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Controllability Radius of Linear Systems under Structured Perturbations

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Abstract. In this short note, we formulate some new results on robustness measure of controllability of linear systems whose coefficient matrices are subjected to structured pertubations. The notion of controllability radius is introduced and some formulas for its computation are derived. Examples are given to illustrate the obtained results.

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1. Introduction

It is well-known, by a classical result of Hautus, that a linear time-invariant system $\dot{x} = Ax + Bu, t \geq 0,\ A \in \mathbb{K}^{n\times n}, B \in \mathbb{K}^{n\times m},\ \mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ is *controllable* if and only if $\operatorname{rank}[A - \lambda I, B] = n$ for all $\lambda \in \mathbb{C}$. In this case, we shall say that the matrix pair (A, B) is controllable (Hautus controllability test, [2]). As pointed out by Lee and Markus [10], the set of all controllable pairs $(A, B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$ is open. Therefore, for a given controllable pair(A, B), one can consider the problem of computing the *controllability radius* $r(A, B)$ when the matrices A, B are subjected to perturbations:

$$r(A,B) = \inf\Big\{\|[\Delta_1, \Delta_2]\| : [\Delta_1, \Delta_2] \in \mathbb{K}^{n\times(n+m)},$$
$$(A + \Delta_1, B + \Delta_2) \text{ is not controllable}\Big\}$$

Here, $\|\cdot\|$ denotes any matrix norm. Due to the practical importance of robustness and sensitivity analysis of systems control, the above mentioned problem has attracted a good deal of attention from researchers over recent decades. The most well-known result in dealing with this problem was due to Eising [3] who has proved the formula

$$r(A,B) = \inf_{\lambda\in\mathbb{C}} \sigma_{\min}([A-\lambda I \;\; B])$$

where $\sigma_{\min}$ denotes the minimal singular value of a matrix. Motivated by a similar problem of computing the stability radius (see e.g. [8, 9], and the extensive literature therein), it is natural to put a problem of computing the *structured controllability radius* when the matrix pair (A,B) is subjected to affine perturbations

$$[A,B] \rightsquigarrow [A + D_1\Delta_1E_1, B + D_2\Delta_2E_2].$$

To our knowledge, there has not been so far any results for such a class of structured perturbations and other more general multi-perturbations. In this paper we shall formulate some of our new results in solving this problem. The key technique is to use some well-known facts from the theory of linear multivalued operators in representing equations and matrices involved in the calculation.

For the reader's convenience, we recall here some notations and known results on linear multivalued operators which will be used in the sequel, see e.g. [1]. Let $\mathcal{F}:\mathbb{K}^n \rightrightarrows \mathbb{K}^m$ be a multivalued operator, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. If the graph of $\mathcal{F}$, defined by

$$\mathrm{gr}\mathcal{F} = \big\{(x,y)\in\mathbb{K}^n\times\mathbb{K}^m : y\in\mathcal{F}(x)\big\}, \tag{1}$$

is a linear subspace of $\mathbb{K}^n\times\mathbb{K}^m$ then $\mathcal{F}$ is called a linear multivalued operator. The norm of $\mathcal{F}$, defined by

$$\|\mathcal{F}\| = \sup\big\{d(0,\mathcal{F}(x)) : x\in\mathrm{dom}(\mathcal{F}), \|x\|=1\big\}, \tag{2}$$

where $\mathrm{dom}(\mathcal{F}) = \big\{x\in\mathbb{K}^n : \mathcal{F}(x)\neq\emptyset\big\}$. For a linear multivalued operator $\mathcal{F}:\mathbb{K}^n\rightrightarrows\mathbb{K}^m$, its adjoint operator $\mathcal{F}^*:(\mathbb{K}^m)^*\rightrightarrows(\mathbb{K}^n)^*$ and its inverse operator $\mathcal{F}^{-1}:\mathrm{Im}\mathcal{F}\rightrightarrows\mathbb{K}^n$ are defined, correspondingly, by

$$\mathcal{F}^*(v) = \big\{u\in(\mathbb{K}^m)^* : \langle x,u\rangle = \langle y,v\rangle, \text{ for all } (x,y)\in\mathrm{gr}\mathcal{F}\big\} \tag{3}$$

$$\mathcal{F}^{-1}(y) = \big\{x\in\mathbb{K}^n : y\in\mathcal{F}(x)\big\}. \tag{4}$$

Obviously, $\mathcal{F}^*$ and $\mathcal{F}^{-1}$ are also linear multivalued operators. One has

$$\|\mathcal{F}\| = \|\mathcal{F}^*\|. \tag{5}$$

It can be proved that $\mathcal{F}$ is surjective (i.e. $\mathcal{F}(\mathbb{K}^n)=\mathbb{K}^m$) if and only if $\mathcal{F}^*$ is injective (i.e. $\mathcal{F}^{*-1}(0)=\{0\}$). Let $\mathcal{F}:\mathbb{K}^n\rightrightarrows\mathbb{K}^m, \mathcal{G}:\mathbb{K}^m\rightrightarrows\mathbb{K}^l$ be linear multivalued operators. Then, we have $\mathcal{F}\mathcal{G}:\mathbb{K}^n\rightrightarrows\mathbb{K}^l$, determined by $\mathcal{F}\mathcal{G}(x) = \mathcal{F}(\mathcal{G}(x))$ for all $x\in\mathbb{K}^n$, is a linear multivalued operator and

$$(\mathcal{F}\mathcal{G})^* = \mathcal{G}^*\mathcal{F}^*, \qquad \|\mathcal{F}\mathcal{G}\| \le \|\mathcal{F}\|\,\|\mathcal{G}\|. \tag{6}$$

2. Main Results

Assume that the matrix pair $(A, B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$ is subjected to structured perturbations of the form:

$$[A, B] \rightsquigarrow [\widetilde{A}, \widetilde{B}] = [A, B] + D\Delta E, \tag{7}$$

where $D \in \mathbb{K}^{n\times r}, E \in \mathbb{K}^{l\times(n+m)}$ are given matrices defining the structure of perturbations, $\Delta \in \mathbb{K}^{r\times l}$ is unknown disturbance matrix, $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. The controllability radius of the linear system (A, B) under structured pertubations of the form (7) is defined by

$$r^{D,E}_{\mathbb{K}}(A, B) = \inf\big\{\|\Delta\| : \Delta \in \mathbb{K}^{r\times l} \text{ s.t. } [\widetilde{A}, \widetilde{B}] = [A, B]+D\Delta E, \text{ uncontrollable } \big\}, \tag{8}$$

Denote $W_\lambda = [A - \lambda I, B]$, for all $\lambda \in \mathbb{C}$.

Theorem 2.2. *The complex controllability radius of the linear system (A, B) under structured perturbations of the form (7) is given by the formula*

$$r^{D,E}_{\mathbb{C}}(A, B) = \inf_{\lambda\in\mathbb{C}} \|\mathcal{F}(D, W_\lambda, E)\|^{-1}, \tag{9}$$

where $\mathcal{F}(D, W_\lambda, E) := EW_\lambda^{-1}D$ is a linear multivalued operator.

Proof. Let the pair (A, B) be controllable, that is

$$\text{rank}W_\lambda = \text{rank}[A - \lambda I, B] = n, \text{ for all } \lambda \in \mathbb{C}. \tag{10}$$

This is equivalent to that W^*_λ is injective for all $\lambda \in \mathbb{C}$. Assume $(\widetilde{A}, \widetilde{B})$with $[\widetilde{A}, \widetilde{B}] = [A, B] + D\Delta E$ is uncontrollable. By Hautus controllability test, there exists $\lambda_0 \in \mathbb{C}$ such that $W_{\lambda_0} + D\Delta E$ is not surjective. This implies that there exists $x_0 \in \mathbb{C}^n, x \neq 0$ such that $W^*_{\lambda_0}(x_0) + E^*\Delta^*D^*(x_0) = 0$. Hence, $x_0 = -(W^*_{\lambda_0})^{-1}E^*\Delta^*D^*(x_0)$ and $D^*(x_0) \neq 0$. By multiplying D^* from the left, we obtain

$$D^*(x_0) = -D^*(W^*_{\lambda_0})^{-1}E^*\Delta^*D^*(x_0).$$

Therefore

$$\|D^*(x_0)\| \le \|D^*(W^*_{\lambda_0})^{-1}E^*\|\ \|\Delta^*\|\ \|D^*(x_0)\|. \tag{11}$$

By using (5) and (6), we get

$$\|\Delta\| \ge \|\mathcal{F}(D, W_{\lambda_0}, E)\|^{-1} \ge \inf_{\lambda\in\mathbb{C}} \|\mathcal{F}(D, W_\lambda, E)\|^{-1}.$$

Since the above inequality holds for any disturbance matrix $\Delta \in \mathbb{C}^{r\times l}$ destroying controllability, we obtain by definition,

$$r_{\mathbb{C}}^{D,E}(A,B) \geq \inf_{\lambda \in \mathbb{C}} \|\mathcal{F}(D, W_\lambda, E)\|^{-1}. \tag{12}$$

To prove the converse inequality, consider a sequence $\{\lambda_n\}$ such that

$$\|\mathcal{F}(D, W_{\lambda_n}, E)\|^{-1} \leq \inf_{\lambda \in \mathbb{C}} \|\mathcal{F}(D, W_\lambda, E)\|^{-1} + \frac{1}{n}. \tag{13}$$

This implies $\|\mathcal{F}(D, W_{\lambda_n}, E)\| > 0$, and therefore, there exists

$$x_n \in \mathrm{dom}(D^*(W_{\lambda_n}^*)^{-1}E^*)$$

such that $\|x_n\| = 1$ and

$$\|D^*(W_{\lambda_n}^*)^{-1}E^*(x_n)\| = \|D^*(W_{\lambda_n}^*)^{-1}E^*\| = \|\mathcal{F}(D, W_{\lambda_n}, E)\|.$$

Let $y_n = -(W_{\lambda_n}^*)^{-1}E^*(x_n) \neq 0$, we construct

$$\Delta_n^*(u) = \frac{\langle u, D^* y_n \rangle}{\|D^* y_n\|^2}\ x_n, \quad \Delta_n := (\Delta_n^*)^*.$$

Then, it is obvious that $\|\Delta_n\| = \|\mathcal{F}(D, W_{\lambda_n}, E)\|^{-1}$ and $W_{\lambda_n}^*(y_n) = -E^*\Delta_n^* D^*(y_n)$. Therefore,

$$r_{\mathbb{C}}^{D,E}(A,B) \leq \|\Delta_n\| = \|\mathcal{F}(D, W_{\lambda_n}, E)\|^{-1} \leq \inf_{\lambda \in \mathbb{C}} \|\mathcal{F}(D, W_\lambda, E)\|^{-1} + \frac{1}{n}. \tag{14}$$

Letting $n \to \infty$, we obtain the converse inequality. The proof is complete. ■

We note that the above result has been established for operator norms of matrices induced by any vector norms in $\mathbb{K}^n$ and $\mathbb{K}^m$. The following corollary generalizes Eising's formula, [3]:

Corollary 2.3. *Assume $D \in \mathbb{K}^{n\times n}, E \in \mathbb{K}^{(n+m)\times(n+m)}$ are non-singular matrices and all vector spaces under consideration are equipped with Euclidean norms. Then, the complex controllability radius of the linear system (A, B) under structured perturbation of the form (7) is given by the formula*

$$r_{\mathbb{C}}^{D,E}(A,B) = \inf_{\lambda \in \mathbb{C}} \sigma_{\min}(D^{-1}W_\lambda E^{-1}). \tag{15}$$

Under some additional assumption, we get the following result formulated in [11], the proof being, unfortunately, incomplete.

Corollary 2.4. *Assume $E^*E(\ker(W_\lambda)^\perp) \subset \ker(W_\lambda)^\perp$ for all $\lambda \in \mathbb{C}$. Then, the complex controllability radius of the linear system (A, B) under structured perturbations of the form (7) is given by the formula*

$$r_{\mathbb{C}}^{D,E}(A,B) = \inf_{\lambda \in \mathbb{C}} \|EW_\lambda^\dagger D\|^{-1}, \tag{16}$$

where $W_\lambda^\dagger = W_\lambda^*(W_\lambda W_\lambda^*)^{-1}$ denotes the Morre-Penrose pseudoinverse of W_λ.

Proof. We have $W_\lambda^\dagger D(u) \in (W_\lambda)^{-1} D(u)$ for all $u \in \mathbb{K}^r$. This implies $EW_\lambda^\dagger D(u) \in E(W_\lambda)^{-1} D(u)$ for all $u \in \mathbb{K}^r$. Moreover, $W_\lambda^\dagger D(u) \in \ker(W_\lambda)^\perp$. By the assumptions, we get $\|EW_\lambda^\dagger D(u)\| = d(0, \mathcal{F}(D, W_\lambda, E)(u))$ for all $u \in \mathbb{K}^r$, and the result follows directly from Theorem 2.2. ■

We consider the linear system (A, B) with constrained control:

$$\begin{cases} \dot{x} = Ax + Bu \\ u \in S, \end{cases} \tag{17}$$

where S is a subspace of $\mathbb{K}^m$. Assume that the system (17) is perturbed as

$$(A, B) \rightsquigarrow (A + \Delta_1, B + \Delta_2). \tag{18}$$

The following corollary is well known by A. Lewis et al. [7] .

Corollary 2.5. *The complex controllability radius of the system* (17) *under perturbations of the form* (7) *is given by the formula*

$$r_{\mathbb{C}}^S(A, B) = r_{\mathbb{C}}(A, BQ) = \inf_{\lambda \in \mathbb{C}} \sigma_{\min}[A - \lambda I, BQ], \tag{19}$$

where $Q \in \mathcal{L}(\mathbb{K}^r, \mathbb{K}^m)$ is any orthonormal map having S as range.

Next, let us consider the controllability radius with respect to perturbations of the form

$$(A, B) \rightsquigarrow (A + D\Delta E,\ B), \tag{20}$$

$$(A, B) \rightsquigarrow (A,\ B + D\Delta E). \tag{21}$$

Corollary 2.6. *The complex controllability radius of the linear system (A, B) with respect to perturbations of the form* (20) *is given by the formula*

$$r_{\mathbb{C}}^{D,E}(A) = \inf_{\lambda \in \mathbb{C}} \|E(A - \lambda I)^{-1} \mathcal{F}_B(D)\|, \tag{22}$$

and the one of the form (21) *is given by the formula*

$$r_{\mathbb{C}}^{D,E}(B) = \inf_{\lambda \in \mathbb{C}} \|EB^{-1} \mathcal{F}_{A - \lambda I}(D)\|, \tag{23}$$

where $\mathcal{F}_G(H)$ is defined by $\mathcal{F}_G(H)(x) = H(x) + \mathrm{Im}(G)$.

3. Remarks and Examples

We illustrate the main result by the following example. Consider the linear control system (A, B) described by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{24}$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. It is clear, by Kalman's criterion, that the system is controllable. Assume that, the control matrix $[A, B]$ is subjected to multi-perturbations of the form $[A, B] \rightsquigarrow [A, B] + D\varDelta E$ where

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then

$$E([A - \lambda I, B])^{-1} D(x) = \left\{ \begin{pmatrix} x + (\lambda + 1)v \\ (\lambda + 1)x + (\lambda^2 - 1)v \end{pmatrix} : v \in \mathbb{C} \right\}.$$

This implies

$$\|E([A - \lambda I, B])^{-1} D\|^{-1} = \begin{cases} \dfrac{\sqrt{|\lambda - 1|^2 + 1}}{2} & \text{if } \lambda \neq -1, \\ 1 & \text{if } \lambda = -1. \end{cases}$$

Thus

$$r_{\mathbb{C}}^{D,E}(A, B) = \frac{1}{2}.$$

We conclude this short note by remarking that the technique used in this paper can be applied to deal with the case when the system matrices are subjected to multi-perturbations of the forms

$$[A, B] \rightsquigarrow [\widetilde{A}, \widetilde{B}] = [A, B] + \sum_{i=1}^{N} D_i \varDelta_i E_i, \tag{25}$$

and

$$[A, B] \rightsquigarrow [\widetilde{A}, \widetilde{B}] = [A, B] + \sum_{i=1}^{N} \delta_i P_i. \tag{26}$$

However, as in the case of stability radius (see, e.g. [9]), one can get only some upper bounds and lower bounds for the complex controllability radius. The results will be published in a separate paper.

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