal solution of the problem (20) with $\beta = 1$. For any $\beta > 1$ we consider the solution $(\beta x^{*1}, \beta x^{*2}, \beta\theta_1^*, \beta\theta_2^*)$ that will be shown below feasible to the problem (20). Indeed, $h_j(\beta x^{*1} + \beta x^{*2}) = \beta h_j(x^{*1} + x^{*2}) \leq \beta d_j^* \quad j = 1, 2, \dots, m$.

Furthermore,

$$\sum_{i=1}^n \beta x_i^{*1} + p\left(\sum_{i=1}^n \beta x_i^{*2}\right) \ = \ \beta\theta_1^* + p(\beta\theta_2^*).$$

Since $\theta_1^* + p(\theta_2^*) \ = \ t$, we have

$$t(\beta) = \frac{p(\beta\theta_2^*)}{p(\theta_2^*)} \ t = \frac{p(\beta\theta_2^*)}{p(\theta_2^*)} \ (\theta_1^* + p(\theta_2^*)) = \frac{p(\beta\theta_2^*)}{p(\theta_2^*)} \ \theta_1^* + p(\beta\theta_2^*) \leq \beta\theta_1^* + p(\beta\theta_2^*).$$

So, the solution $(\beta x^{*1}, \beta x^{*2}, \beta\theta_1^*, \beta\theta_2^*)$ is feasible to the problem (20). Therefore,

$$s(\beta) = \alpha_0\theta_2(\beta) - \beta\gamma \geq \alpha_0\beta\theta_2^* - \beta\gamma = (\alpha_0\theta_2^* - \gamma)\beta = \delta\beta.$$

Thus, $s(\beta)$ is upper-linear. Since $s(\cdot)$ is concave and $s(\beta) \to +\infty$ as $\beta \to +\infty$, it follows that $s(\cdot)$ is increasing. ■

6. Concluding Remarks

In this article we have studied the economy of a firm in which the activities can be partitioned into the activities for private goods and the activities for public goods. The economy can be represented by indicating the value of the activities for private goods and the value of the activities for public goods that maximize the total value of the activities and assign the value t for the firm utility. The conscious co-operativeness of the economy can be interpreted in several ways. In the first alternative, the economy can be formulated as a problem of maximizing the value of the activities for public goods with the constraint in which the firm utility is not less than the value t. In the second alternative, the economy can be formulated as a problem of maximizing the sum of the firm utility and the weighted value of the activities for public goods. In order to maintain its activities the economy consumes a given set of resources. According to the first alternative, the set of resources is balanced if and only if the value of the activities for public goods is maximized. According to the second alternative, the set of resources is balanced if and only if the weight of the value of the activities for public goods in the objective is maximized. The imbalance of the set of resource yields the consequences that range from the infeasibility to the inefficiency of the economy. With the increase of the firm utility compatible to the increase of the resource the economy yields the property of increasing upper-linear return. The activities for the private goods remain bounded even when the set of resources approaches the unboundedness, because θ_1 is not greater than the value $tp(\beta\theta_2^*)/p(\theta_2^*)$ which is bounded by $t^2/p(\theta_2^*)$. So, the value of the activities for private goods can be negligeable w.r.t. the value of the activities for public goods with great enough resources.

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