Vietnam Journal of Mathematics **36**:4(2008) 463-472

Vietnam Journal
of
MATHEMATICS

Distance from an Exactly Controllable System to Not Approximately Controllable Systems

B. T. Anh[1], D. C. Khanh[2] and D. D. X. Thanh[3]

[1]*Department of Mathematics, University of Pedagogy,*
280 An Duong Vuong Str., HoChiMinh City, Vietnam
[2]*Department of Information Technology, HCM University of Technology,*
144/24 Dien Bien Phu Str., HoChiMinh City, Vietnam [3]*Department of Mathematics, Ton Duc Thang University,*
98 Ngo Tat To Str., HoChiMinh City, Vietnam

Received April 02, 2008

Abstract. Given an exactly controllable time-invariant linear control system on a Hilbert space, the distance from the given system to the set of not approximately controllable systems is the norm of the smallest perturbation that makes the given system not approximately controllable. In this paper, the distances when both or only one of the system operators is perturbed are formulated in terms of optimization problems depending on a complex variable. In some cases, these optimization problems can be reduced to depend on one real variable, as well as the real and complex radii are shown to be equal. The obtained results in this paper also generalize the recent work of [6, 12].

2000 Mathematics Subject Classification: 93B05, 93C25, 93C05.

Key words: Controllability, time-invariant linear systems, distance, Hilbert spaces.

1. Introduction

Let X and U be some Hilbert spaces. In this paper, we consider the following time-invariant linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t \geq 0, \tag{1}$$

where $A \in \mathcal{L}(X)$, and $B \in \mathcal{L}(U, X)$. The problem of the controllability of dynamical systems has attracted attention since 1960s. Exact controllability refers to the ability of a system to transfer the state vector from one specified vector value to another in finite time. In particular, the systems (1) is called *exactly controllable* if for every $x_0, x_1 \in X$, there exist an input vector $u(t)$ and a real $t_1 > 0$ such that $x(0) = x_0$ and $x(t_1) = x_1$. First important contributions are of Kalman and Hautus, see e.g. [1, 8, 11, 18], that the system (1) is exactly controllable if and only if rank $[A - \lambda I,\ B] = n$ for all $\lambda \in \mathbb{C}$. This result is often called *Hautus test*.

It is worth to remark that there is another notion of controllability, *approximate controllability*, which means that for every $x_0, x_1 \in X$ and arbitrary $\varepsilon > 0$, there exist an input vector $u(t)$ and a real $t_1 > 0$ such that $x(0) = x_0$ and $\|x(t_1) - x_1\| < \varepsilon$. From the definitions, it is easy to see that exact controllability implies approximate controllability. For finite dimensional systems, approximate controllability is equivalent to exact controllability. However, for infinite dimensional systems, the equivalence is not true. And some researchers generalized the Hautus test for the case of infinite dimensional systems in [5, 10, 17].

It is proved in [14] that the subset of the operators (A, B) such that the system (1) is approximately controllable is not closed as well as not open in the product topology, but the subset of the operators (A, B) such that the system (1) is exactly controllable is open. This means that the exact controllability property is robust, but the approximate controllability property is not.

Over the past decades, problems of robustness of controllability of the system (1) have attracted a good deal of attention in finite dimensional control theory. One of the most effective and flexible approaches towards those problems is based on the concept of distance from an controllable system to the set of not controllable systems, *shortly, controllability distance*. This concept was first introduced in [6], in which Eising did research on system (1) when both matrices A and B are subjected to unstructured perturbations of the form

$$A \mapsto A + \Delta_A,\ B \mapsto B + \Delta_B.$$

In [2], Boley discussed the problem of computation of controllable space. Then Boley and Wu-Sheng Lu considered the problem of how to extend the small controllability distance by feedback gain in [3]. Guangdi Hu and Edward J. Davison gave formulae for the real controllability distance of the system in [9]. In [12], we studied the controllability distances when only the matrix A or B is perturbed.

However, the investigation of extension of the controllability distance for infinite dimensional systems has not been studied yet. And in this paper, we generalize the results of controllability distances for time-invariant linear systems on Hilbert spaces. The organization of this work is as follows. In the next section, we summarize some notations and give some results which will be used in the remainder. The main results of the work will be presented in Sec. 3. An example is given in Sec. 4 to illustrate the obtained results.

2. Preliminaries

Let X, Y be Hilbert spaces, V be a subspace of X and $M \in \mathcal{L}(X,Y)$, we denote the spectrum of M by $\sigma(M)$, the adjoint operator by M^*, the null subspace by $\mathcal{N}(M)$, the range subspace by $\mathcal{R}(M)$, the projection of X onto V by P_V and

$$s_{\min}(M) := \min\left\{\sqrt{\lambda} : \lambda \in \sigma(MM^*)\right\}.$$

If X, Y are finite dimension spaces, $s_{\min}(M)$ is the smallest singular value of the matrix M. Throughout the paper, we always assume that all spaces considered are Hilbert spaces. Let $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(U,Y)$ then the operator $[A,\ B] \in L(X \times U, Y)$ is defined by

$$[A,\ B]\,(x,u) := Ax + Bu,\ \forall (x,u) \in X \times U.$$

Hautus test is a common approach to derive the controllability distances for finite dimensional systems, see [6, 9, 12]. The following theorem generalizes the Hautus test for systems on infinite dimensional spaces.

Theorem 2.1. [5, 10, 17] *The system* (1) *is*

(*a*) *exactly controllable if and only if*

$$\mathcal{R}\left([A - \lambda I,\ B]\right) = X,\ \forall \lambda \in \mathbb{C};$$

(*b*) *approximately controllable if and only if*

$$\overline{\mathcal{R}\left([A - \lambda I,\ B]\right)} = X,\ \forall \lambda \in \mathbb{C}.$$

From the theory of adjoint operator, we generalize a characteristic of the smallest singular value of matrix M for bounded operator in the following result.

Lemma 2.2. *Let* $A \in \mathcal{L}(X,Y)$*, then*

$$\inf_{\|y\|=1} \|A^*y\| = s_{\min}(A).$$

Proof. By the definition of adjoint operator,

$$\|A^*y\| =< A^*y, A^*y >^{1/2} =< AA^*y, y >^{1/2}.$$

Using a well-known fact that $\inf\{< AA^*y, y >: \|y\| = 1\} = \min\{\lambda : \lambda \in \sigma(AA^*)\}$, see e.g. [16, 4], we get the proof. ■

The continuity of $s_{\min}(.)$ is obtained as a corollary of the above lemma as follows.

Corollary 2.3. *Let A and $B \in \mathcal{L}(X,Y)$, then*

$$s_{\min}(A) - s_{\min}(B) \le \|A - B\|.$$

Proof. For $y \in Y$ with $\|y\| = 1$, we have

$$\|A^*y\| \le \|A^*y - B^*y\| + \|B^*y\| \le \|A^* - B^*\| + \|B^*y\|.$$

Hence,

$$\inf_{\|y\|=1} \|A^*y\| - \inf_{\|y\|=1} \|B^*y\| \le \|A^* - B^*\| = \|A - B\|.$$

Thanks to Lemma 2.2, we are done. ∎

The following theorem characterizes the value $s_{\min}(.)$ via the range subspace of its operator argument.

Theorem 2.4. *Let $A \in \mathcal{L}(X,Y)$, then $s_{\min}(A) > 0$ if and only if $\mathcal{R}(A) = Y$.*

Proof. If $\mathcal{R}(A) \ne Y$, then $\mathcal{R}(AA^*) \ne Y$. This means $0 \in \sigma(AA^*)$, or $s_{\min}(A) = 0$.

Conversely, if $\mathcal{R}(A) = Y$, decompose $A = A_1\pi$ with $\pi : X \longrightarrow X/\mathcal{N}(A)$ being the canonical quotient map and $A_1 : X/\mathcal{N}(A) \longrightarrow Y$ defined by $A_1(\tilde{x}) = Ax$ for any $x \in \tilde{x} \in X/\mathcal{N}(A)$. Then A_1 has bounded inversion by the open mapping theorem, so does A_1^*. Hence, $AA^* = A_1\pi\pi^*A_1^* = A_1A_1^*$ has bounded inversion. And this means $0 \notin \sigma(AA^*)$, or $s_{\min}(A) > 0$. ∎

It is remarkable that when $X = \mathbb{C}^m$ and $Y = \mathbb{C}^n$, Theorem 2.4 collapses to a well-known result that $s_{\min}(A) > 0$ if and only if $\operatorname{rank}(A) = n$. And to give the formulae for controllability distance, we need the following lemma.

Lemma 2.5. *Let $A \in \mathcal{L}(X,Y)$ satisfying $\mathcal{R}(A) = Y$, then*

$$\inf_{\Delta \in \mathcal{L}(X,Y)} \left\{ \|\Delta\| : \overline{\mathcal{R}(A+\Delta)} \ne Y \right\} = s_{\min}(A).$$

Proof. First, the condition $\overline{\mathcal{R}(A+\Delta)} \ne Y$ means $\mathcal{N}(A^* + \Delta^*) \ne \{0\}$. Then, there exists a vector $y \in Y$ with $\|y\| = 1$ satisfying $A^*y = -\Delta^*y$. From this, we get

$$\|\Delta\| = \|\Delta^*\| \ge \inf_{\|y\|=1} \|A^*y\|. \tag{2}$$

By Lemma 2.2, equation (2) implies

$$\inf_{\Delta \in \mathcal{L}(X,Y)} \left\{ \|\Delta\| : \overline{\mathcal{R}(A+\Delta)} \ne Y \right\} \ge s_{\min}(A). \tag{3}$$

Second, for arbitrary $\varepsilon > 0$, Lemma 2.2 implies there exists a vector $y_\varepsilon \in Y$ with $\|y_\varepsilon\| = 1$ satisfying $\|A^*y_\varepsilon\| < s_{\min}(A) + \varepsilon$. By Hahn-Banach theorem, there exists $\Lambda_\varepsilon \in Y^*$ such that $\|\Lambda_\varepsilon\| = 1$ and $\Lambda_\varepsilon y_\varepsilon = 1$. Choose $\Delta_\varepsilon \in \mathcal{L}(X,Y)$ such that $\varDelta_\varepsilon^* y = -(A^* y_\varepsilon)\Lambda_\varepsilon y$ for all $y \in Y$. It is easy to verify that $\|\varDelta_\varepsilon^*\| = \|A^* y_\varepsilon\|$, and $(A^* + \varDelta_\varepsilon^*)y_\varepsilon = 0$ which implies $\overline{\mathcal{R}(A+\varDelta_\varepsilon)} \neq Y$. This means

$$\inf_{\varDelta \in \mathcal{L}(X,Y)} \left\{ \|\varDelta\| : \overline{\mathcal{R}(A+\varDelta)} \neq Y \right\} \leq s_{\min}(A). \tag{4}$$

From (3) and (4) the proof is complete. ■

It is noticeable that Lemma 2.5 collapses to the well-known rank reduction problem when X and Y are finite dimension spaces, for example see [13, p.415], that

$$\inf_{\varDelta \in \mathbb{C}^{n \times m}} \{\|\varDelta\| : \operatorname{rank}(A+\varDelta) \leq n\} = s_{\min}(A).$$

This result is received via singular value decomposition technique, and is the key tool to derive controllability distance in [6, 9].

Using the same approach to solve the problem when the perturbation occurs only on operator A or B of the system (1) as in the paper [12], we generalize Lemma 1.1 in [12] to the following lemma with some change of the format for being suitable with Hilbert space context.

Lemma 2.6. *Let $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(U,Y)$ satisfying $\mathcal{R}[A,\ B] = Y$, then*

$$\inf_{\varDelta \in \mathcal{L}(X,Y)} \left\{ \|\varDelta\| : \overline{\mathcal{R}[A+\varDelta,\ B]} \neq Y \right\} = s_{\min}\left(P_{\mathcal{N}(B^*)}A\right).$$

Proof. The condition $\overline{\mathcal{R}[A+\varDelta,\ B]} \neq Y$ is equivalent to $\overline{\mathcal{R}\left[P_{\mathcal{N}(B^*)}(A+\varDelta)\right]} \neq \mathcal{N}(B^*)$. By setting $\varDelta_{pr} = P_{\mathcal{N}(B^*)}\varDelta \in \mathcal{L}(X, \mathcal{N}(B^*))$, we can rewrite the condition as $\overline{\mathcal{R}\left[P_{\mathcal{N}(B^*)}A + \varDelta_{pr}\right]} \neq \mathcal{N}(B^*)$. Noting that $\|\varDelta_{pr}\| \leq \|\varDelta\|$ and applying Lemma 2.5, we get the proof. ■

3. Controllability Distances

3.1. Perturbation on both *A* and *B*

In this subsection, we assume that both operators A and B are subjected to perturbation of the form

$$A \mapsto A + \varDelta_A, \qquad B \mapsto B + \varDelta_B,$$

where $\varDelta_A \in \mathcal{L}(X)$ and $\varDelta_B \in \mathcal{L}(U,X)$ are unknown operators. Then, the perturbed system of system (1) is

$$\dot{x}(t) = (A+\varDelta_A)x(t) + (B+\varDelta_B)u(t),\ t \geq 0. \tag{5}$$

Definition 3.1. The controllability distance of the system (1) with respect to the perturbation on both A and B is defined by

$$d_{AB} = \inf_{\substack{\Delta_A \in \mathcal{L}(X) \\ \Delta_B \in \mathcal{L}(U,X)}} \{\|\Delta_A,\ \Delta_B\|\} : \text{the perturbed system (5) is not approximately controllable}\}.$$

Theorem 3.2. *Let the system* (1) *be exactly controllable, then*

$$d_{AB} = \min_{\lambda \in C} s_{\min}\left[A - \lambda I,\ B\right].$$

Proof. Using Theorem 2.1 and Lemma 2.5, we get

$$\begin{aligned} d_{AB} = & \inf_{\substack{\Delta_A \in \mathcal{L}(X) \\ \Delta_B \in \mathcal{L}(U,X)}} \left\{\|\Delta_A,\ \Delta_B]\| : \exists \lambda \in C, \overline{\mathcal{R}\left[A + \Delta_A - \lambda I,\ B + \Delta_B\right]} \neq X\right\} \\ = & \inf_{\lambda \in C} \inf_{\substack{\Delta_A \in \mathcal{L}(X) \\ \Delta_B \in \mathcal{L}(U,X)}} \left\{\|\Delta_A,\ \Delta_B]\| : \overline{\mathcal{R}\left[A + \Delta_A - \lambda I,\ B + \Delta_B\right]} \neq X\right\} \\ = & \inf_{\lambda \in C} s_{\min}\left[A - \lambda I,\ B\right]. \end{aligned}$$

Moreover, by Corollary 2.3, we obtain

$$\left|s_{\min}\left[A - \lambda_1 I,\ B\right] - s_{\min}\left[A - \lambda_2 I,\ B\right]\right| \leq \left|\lambda_1 - \lambda_2\right|.$$

This implies the continuity of $s_{\min}\left[A - \lambda I,\ B\right]$ in λ. Noting that $s_{\min}\left[A - \lambda I,\ B\right]$ tends to infinity when $|\lambda|$ tends to infinity, we complete the proof. ■

So the controllability distance is formulated as an optimization problem depending on one complex variable. Employing some special structure of the system operator A, the optimization problem is reduced to depend on one real variable as follows.

Theorem 3.3. *Let the system* (1) *be exactly controllable.*

(a) *If A is Hermitian then* $d_{AB} = \min_{\lambda \in \mathbb{R}} s_{\min}\left[A - \lambda I,\ B\right].$

(b) *If A is skew-Hermitian then* $d_{AB} = \min_{\lambda \in \mathbb{R}} s_{\min}\left[A - i\lambda I,\ B\right].$

Proof.

(a) By Theorem 3.2 and Lemma 2.2, we get

$$d_{AB} = \min_{\lambda \in \mathbb{C}} \min_{\|y\|=1} \sqrt{\|(A^* - \lambda I)y\|^2 + \|B^* y\|^2}.$$

Since A is Hermitian, we have

$$\begin{aligned} \|(A^* - \lambda I)y\|^2 &= \|(A^* - \Re\lambda I)y\|^2 + (\Im\lambda)^2 \|y\|^2 \\ &\geq \|(A^* - \Re\lambda I)y\|^2. \end{aligned}$$

Thus

$$d_{AB} = \min_{\lambda\in\mathbb{R}} \min_{\|y\|=1} \sqrt{\|(A^* - \lambda I)y\|^2 + \|B^* y\|^2} = \min_{\lambda\in\mathbb{R}} s_{\min}\left[A - \lambda I,\ B\right].$$

(b) From the fact that A is skew-Hermitian if and only if iA is Hermitian, we obtain (b).

■

3.2. Perturbation on only *A*

In this subsection, we assume that only the system operator A is subjected to perturbation. Then, the perturbed system of system (1) is

$$\dot{x}(t) = (A + \Delta_A)x(t) + Bu(t), \qquad t \geq 0. \tag{6}$$

Definition 3.4. The controllability distance of the system (1) with respect to perturbation on only A is defined by

$$d_A = \inf_{\Delta_A \in \mathcal{L}(X)} \{\|\Delta_A\| : \text{the perturbed system (6) is not approximately controllable}\}.$$

Theorem 3.5. *Let the system* (1) *be exactly controllable, then*

$$d_A = \min_{\lambda\in C} s_{\min}\left[P_{\mathcal{N}(B^*)}(A - \lambda I)\right].$$

Proof. From Theorem 2.1, we get

$$\begin{aligned} d_A &= \inf_{\Delta_A\in\mathcal{L}(X)} \left\{\|\Delta_A\| : \exists\lambda \in C, \overline{\mathcal{R}\left[A + \mathrm{d}_A - \lambda I,\ B\right]} \neq X\right\} \\ &= \inf_{\lambda\in C} \inf_{\Delta_A\in\mathcal{L}(X)} \left\{\|\Delta_A\| : \overline{\mathcal{R}\left[A + \Delta_a - \lambda I,\ B\right]} \neq X\right\}. \end{aligned}$$

Using Lemma 2.6, we can get the proof. ■

Here, we can also reduce the controllability distance to an optimization problem depending on just one real variable as in Theorem 3.3.

Theorem 3.6. *Let the system* (1) *be exactly controllable.*

(a) *If A is Hermitian then* $d_A = \min_{\lambda\in\mathbb{R}} s_{\min}\left[P_{\mathcal{N}(B^*)}(A - \lambda I)\right]$.

(b) *If A is skew-Hermitian then* $d_A = \min_{\lambda\in\mathbb{R}} s_{\min}\left[P_{\mathcal{N}(B^*)}(A - i\lambda I)\right]$.

3.3. Perturbation on only B

In this subsection, we assume that only the system operator B is subjected to perturbation. Then, the perturbed system of system (1) is

$$\dot{x}(t) = Ax(t) + (B + \varDelta_B)u(t), \qquad t \geq 0, \tag{7}$$

Definition 3.7. The controllability distance of system (1) with respect to perturbation on only B is defined by

$$d_B = \inf_{\varDelta_B \in \mathcal{L}(U,X)} \{\|\varDelta_B\| : \text{the perturbed system (7) is not approximately controllable}\}.$$

Theorem 3.8. *Let the system* (1) *be exactly controllable, then*

$$d_B = \min_{\lambda \in \sigma(A)} s_{\min}\left[P_{\mathcal{N}(A^* - \overline{\lambda} I)} B\right].$$

Proof. From Theorem 2.1, we have

$$\begin{aligned} d_A &= \inf_{\varDelta_B \in \mathcal{L}(U,X)} \left\{\|\varDelta_B\| : \exists \lambda \in C, \overline{\mathcal{R}\left[A - \lambda I,\ B + \mathrm{d}_B\right]} \neq X\right\} \\ &= \inf_{\lambda \in C} \inf_{\varDelta_B \in \mathcal{L}(U,X)} \left\{\|\varDelta_B\| : \overline{\mathcal{R}\left[A - \lambda I,\ B + \varDelta_B\right]} \neq X\right\}. \end{aligned}$$

Noting that $\mathcal{R}(A - \lambda I) = X$ for all $\lambda \notin \sigma(A)$ and using Lemma 2.6 we can get the proof. ■

We also reduce the problem to an optimization problem on just the real case as in Theorems 3.3 and 3.6. However, if A is Hermitian then all eigenvalues of A^* are necessarily real, and if A is skew-Hermitian then all eigenvalues of A^* have to be pure imaginary. So, the reduction is trivial.

3.4. Real and complex controllability distance

Let X be a Hilbert separable space with orthogonal basis $\{x_i\}_{i=1}^{\infty}$, a vector $x \in X$ is said to be *nonnegative* if $< x, x_i >$ is nonnegative for all $i = 1, 2, ...$ So, we can consider X as a complex Banach lattice, (see [15]), with the order defined by $x \geq y$ if and only if $x - y$ is nonnegative. Hence, we can consider perturbations of the two different types:

$$\varDelta \in \mathcal{L}(X,Y) \ \varDelta \in \mathcal{L}^{\mathbb{R}}(X,Y),$$

for some Hilbert separable spaces X and Y with their specific orthogonal bases, see [7] for more about these kinds of perturbations. In particular, the complex controllability distance $d^{\mathbb{C}}_{AB}, d^{\mathbb{C}}_{A}, d^{\mathbb{C}}_{B}$ are defined by Definitions 3.1, 3.4 and 3.7. When Δ_A and Δ_B are restricted to belong to $\mathcal{L}^{\mathbb{R}}(X)$ and $\mathcal{L}^{\mathbb{R}}(U, X)$, we get the notation of real controllability distance $d^{\mathbb{R}}_{AB}, d^{\mathbb{R}}_{A}, d^{\mathbb{R}}_{B}$.

By choosing y_ε to be real and using Hahn-Banach theorem for real operators, see [20, p.249], instead of using Hahn-Banach theorem to construct $\Delta_\varepsilon \in \mathcal{L}^{\mathbb{R}}(X, Y)$ in the proof of Lemma 2.5, we get the following result.

Lemma 3.9. *Let X, Y be some Hilbert separable spaces with their particular orthogonal bases and $A \in \mathcal{L}^{\mathbb{R}}(X, Y)$ satisfying $\mathcal{R}(A) = Y$, then for arbitrary $\varepsilon > 0$ there exists a perturbation $\Delta_\varepsilon \in \mathcal{L}^{\mathbb{R}}(X, Y)$ satisfying $\mathcal{R}(A + \Delta_\varepsilon) \neq Y$ and $s_{\min}(A) \leq \|\Delta_\varepsilon\| \leq s_{\min}(A) + \varepsilon$.*

We also get the same result for the case when apart of the operator is perturbed as in Lemma 2.6.

Lemma 3.10. *Let X, Y, U be some Hilbert separable spaces with their particular orthogonal bases, and $A \in \mathcal{L}^{\mathbb{R}}(X, Y), B \in \mathcal{L}^{\mathbb{R}}(U, Y)$ satisfying $\mathcal{R}[A,\ B] = Y$, then for arbitrary $\varepsilon > 0$ there exists a perturbation $\Delta_\varepsilon \in \mathcal{L}^{\mathbb{R}}(X, Y)$ satisfying $\mathcal{R}[A + \Delta_\varepsilon,\ B] \neq Y$ and $s_{\min}[P_{\mathcal{N}(B^*)}A] \leq \|\Delta_\varepsilon\| \leq s_{\min}[P_{\mathcal{N}(B^*)}A] + \varepsilon$.*

From the definition of complex and real controllability distances, we get $d^{\mathbb{C}}_{AB} \leq d^{\mathbb{R}}_{AB}, d^{\mathbb{C}}_{A} \leq d^{\mathbb{R}}_{A}$ and $d^{\mathbb{C}}_{B} \leq d^{\mathbb{R}}_{B}$. And the equality can attain in the following theorem.

Theorem 3.11. *Let X, U be some Hilbert separable spaces with their particular orthogonal bases and the system (1), with $A \in \mathcal{L}^{\mathbb{R}}(X)$ symmetric and $B \in \mathcal{L}^{\mathbb{R}}(U, X)$, exactly controllable, then $d^{\mathbb{C}}_{AB} = d^{\mathbb{R}}_{AB}, d^{\mathbb{C}}_{A} = d^{\mathbb{R}}_{A}$ and $d^{\mathbb{C}}_{B} = d^{\mathbb{R}}_{B}$.*

Proof. By Lemma 3.9, the formula of complex controllability distance in (a) of Theorem 3.3 means there exists a sequence of couple real operators (Δ^k_A, Δ^k_B), $k = 1, 2, \ldots$, which make the perturbed system (5) not approximately controllable and $\|[\Delta^k_A,\ \Delta^k_B]\|$ converge to $d^{\mathbb{C}}_{AB}$. This implies $d^{\mathbb{C}}_{AB} \geq d^{\mathbb{R}}_{AB}$. However, from the definition, we get $d^{\mathbb{C}}_{AB} \leq d^{\mathbb{R}}_{AB}$. So, we have $d^{\mathbb{C}}_{AB} = d^{\mathbb{R}}_{AB}$. And using the same arguments and Lemma 3.10 for the cases of d_A and d_B, we complete the proof. ∎

4. Example

Let $X = \mathbb{C}^2$ and $U = l^2(\mathbb{N}, \mathbb{C})$ with their normal orthogonal bases. Consider system (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B(u_1, u_2, \ldots) = \begin{bmatrix} \sum_{i=1}^{+\infty} \dfrac{u_{2i-1}}{\sqrt{2^i}} \\ \sum_{i=1}^{+\infty} \dfrac{2u_{2i}}{\sqrt{2^i}} \end{bmatrix}.$$

It is easy to check that this system is exactly controllable. By Theorems 3.11, the real and complex controllability distances are equal. Using Theorems 3.3, 3.6 and 3.8, we get $d_{AB} = \sqrt{2}, d_A = +\infty, d_B = \sqrt{\frac{5}{2}}$. This illustrates for the fact that $d_{AB} \leq \min\{d_A, d_B\}$, and the strict inequality may occur.

References

1. J. S. Bay, *Fundamentals of Linear State Space Systems*, McGraw-Hill, 1998.
2. D. Boley, A perturbation result for linear control problems, *SIAM J. Algebraic Discrete Methods* **6** (1985), 66–72.
3. D. Boley and Lu Wu-Sheng, Measuring how far a controllable system is from an not approximately controllable one, *IEEE Trans. Automat. Control* **31** (1986), 249–251.
4. Conway and John B, *A Course in Functional Analysis*, Second edition, Graduate Texts in Mathematics, 96. Springer-Verlag, New York, 1990.
5. G. Eckstein, Exact controllability and spectrum assignment, *Operator Theory: Advances and Applications* **2** (1981), 81–94.
6. R. Eising, Between controllable and not approximately controllable, *Systems & Control Lett.* **4** (1984), 263–264.
7. A. Fischer, D. Hinrichsen, and N. K. Son, Stability radii of Metzler operators, *Vietnam J. Math.* **26** (1998), 147–163.
8. M. L. J. Hautus, Controllability and observability conditions of linear autonomous systems, *Proc. Koninklijke Nederlnadse Akademie van Weten- schappen* **7** (1969), 443 – 448.
9. G. Hu and E. J. Davison, Real Controllability/Stabilizability Radius of LTI Systems, *IEEE Trans. Automat. Control* **49** (2004), 254–257.
10. M. A. Kaashoek, C. V. M. van der Mee, and L. Rodman, Analytic operator functions with compact spectrum, III. Hilbert space case: inverse problem and applications, *J. of Operator Theory* **10** (1983), 219–250.
11. R. E. Kalman, Contribution to the theory of optimal control, *Bol. Soc. Math. Mexicana* **5** (1960), 102–119.
12. D. C. Khanh, D. D. X. Thanh, Controllability radii and stabilizability radii of time-invariant linear systems, *Vietnam J. Math.* **34** (2006), 495–499.
13. S. J. Leon, *Linear Algebra with Applications*, Prentice-Hall, Inc., Upper Saddle River Publisher, New Jersey, 1998.
14. M. Megan and V. Hiriş, On the space of linear controllable systems in Hilbert spaces, *Glasnik Mat. Ser. III*, **10** (30) (1975), 161–167.
15. P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin, 1991.
16. W. Rudin, *Functional Analysis*, McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.
17. K. Takahashi, Exact controllability and spectrum assignment, *J. Math. Anal. Appl.* **104** (1984), 537–545.
18. M. Wonham, *Linear Multivariable Control: Geometric Approach*, Springer -Verlag, New York, 1979.
19. K. Yosida, *Functional analysis*, Springer-Verlag, Berlin, 1974.
20. A. C. Zaanen, *Introduction to Operator Theory in Riesz Spaces*, Springer-Verlag, Berlin, 1997.