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Complemented Subspaces in $L^\infty(\mathbb{D})$

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Abstract. In this paper we have shown that the little Bloch space $\mathcal{B}_0$ cannot be complemented in $\mathcal{B}$ and hence $C(\overline{\mathbb{D}})$ cannot be complemented in $L^\infty(\mathbb{D})$. Further, we have obtained some closed subspaces of $L^\infty(\mathbb{D})$ that can be complemented in $L^\infty(\mathbb{D})$. As a consequence of these results we have shown $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$ can be complemented in $\mathcal{L}(L^2_a)$ and $\{h_\phi : \phi \in h^\infty(\mathbb{D})\}$ cannot be complemented in $\mathcal{L}(L^2_a, (\overline{L^2_a})_0)$. Here T_ϕ is the Toeplitz operator on the Bergman space L^2_a, h_ϕ is the little Hankel operator from L^2_a into $(\overline{L^2_a})_0 = \{\bar{f} : f \in L^2_a, f(0) = 0\}$ and $h^\infty(\mathbb{D})$ is the space of bounded harmonic functions on the unit disk $\mathbb{D}$.

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The Bloch space $\mathcal{B}$ of $\mathbb{D}$ is defined to be the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}} = \sup\left\{(1 - |z|^2)|f'(z)| : z \in \mathbb{D}\right\} < \infty.$$

It is not difficult to check that $\|\|_{\mathcal{B}}$ is a complete semi-norm on $\mathcal{B}$. The space $\mathcal{B}$ can be made [9] into a Banach space by introducing the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}}.$$

Let $dA(z) = \dfrac{1}{\pi}dxdy$ be the normalized area measure on $\mathbb{D}$. For $1 \leq p < +\infty$, $L^p(\mathbb{D}, dA)$ will denote the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\|f\|_p = \left[\int_{\mathbb{D}} |f(z)|^p dA(z)\right]^{\frac{1}{p}} < +\infty.$$

$L^\infty(\mathbb{D}, dA)$ will denote the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\|f\|_\infty = \text{ ess sup } \{|f(z)| : z \in \mathbb{D}\} < +\infty.$$

For $1 \leq p < \infty$, the space $L_a^p(\mathbb{D})$ is defined to be the subspace of $L^p(\mathbb{D}, dA)$ consisting of analytic functions. For $1 \leq p < \infty$, L_a^p is a closed subspace [9] of $L^p(\mathbb{D}, dA)$. The Bergman space $L_a^2(\mathbb{D})$ (in case $p = 2$) is a Hilbert space and the inner product on $L_a^2(\mathbb{D})$ is given by the formula

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z), f, g \in L_a^2(\mathbb{D}).$$

Let P be the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L_a^2(\mathbb{D})$. The operator P is called the Bergman projection. It is shown in [9] that the Bergman projection P is a bounded linear operator from $L^\infty(\mathbb{D})$ onto $\mathcal{B}$. The little Bloch space of $\mathbb{D}$, denoted by $\mathcal{B}_0$, is the closed subspace of $\mathcal{B}$ consisting of functions f with $(1-|z|^2)f'(z) \to 0$ as $|z| \to 1^-$. The space of bounded analytic functions on $\mathbb{D}$ will be denoted by $H^\infty(\mathbb{D})$. It is not so difficult to verify that $H^\infty \subset \mathcal{B}$, $\|f\|_{\mathcal{B}} \leq \|f\|_\infty$ for all f in H^∞ and neither $\mathcal{B}_0$ is contained in H^∞ nor is H^∞ contained in $\mathcal{B}_0$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then $f \in \mathcal{B}$ implies $|a_n| \leq 2\|f\|_{\mathcal{B}}, n = 1, 2, \ldots$ and $f \in \mathcal{B}_0$ implies $a_n \to 0$ as $n \to \infty$ (see [9]).

2. Uncomplemented Subspaces

In this section we show that the little Bloch space $\mathcal{B}_0$ cannot be complemented in $\mathcal{B}$, hence $C(\overline{\mathbb{D}})$ cannot be complemented in $L^\infty(\mathbb{D})$. We shall use the following lemma to prove these results.

Lemma 2.1. [*1*] *If* $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ *is a Hadamard gap series, that is,* $\lambda_{n+1} \geq c\lambda_n$ *for some constant* $c > 1$ *and all* n, *then* $f \in \mathcal{B}$ *if and only if* $\{a_n\}$ *is bounded;* $f \in \mathcal{B}_0$ *if and only if* $a_n \to 0$ *as* $n \to \infty$.

Theorem 2.2. *The little Bloch space* $\mathcal{B}_0$ *cannot be complemented in* $\mathcal{B}$.

Proof. Define a map $T : \mathcal{B} \to l^\infty$ by $T(f) = (a_{2^n})_{n=1}^{\infty}$ for $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. The map T is well-defined [9] and is a bounded linear operator from $\mathcal{B}$ to the Banach space l^∞ of bounded complex sequences with the supremum norm. Further from [9] it follows that if $f \in \mathcal{B}_0$, then $Tf \in c_0$, the subspace of l^∞ consisting of sequences which tend to zero. By Lemma 2.1, the linear map $S : l^\infty \to \mathcal{B}$ defined by $S((a_k)_{k=1}^\infty) = \sum_{k=1}^\infty a_k z^{2^k}$ is a well-defined, bounded linear operator and S maps c_0 into $\mathcal{B}_0$. Now suppose there exists a projection Q from $\mathcal{B}$ onto $\mathcal{B}_0$. Then $M = T \circ Q \circ S$ is a bounded linear operator from l^∞ into c_0 and $M^2 = M$. To see that M is onto, let $(a_k)_{k=1}^\infty \in c_0$. Let $f(z) = \sum_{k=1}^\infty a_k z^{2^k}$. Then $f \in \mathcal{B}_0$ by Lemma 2.1 and $Tf = (a_k)_{k=1}^\infty, Qf = f$. Moreover, $S((a_k)_{k=1}^\infty) = f$. Hence $(T \circ Q \circ S)(a_k)_{k=1}^\infty = (a_k)_{k=1}^\infty$. Thus M is a bounded projection from l^∞ onto c_0. But it is known [5] that c_0 cannot be complemented in l^∞. Hence $\mathcal{B}_0$ cannot be complemented in $\mathcal{B}$. ■

Corollary 2.3. *The algebra $C(\overline{\mathbb{D}})$ of complex valued continuous functions on the closure of $\mathbb{D}$ cannot be complemented in $L^\infty(\mathbb{D})$.*

Proof. Suppose there exists a bounded projection Q from $L^\infty(\mathbb{D})$ onto $C(\overline{\mathbb{D}})$. Let i be the inclusion map from $\mathcal{B}$ into $L^\infty(\mathbb{D})$. It is known [9] that the Bergman projection P maps $C(\overline{\mathbb{D}})$ onto $\mathcal{B}_0$. Thus $M = P \circ Q \circ i$ is a bounded linear operator from $\mathcal{B}$ onto $\mathcal{B}_0$ and $M^2 = M$. Hence M is a bounded projection from $\mathcal{B}$ onto $\mathcal{B}_0$. By Theorem 2.2, such projection does not exist. It therefore follows that there is no bounded projection from $L^\infty(\mathbb{D})$ onto $C(\overline{\mathbb{D}})$ and $C(\overline{\mathbb{D}})$ cannot be complemented in $L^\infty(\mathbb{D})$. ■

Remark 2.4. Let $C_0(\mathbb{D})$ be the subalgebra of $C(\overline{\mathbb{D}})$ consisting of functions f with $f(z) \to 0$ as $|z| \to 1^-$. It is known [9] that the Bergman projection maps $C_0(\mathbb{D})$ onto $\mathcal{B}_0$. By a similar argument as in Corollary 2.3, one can show that $C_0(\mathbb{D})$ cannot be complemented in $L^\infty(\mathbb{D})$.

Let $h^\infty(\mathbb{D})$ be the space of bounded harmonic functions on $\mathbb{D}$. Let $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$. Suppose $f \in L^2(\mathbb{T})$ and I is an interval contained in $\mathbb{T}$. We write the mean of f over I as

$$f_I = \frac{1}{|I|}\int_I f(\theta)d\theta,$$

where $|I|$ denotes the length of I. The function f is said to have bounded mean oscillation on $\mathbb{T}$ if

$$\|f\|_{BMO} = \sup_I \left[\frac{1}{|I|}\int_I |f(\theta) - f_I|^2 d\theta\right]^{\frac{1}{2}} < +\infty.$$

Let BMO (abbreviated for $BMO(\mathbb{T})$) denote the space of all functions $f \in L^2(\mathbb{T})$ having bounded mean oscillation. It can be checked that BMO is a Banach space modulo constants. Let $BMOA$ be the intersection of BMO with the Hardy space H^2 and let $BMOA(\mathbb{D})$ be the space consisting of harmonic extensions of functions in $BMOA$. It is known [9] that an analytic function on $\mathbb{D}$ is in $BMOA(\mathbb{D})$ if and only if it is the Bergman projection of a bounded harmonic function on $\mathbb{D}$.

Theorem 2.5. *$BMOA(\mathbb{D})$ can be complemented in $\mathcal{B}$.*

Proof. Let i be the inclusion map from $\mathcal{B}$ into $L^\infty(\mathbb{D})$. It is shown in [6, 7] that $h^\infty(\mathbb{D})$ can be complemented in $L^\infty(\mathbb{D})$. Let Q be the bounded projection from $L^\infty(\mathbb{D})$ onto $h^\infty(\mathbb{D})$. By [9], the Bergman projection P maps $h^\infty(\mathbb{D})$ onto $BMOA(\mathbb{D})$. Thus $M = P \circ Q \circ i$ maps $\mathcal{B}$ onto $BMOA(\mathbb{D})$ and $M^2 = M$. Thus M is a bounded projection from $\mathcal{B}$ onto $BMOA(\mathbb{D})$ and therefore $BMOA(\mathbb{D})$ can be complemented in $\mathcal{B}$. ∎

Given $f \in L^1(\mathbb{T})$, the harmonic extension of f to $\mathbb{D}$ denoted by $\widehat{f}(z)$ is defined as

$$\widehat{f}(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)P_z(\theta)d\theta, z \in \mathbb{D},$$

where $P_z(\theta) = \dfrac{1-|z|^2}{|1-\bar{z}e^{i\theta}|^2} = \mathrm{Re}\left(\dfrac{e^{i\theta}+z}{e^{i\theta}-z}\right)$ is the Poisson kernel of $\mathbb{D}$. It follows that $\widehat{f}(z)$ is harmonic in $\mathbb{D}$. Moreover, if a_n are the Fourier coefficients of f, then

$$\widehat{f}(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n}\bar{z}^n.$$

Conversely, if $f(z)$ is an analytic function on $\mathbb{D}$ with

$$\sup_{r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta < +\infty,$$

then Fatou's theorem [3] implies that the limit $f(\theta) = \lim_{r\to 1^-} f(re^{i\theta})$ exists for almost every θ (with respect to $d\theta$), $f(\theta) \in H^p$, and the harmonic extension of $f(\theta)$ to $\mathbb{D}$ is precisely $f(z)$. Thus, the harmonic extension (the Poisson extension) establishes an one-to-one correspondence between H^p of $\mathbb{T}$ and the space of analytic functions $f(z)$ on $\mathbb{D}$ with

$$\sup_{r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta < \infty$$

and

$$\|f\|^p_{L^p(\mathbb{T})} = \sup_{r<1}\frac{1}{2\pi}\int_0^{2\pi}|\widehat{f}(re^{i\theta})|^p d\theta.$$

We let $H^p(\mathbb{D})$ denote the space of analytic functions on $\mathbb{D}$ which are harmonic extensions of functions in H^p. We shall not distinguish between $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$. Since $H^2(\mathbb{T})$ is a closed subspace of the Hilbert space $L^2(\mathbb{T})$, there exists an orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. We shall denote this projection by $\widetilde{P}$. Let $\widehat{\widetilde{P}}$ be the composition of $\widetilde{P}$ with the harmonic extension; that is, $\widehat{\widetilde{P}}f = \widehat{\widetilde{P}f}$ for all $f \in L^2(\mathbb{T})$. Clearly $\widehat{\widetilde{P}}$ maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$ and $\widehat{\widetilde{P}}$ is a projection in the sense that $\widehat{\widetilde{P}}$ when applied to functions in $H^2(\mathbb{T})$ is simply the

Poisson extension and $\widetilde{\widetilde{P}}$ is called the Szegō projection. Let $L^\infty(\mathbb{T})$ be the space of all essentially bounded, measurable functions on $\mathbb{T}$ and

$$H^\infty(\mathbb{T}) = \left\{ \phi \in L^\infty(\mathbb{T}) : \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta}) e^{in\theta} d\theta = 0, n = 1, 2, 3, \dots \right\}.$$

Let $h^\infty_C(\mathbb{D})$ be the space of all bounded harmonic functions on $\mathbb{D}$ which can be extended to $\partial\mathbb{D} = \mathbb{T}$ continuously. Let $A(\mathbb{D})$ be the disc algebra consisting of all functions which are continuous on the closed unit disk and analytic at each interior point. The space $A(\mathbb{D})$ is a Banach space under the supremum norm $\|f\|_\infty = \sup_{|z|\le 1} |f(z)|$.

Theorem 2.6. *There exists no bounded projection from $h^\infty_C(\mathbb{D})$ onto $A(\mathbb{D})$.*

Proof. It is possible to identify the functions in $A(\mathbb{D})$ with their boundary values, thus obtaining an isomorphism between $A(\mathbb{D})$ and the Banach space of continuous functions on the unit circle such that $\int_0^{2\pi} f(\theta) e^{in\theta} d\theta = 0, n = 1, 2, 3, \dots$. This algebra of continuous functions is also denoted by A. It is known [4] that there exists no bounded projection from $C(\mathbb{T})$ onto A. Notice that by taking harmonic extensions of functions in $C(\mathbb{T})$ and A, we shall obtain the classes $h^\infty_C(\mathbb{D})$ and $A(\mathbb{D})$ respectively.

Let T be a map from $C(\mathbb{T})$ onto $h^\infty_C(\mathbb{D})$ such that $Tf = \widehat{f}$. Suppose there exists a bounded projection Q from $h^\infty_C(\mathbb{D})$ onto $A(\mathbb{D})$. Let S be the map from $A(\mathbb{D})$ onto A such that $Sg = \widetilde{g}$ where $\widetilde{g}(e^{i\theta}) = \lim_{r\to 1^-} g(re^{i\theta})$. Then $M = S \circ Q \circ T$ maps $C(\mathbb{T})$ onto A and $M^2 = M$. Hence M is a bounded projection from $C(\mathbb{T})$ onto A. But such map M does not exist (see [4]). ■

It is not so difficult to check that if $f \in L^2(\mathbb{T})$ then the Bergman projection of the harmonic extension of f is equal to the harmonic extension of the Szegō projection of f. Thus if the Szegō projection $\widetilde{P}$ would have mapped $C(\mathbb{T})$ onto A then $\widehat{\widetilde{P}f} = P\widehat{f}$ where P is the Bergman projection and P would have been the projection from $h^\infty_C(\mathbb{D})$ onto $A(\mathbb{D})$.

The space VMO (vanishing mean oscillation) is the subspace of BMO consisting of functions f such that

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_I |f(\theta) - f_I|^2 d\theta = 0.$$

Clearly, VMO is closed in BMO and VMO contains $C(\mathbb{T})$, the space of continuous functions on $\mathbb{T}$. Let $VMOA$ be the intersection of VMO with H^2. Let $VMOA(\mathbb{D})$ be the space of harmonic extensions of functions in $VMOA$. It is shown in [9] that $BMOA = \widetilde{P}L^\infty(\mathbb{T}), VMOA = \widetilde{P}C(\mathbb{T}), \widetilde{\widetilde{P}}L^\infty(\mathbb{T}) = BMOA(\mathbb{D})$ and $\widetilde{\widetilde{P}}C(\mathbb{T}) = VMOA(\mathbb{D})$. We shall show below that $VMOA(\mathbb{D})$ cannot be complemented in $BMOA(\mathbb{D})$ and $h^\infty_C(\mathbb{D})$ cannot be complemented in $h^\infty(\mathbb{D})$. But first we shall prove the following lemma.

Lemma 2.7. *$C(\mathbb{T})$ cannot be complemented in $L^\infty(\mathbb{T})$.*

Proof. Since $h^\infty(\mathbb{D})$ can be complemented (see [6, 7]) in $L^\infty(\mathbb{D})$, there exists a bounded projection M from $L^\infty(\mathbb{D})$ onto $h^\infty(\mathbb{D})$. Let $S : h^\infty(\mathbb{D}) \to L^\infty(\mathbb{T})$ be such that $Sf = \widetilde{f}$ where $\widetilde{f}(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$. Suppose there exists a bounded projection Q from $L^\infty(\mathbb{T})$ onto $C(\mathbb{T})$. Define T from $C(\mathbb{T})$ onto $C(\overline{\mathbb{D}})$ as $Tg = \widehat{g}$ where $\widehat{g}$ is the harmonic extension of g into $\mathbb{D}$. Then it follows that $U = T\circ Q\circ S\circ M$ is a map from $L^\infty(\mathbb{D})$ onto $C(\overline{\mathbb{D}})$ and $U^2 = U$. It happens since if f is continuous on the unit circle, then $\widehat{f}$, the harmonic extension of f is continuous on the closed disk, harmonic in the interior. That is, $\widehat{f} \in C(\overline{\mathbb{D}}) \cap h^\infty(\mathbb{D})$ if $f \in C(\mathbb{T})$. Thus we obtain a bounded projection from $L^\infty(\mathbb{D})$ onto $C(\overline{\mathbb{D}})$ and by Corollary 2.3 such map does not exist. Hence there exists no bounded projection from $L^\infty(\mathbb{T})$ onto $C(\mathbb{T})$. ■

Corollary 2.8. *$VMOA(\mathbb{D})$ cannot be complemented in $BMOA(\mathbb{D})$.*

Proof. Since $\widehat{\widetilde{P}}L^\infty(\mathbb{T}) = BMOA(\mathbb{D})$ and $\widehat{\widetilde{P}}C(\mathbb{T}) = VMOA(\mathbb{D})$, hence $VMOA(\mathbb{D})$ can be complemented in $BMOA(\mathbb{D})$ if and only if $C(\mathbb{T})$ can be complemented in $L^\infty(\mathbb{T})$. By Lemma 2.7, there exists no bounded projection from $L^\infty(\mathbb{T})$ onto $C(\mathbb{T})$. Thus $VMOA(\mathbb{D})$ cannot be complemented in $BMOA(\mathbb{D})$. ■

Corollary 2.9. *The space $h^\infty_C(\mathbb{D})$ cannot be complemented in $h^\infty(\mathbb{D})$.*

Proof. It is established in [9] that an analytic function on $\mathbb{D}$ is in $BMOA(\mathbb{D})$ if and only if it is the Bergman projection of a bounded harmonic function on $\mathbb{D}$ and an analytic function on $\mathbb{D}$ is in $VMOA(\mathbb{D})$ if and only if it is the Bergman projection of a harmonic function on $\mathbb{D}$ which extends to $\partial\mathbb{D}$ continuously. That is, $Ph^\infty(\mathbb{D}) = BMOA(\mathbb{D})$ and $Ph^\infty_C(\mathbb{D}) = VMOA(\mathbb{D})$ where P is the Bergman projection. Hence $h^\infty_C(\mathbb{D})$ can be complemented in $h^\infty(\mathbb{D})$ if and only if $VMOA(\mathbb{D})$ can be complemented in $BMOA(\mathbb{D})$. By Corollary 2.8 $VMOA(\mathbb{D})$ cannot be complemented in $BMOA(\mathbb{D})$. Thus $h^\infty_C(\mathbb{D})$ cannot be complemented in $h^\infty(\mathbb{D})$. ■

3. Complemented Subspaces of the Space of Bounded Linear Operators

For $1 < p < \infty$, the Besov space B_p of $\mathbb{D}$ is defined to be the space of analytic functions f in $\mathbb{D}$ such that

$$\|f\|_{B_p} = \left[\int_{\mathbb{D}} (1-|z|^2)^p |f'(z)|^p d\lambda(z)\right]^{\frac{1}{p}} < \infty,$$

where $d\lambda(z) = \dfrac{dA(z)}{(1-|z|^2)^2}$ is the Mobius invariant measure on $\mathbb{D}$. It is easy to show that $\|.\|_{B_p}$ is a complete semi-norm on B_p. The space B_p becomes a Banach space with the norm

$$\|f\| = |f(0)| + \|f\|_{B_p}.$$

Let $\mathcal{L}(H^2, (H^2)^\perp)$ be the space of all bounded linear operators from the Hardy space H^2 into $(H^2)^\perp$ (the orthogonal complement of H^2 in L^2). For $\phi \in L^\infty(\mathbb{T})$, define $H_\phi : H^2(\mathbb{T}) \to (H^2(\mathbb{T}))^\perp$ as $H_\phi f = (I - \widetilde{P})(\phi f)$. The operator H_ϕ is called the Hankel operator with symbol ϕ. Notice that H_ϕ is linear, bounded and $\|H_\phi\| \leq \|\phi\|_\infty$. The Hankel operator H_f can also be defined with symbol $f \in L^2(\mathbb{T})$. In general, H_f is only densely defined with domain containing H^∞. One can show that H_f is bounded precisely when $H_f = H_g$ for some $g \in L^\infty(\mathbb{T})$.

Given $1 \leq p < \infty$, we define the Schatten p-class of the Hilbert space H, denoted by $S_p(H)$ or simply S_p, to be the space of all compact operators T on H with its singular value sequence $\{\lambda_n\}$ belonging to l^p (the p-summable sequence space). It is known that S_p is a Banach space [9] with the norm

$$\|T\|_p = \left(\sum_n |\lambda_n|^p \right)^{\frac{1}{p}}.$$

Theorem 3.1. *The linear space* $\{H_{\bar{f}} : f \in B_p\}$ *can be complemented in the Schatten* p*-class* $S_p, 1 < p < \infty$ *in* $\mathcal{L}(H^2, (H^2)^\perp)$.

Proof. The functions $e_n(t) = e^{int}, n \geq 0$ is the standard orthonormal basis for $H^2(\mathbb{T})$ and $\overline{e_n}(t) = e^{-int}, n \geq 1, n \in \mathbb{Z}$, is an orthonormal basis for $(H^2)^\perp$. Let $T \in \mathcal{L}(H^2, (H^2)^\perp)$. For $n \geq 1$, let

$$t_n = \frac{1}{n} \sum_{k=1}^{n} \overline{\langle T e_{n-k}, \overline{e_n} \rangle}.$$

The boundedness of T implies that $\{t_n\}$ is a bounded sequence; thus the following series defines an analytic function on $\mathbb{D}$

$$\widehat{T}(z) = \sum_{n=1}^{\infty} t_n z^n.$$

If T is a bounded linear operator from H^2 into $(H^2)^\perp$, then $\widehat{T}(z)$ is in the Bloch space $\mathcal{B}$ of $\mathbb{D}$ and if $1 < p < \infty$, then the mapping σ defined by $\sigma(T) = \widehat{T}$ maps the Schatten class S_p into the Besov space B_p and the map σ is bounded (see [9]). Further if $p > 1$ and f is in H^2, then $H_{\bar{f}}$ is in the Schatten class S_p if and only if f is in the Besov space B_p [9]. Define a map ρ from B_p onto $\{H_{\bar{f}} : f \in B_p\}$ by $\rho(g) = H_{\bar{g}}$. It is not difficult [9] to see that ρ is bounded and $\|H_{\bar{f}}\|_p \leq c\|f\|_{B_p}$ where c is a constant. Hence $\rho \circ \sigma$ maps S_p in $\mathcal{L}(H^2, (H^2)^\perp)$ onto $\{H_{\bar{f}} : f \in B_p\}$. It also follows easily that $(\rho \circ \sigma)^2 = \rho \circ \sigma$. Thus $Q = \rho \circ \sigma$ is a bounded projection from S_p onto $\{H_{\bar{f}} : f \in B_p\}$ and $\{H_{\bar{f}} : f \in B_p\}$ can be complemented in S_p for $1 < p < \infty$. ∎

Let $\mathcal{L}(L^2_a)$ be the space of bounded linear operators from the Bergman space L^2_a into itself. For $\phi \in h^\infty(\mathbb{D})$, define the Toeplitz operator $T_\phi : L^2_a \to L^2_a$ as $T_\phi f = P(\phi f)$ where P is the Bergman projection. Notice that T_ϕ is bounded with $\|T_\phi\| \leq \|\phi\|_\infty$. Similarly for $\psi \in L^\infty(\mathbb{T})$, one can define the Toeplitz operator B_ψ from $H^2(\mathbb{T})$ into itself as $B_\psi f = \widetilde{P}(\psi f)$, where $\widetilde{P}$ is the Szegō projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. Functions in $L^\infty(\mathbb{T})$ correspond via Poisson integral to bounded harmonic functions on $\mathbb{D}$ and the radial limits of functions in $h^\infty(\mathbb{D})$ belong to $L^\infty(\mathbb{T})$ (see [9]).

Hence there is an one-to-one correspondence between $h^\infty(\mathbb{D})$ and $L^\infty(\mathbb{T})$. Let $\mathcal{L}(H^2)$ be the set of all bounded linear operators from $H^2(\mathbb{T})$ into itself. The functions $u_n(z) = \sqrt{n+1}z^n, n = 0, 1, 2, \cdots$ form an orthonormal basis for $L^2_a(\mathbb{D})$ and $\{e_n(t)\}_{n=0}^\infty = \{e^{int}\}_{n=0}^\infty$ form an orthonormal basis for $H^2(\mathbb{T})$. In the following theorem we shall prove the existence of a bounded projection from $\mathcal{L}(L^2_a)$ onto $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$.

Theorem 3.2. *There exists a bounded projection from $\mathcal{L}(L^2_a)$ onto $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$.*

Proof. The set of functions $u_n(z) = \sqrt{n+1}z^n, n \in \mathbb{Z}_+$(the set of nonnegative integers), $z \in \mathbb{D}$ form an orthonormal basis for $L^2_a(\mathbb{D})$. Also every bounded harmonic function f on $\mathbb{D}$ can be written in the form $f = f_1 + \bar{f_2} = \sum_{n=0}^\infty a_n z^n + \sum_{n=1}^\infty a_{-n}\bar{z}^n, f_1, f_2 \in H^2(\mathbb{D})$ (see [9]).

This implies

$$f(z) = \sum_{n=0}^\infty \frac{a_n}{\sqrt{n+1}} u_n + \sum_{n=1}^\infty \frac{a_{-n}}{\sqrt{n+1}} \bar{u}_n.$$

Then we have

$$\begin{aligned}
&\langle T_f u_n, u_m\rangle \\
&= \left\langle u_n \sum_{k=0}^\infty \frac{a_k}{\sqrt{k+1}} u_k, u_m \right\rangle + \left\langle u_n \sum_{k=1}^\infty \frac{a_{-k}}{\sqrt{k+1}} \bar{u}_k, u_m \right\rangle \\
&= \left\langle \sum_{k=0}^\infty a_k u_{n+k} \sqrt{\frac{n+1}{n+k+1}}, u_m \right\rangle + \left\langle \sum_{k=1}^\infty a_{-k} \bar{u}_{k+m} \sqrt{\frac{m+1}{m+k+1}}, \bar{u}_n \right\rangle \\
&= \begin{cases} a_0 \text{ if } m = n; \\ a_{m-n}\sqrt{\dfrac{n+1}{m+1}} \text{ if } m > n; \\ a_{-(n-m)}\sqrt{\dfrac{m+1}{n+1}} \text{ if } m < n. \end{cases}
\end{aligned}$$

Thus the matrix of a Toeplitz operator with bounded harmonic symbol f is of the above form where $a_k, k \in \mathbb{Z}$ is the kth Fourier coefficient of $\widetilde{f}(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$.

Define W from $H^2(\mathbb{T})$ onto $L^2_a(\mathbb{D})$ as $We_n = u_n, n = 0, 1, 2, \ldots$ where $\{e_n\}_{n=0}^\infty$ is the standard orthonormal basis for $H^2(\mathbb{T})$ and $\{u_n\}_{n=0}^\infty$ is the standard orthonormal basis for $L^2_a(\mathbb{D})$. It is not difficult to show that W is a unitary operator from $H^2(\mathbb{T})$ onto $L^2_a(\mathbb{D})$ and it induces a map σ from $\mathcal{L}(L^2_a(\mathbb{D}))$ into $\mathcal{L}(H^2)$ given by $\sigma(T) = W^*TW$.

In [2], it is shown that there is a positive linear projection Ω from $\mathcal{L}(H^2)$ onto $\{B_\psi : \psi \in L^\infty(\mathbb{T})\}$ such that $\Omega(B_\psi) = B_\psi$ for all $\psi \in L^\infty(\mathbb{T})$. Let $\mathbb{N}$ be the additive semigroup of all positive integers and let Λ be a Banach limit on $\mathbb{N}$. Thus Λ is a state on the commutative C^*-algebra $l^\infty(\mathbb{N})$ (whose value at a bounded sequence $(a_n)_{n\geq 1}$ is denoted by $\Lambda_n a_n$) which has the additional property $\Lambda_n a_{n+1} = \Lambda_n a_n, (a_n) \in l^\infty(\mathbb{N})$. Let U denote the bilateral shift defined on the basis $\{e_n\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{T})$ by $Ue_n = e_{n+1}, n \in \mathbb{Z}$. It is well known [8] that U is a unitary operator and for $x, y \in H^2, A \in \mathcal{L}(H^2)$, we may define the form

$$[x, y] = \Lambda_n \langle U^{*n} A U^n x, y\rangle.$$

A straight forward application of the Schwarz lemma yields a unique operator $\Pi(A) \in \mathcal{L}(H^2)$ such that

$$\langle \Pi(A)x, y\rangle = \Lambda_n \langle U^{*n} A U^n x, y\rangle,$$

$U^*\Pi(A)U = \Pi(A)$ and define $\Omega(A) = \Pi(A)$ which is a Toeplitz operator B_ψ on the Hardy space for some $\psi \in L^\infty(\mathbb{T})$. As we have seen if $\phi \in h^\infty(\mathbb{D})$ then the matrix of the Toeplitz operator T_ϕ on the Bergman space has a special form and it follows easily that if $A = W^*T_\phi W, \phi \in h^\infty(\mathbb{D})$ then $\Omega(W^*T_\phi W) = \Omega(A) = \Pi(A) = B_{\widetilde{\phi}}$ where $\widetilde{\phi}(e^{i\theta}) = \lim_{r\to 1^-} \phi(re^{i\theta})$ belonging to $L^\infty(\mathbb{T})$. For more details see [2].

We have already seen that there is an one-to-one map ρ from $\{B_\psi : \psi \in L^\infty(\mathbb{T})\}$ onto $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$ such that $\rho(B_\psi) = T_{\widehat{\psi}}$ where $\widehat{\psi} \in h^\infty(\mathbb{D})$ is the harmonic extension of ψ. Hence $\rho \circ \Omega \circ \sigma$ is a map from $\mathcal{L}(L^2_a(\mathbb{D}))$ onto $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$ and $(\rho \circ \Omega \circ \sigma)^2 = \rho \circ \Omega \circ \sigma$. Hence $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$ can be complemented in $\mathcal{L}(L^2_a(\mathbb{D}))$. It is not difficult to verify that $\{T_\phi : \phi \in h^\infty(\mathbb{D})\}$ is closed in $\mathcal{L}(L^2_a(\mathbb{D}))$. ■

Let $(\overline{L^2_a})_0 = \{\bar{f} : f \in L^2_a, f(0) = 0\}$, the subspace of $L^2(\mathbb{D}, dA)$ consisting of all complex conjugates of functions in $L^2_a(\mathbb{D})$ which vanish at the origin. Let $\overline{P}$ be the orthogonal projection from L^2 onto $(\overline{L^2_a})_0$. For $\psi \in L^\infty(\mathbb{D})$, define the little Hankel operator $h_\psi : L^2_a \to (\overline{L^2_a})_0$ as $h_\psi f = \overline{P}(\psi f)$. The operator h_ψ is linear, bounded and $\|h_\psi\| \leq \|\psi\|_\infty$. Further, it is not difficult to show [9] that $\{h_\phi : \phi \in h^\infty(\mathbb{D})\} = \{h_{\overline{P\psi}} : \psi \in h^\infty(\mathbb{D})\} = \{h_{\overline{g}} : g \in BMOA(\mathbb{D})\}$. There are many equivalent ways of defining little Hankel operator on the Bergman space. For $\phi \in L^\infty(\mathbb{D})$, we can define $S_\phi : L^2_a \to L^2_a$ as $S_\phi f = P(J(\phi f))$ where $J : L^2(\mathbb{D}, dA) \to L^2(\mathbb{D}, dA)$ is such that $Jf(z) = f(\bar{z})$. Notice that $JS_\phi = h_\phi$ and J is unitary. Similarly, one can define $\Gamma_\phi : L^2_a \to L^2_a$ as $\Gamma_\phi f = P(\phi J f)$. It is easy to verify that $S_\phi = \Gamma_{J\phi}$. We shall refer all these operators $h_\phi, S_\phi, \Gamma_\phi$ as little Hankel operators on the Bergman space.

Recall that we had defined the Hankel operator on the Hardy space as follows: For $\psi \in L^\infty(\mathbb{T})$, the Hankel operator $H_\psi : H^2 \to (H^2)^\perp = (\overline{H^2})_0$ is defined as

$H_\psi f = (I - \widetilde{P})(\psi f)$, where $\widetilde{P}$ is the Szegö projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. One can also define Hankel operators on the Hardy space in the following way: For $\phi \in L^\infty(\mathbb{T})$, define $E_\phi : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ as $E_\phi f = \widetilde{P}(\widetilde{J}(\phi f))$ where $\widetilde{J} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined as $\widetilde{J} f(e^{it}) = f(e^{-it})$. The operator E_ϕ is unitarily equivalent to some $H_\psi, \psi \in L^\infty(\mathbb{T})$.

It is also easy [9] to see that $\{H_\psi : \psi \in L^\infty(\mathbb{T})\} = \{H_{\bar{f}} : f \in BMOA\}$ since $\widetilde{P} L^\infty(\mathbb{T}) = BMOA$.

If $\phi \in H^\infty(\mathbb{D})$, then $h_\phi = 0$. In fact, $h_\phi = 0$ if and only if $\phi \in (\overline{L^2_a})^\perp$. For $\phi \in \overline{H^\infty(\mathbb{D})}$, and $\phi(z) = \sum_{k=0}^\infty \widehat{\phi}(-k)\bar{z}^k$, the matrix of h_ϕ with respect to the orthonormal basis $\{u_n\}_{n=0}^\infty = \{\sqrt{n+1} z^n\}_{n=0}^\infty$ of $L^2_a(\mathbb{D})$ and $\{\bar{u}_n\}_{n=1}^\infty = \{\sqrt{n+1}\bar{z}^n\}_{n=1}^\infty$ of $(\overline{L^2_a})_0$ is given by

$$\langle h_\phi u_j, \bar{u}_i\rangle = \frac{\sqrt{i+1}\sqrt{j+1}}{i+j+1}\widehat{\phi}(-(i+j)), j = 0, 1, 2, \ldots, i = 1, 2, 3, \ldots.$$

Here $\overline{\widehat{\phi}(-k)}$ is the kth Taylor coefficient of $\bar{\phi}$. Therefore, $h_\phi = D_2 C_{\widetilde{\psi}} D_2$ where D_2 is the diagonal matrix given by $D_2 = \operatorname{diag}(1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots)$ and $\widetilde{\psi}(e^{i\theta}) = \sum_{k=0}^\infty \frac{1}{k+1}\widehat{\phi}(-k)e^{-ik\theta}$.

The function $\widetilde{\psi}$ is the convolution on the circle of $\widetilde{\phi}(e^{i\theta}) = \sum_{k=0}^\infty \widehat{\phi}(-k)e^{-ik\theta}$ with the function $\widetilde{\phi_1}(e^{i\theta}) = \sum_{k=0}^\infty \frac{1}{k+1}e^{-ik\theta}$ and $C_{\widetilde{\psi}}$ is the operator in $\mathcal{L}(L^2_a, (\overline{L^2_a})_0)$ having a classical Hankel matrix with symbol $\widetilde{\psi} \in L^\infty(\mathbb{T})$. That is,

$$\langle C_{\widetilde{\psi}} u_n, \bar{u}_m\rangle = \widehat{\widetilde{\psi}}(-(m+n)) = \langle H_{\widetilde{\psi}} e^{int}, e^{-imt}\rangle, n = 0, 1, 2, \ldots, m = 1, 2, 3, \ldots,$$

$\widehat{\widetilde{\psi}}(-k)$ is the kth negative Fourier coefficient of $\widetilde{\psi}$ and $H_{\widetilde{\psi}}$ is the Hankel operator on the Hardy space with symbol $\widetilde{\psi}$. Thus we have verified that if $\phi \in \overline{H^\infty(\mathbb{D})}$ then $h_\phi = D_2 C_{\widetilde{\psi}} D_2$.

Now let $\widetilde{\phi}(e^{i\theta}) = \sum_{k=0}^\infty \widehat{\phi}(-k)e^{-ik\theta} \in \overline{H^\infty(\mathbb{T})}$. Let

$$\widetilde{\psi}(e^{i\theta}) = \sum_{k=1}^\infty \frac{1}{k+1}\widehat{\phi}(-k)e^{-ik\theta} = \widetilde{\phi} \bigstar \widetilde{\phi_1}$$

where $\bigstar$ denotes convolution. Let $C_{\widetilde{\psi}}$ be the operator from L^2_a into $(\overline{L^2_a})_0$ such that

$$\langle C_{\widetilde{\psi}} u_n, \bar{u}_m\rangle = \widehat{\widetilde{\psi}}(-(m+n)) = \langle H_{\widetilde{\psi}} e^{in\theta}, e^{-im\theta}\rangle, n = 0, 1, 2, \ldots, m = 1, 2, 3, \ldots.$$

It is not difficult to verify that $D_2 C_{\widetilde{\psi}} D_2$ is a little Hankel operator on the Bergman space. In fact, $D_2 C_{\widetilde{\psi}} D_2 = h_\phi$, where $h_\phi \in \mathcal{L}(L^2_a, (\overline{L^2_a})_0)$ is the little Hankel operator with symbol ϕ, the harmonic extension of $\widetilde{\phi}$ into $\mathbb{D}$. Notice that $\phi \in \overline{H^\infty(\mathbb{D})}$. Such is the case if we replace $\overline{H^\infty(\mathbb{D})}$ by $\overline{BMOA(\mathbb{D})}$ and $H^\infty(\mathbb{T})$ by

$BMOA$. That is, the matrix of $h_\phi, \phi \in BMOA$ has a special form. Let L be a map from $\{h_{\bar\phi} : \phi \in BMOA(\mathbb{D})\}$ onto $\{H_{\overline{\widetilde{\phi}}} : \widetilde{\phi} \in BMOA\}$ such that $L(h_{\bar\phi}) = H_{\overline{\widetilde{\phi}}}$ where $\widetilde{\phi}(e^{i\theta}) = \lim_{r\to 1^-} \phi(re^{i\theta})$ a.e.. Notice that L is a linear, bijective, bounded map and $L^{-1}(H_{\overline{\widetilde{\phi}}}) = h_{\bar\phi}$ where ϕ is the harmonic extension of $\widetilde{\phi}$ into $\mathbb{D}$. Further, $\phi \in BMOA(\mathbb{D})$ if and only if $\widetilde{\phi} \in BMOA$.

Theorem 3.3. *The space $\{h_\phi : \phi \in h^\infty(\mathbb{D})\}$ cannot be complemented in $\mathcal{L}(L_a^2, (\overline{L_a^2})_0)$. That is, there exists no bounded projection from $\mathcal{L}(L_a^2, (\overline{L_a^2})_0)$ onto $\{h_\phi : \phi \in h^\infty(\mathbb{D})\}$.*

Proof. Suppose to the contrary there exists a bounded projection Ω from $\mathcal{L}(L_a^2, (\overline{L_a^2})_0)$ onto $\{h_\phi : \phi \in h^\infty(\mathbb{D})\}$. We have already defined a bijective, bounded linear map L from $\{h_\phi : \phi \in h^\infty(\mathbb{D})\} = \{h_{\bar g} : g \in BMOA(\mathbb{D})\}$ onto $\{H_\psi : \psi \in L^\infty(\mathbb{T})\} = \{H_{\bar f} : f \in BMOA\}$.

For $T \in \mathcal{L}(H^2, (\overline{H^2})_0)$, let $t_{-(m+n)} = \langle Te^{int}, e^{-imt}\rangle, n = 0, 1, 2, \ldots, m = 1, 2, 3, \ldots$. Let $c_{mn} = \dfrac{\sqrt{m+1}\sqrt{n+1}}{m+n+1}, m = 1, 2, 3, \ldots, n = 0, 1, 2, \ldots$. Define a map $\varPhi$ from $\mathcal{L}(H^2, (\overline{H^2})_0)$ into $\mathcal{L}(L_a^2, (\overline{L_a^2})_0)$ as $\varPhi(T) = W$, where $\langle Wu_n, \bar u_m\rangle = c_{mn} t_{-(m+n)}, n = 0, 1, 2, \ldots, m = 1, 2, 3, \ldots$. Since $|c_{mn}| \le 1$ for all $n = 0, 1, 2, \ldots, m = 1, 2, 3, \ldots$, hence $W \in \mathcal{L}(L_a^2, (\overline{L_a^2})_0)$. It therefore follows that $M = L \circ \Omega \circ \varPhi$ is a map from $\mathcal{L}(H^2, (\overline{H^2})_0)$ into $\{H_\phi : \phi \in L^\infty(\mathbb{T})\}$ and $M^2 = M$. The map M is onto. For, suppose $\bar\phi \in L^\infty(\mathbb{T})$. Let $\overline{\widetilde{P}\phi} = \overline{g}, g \in BMOA$. Let $\widehat{\overline{g}}$ be the harmonic extension of $\overline{g}$ into $\mathbb{D}$. Then $L(h_{\widehat{\overline{g}}}) = H_{\overline{g}} = H_{\overline{\widetilde{P}\phi}} = H_{\overline{\phi}}$. Thus M is a bounded projection from $\mathcal{L}(H^2, (\overline{H^2})_0)$ onto $\{H_\phi : \phi \in L^\infty(\mathbb{T})\}$. But such a map M does not exist (see [8]). Hence there exists no bounded projection from $\mathcal{L}(L_a^2, (\overline{L_a^2})_0)$ onto $\{h_\phi : \phi \in h^\infty(\mathbb{D})\}$. ■

4. On Certain Subspaces of $L^\infty(\mathbb{D})$

A function $\phi \in L^\infty(\mathbb{D})$ is said to satisfy the condition (*) if there exists a compact subset M of $\mathbb{D}$ such that ϕ is continuous on $\overline{\mathbb{D}} - M$. Let $F(\mathbb{D}) = \{\phi \in L^\infty(\mathbb{D}) : \phi$ satisfies $(*)\}$.

Notice that every element of $F(\mathbb{D})$ is continuous off a compact subset of $\mathbb{D}$ and also continuous on $\mathbb{T}$. Clearly, $F(\mathbb{D})$ is an algebra but not closed. Let $\overline{F(\mathbb{D})}$ be the closure of $F(\mathbb{D})$ in $L^\infty(\mathbb{D})$. Let

$$I_0 = \{\phi \in L^\infty(\mathbb{D}) : \lim_{\delta\to 0} \text{ ess sup }_{1-\delta<|z|<1} |\phi(z)| = 0\}.$$

Let $V(\mathbb{D}) = \overline{F(\mathbb{D})}/I_0$. In Theorem 4.1, we shall show that $\overline{F(\mathbb{D})}$ cannot be complemented in $h^\infty(\mathbb{D})$.

Theorem 4.1. *There exists an isometrical *-isomorphism from $V(\mathbb{D})$ onto $C(\mathbb{T})$ and there exists no bounded projection from $h^\infty(\mathbb{D})$ onto $\overline{F(\mathbb{D})}$.*

Proof. Notice that for every $\phi \in F(\mathbb{D}), \lim_{r\to 1^-} \phi(re^{i\theta})$ exists for all $\theta \in [0, 2\pi]$. Let $\widetilde{\phi}$ represent the radial limit of ϕ in $F(\mathbb{D})$. Clearly $\widetilde{\phi} \in C(\mathbb{T})$ for every $\phi \in F(\mathbb{D})$. By definition, for every function $\phi \in \overline{F(\mathbb{D})}$, there exists a sequence $\{\phi_n\}$ of functions in $F(\mathbb{D})$ such that $\lim\limits_{n\to\infty} \|\phi_n - \phi\|_\infty = 0$. Hence it follows that $\lim_{m,n\to\infty} \|\phi_n - \phi_m\|_\infty = 0$. Thus we have $\lim_{m,n\to\infty} \|\widetilde{\phi}_n - \widetilde{\phi}_m\|_\infty = 0$ which implies that the sequence $\{\widetilde{\phi}_n\}$ is a Cauchy sequence in $C(\mathbb{T})$, hence converges in $C(\mathbb{T})$. Let it converge to $\widetilde{\phi}$ in $C(\mathbb{T})$. Clearly $\widetilde{\phi}$ is independent of the choice of the sequence $\{\phi_n\}$, thus is well-defined. We shall call $\widetilde{\phi}$ the boundary value function of ϕ in $\overline{F(\mathbb{D})}$. Now, the map $\phi \mapsto \widetilde{\phi}$ is a $*-$ homomorphism of $\overline{F(\mathbb{D})}$ onto $C(\mathbb{T})$ with I_0 as the kernel. Hence by the first isomorphism theorem the map $\phi + I_0 \mapsto \widetilde{\phi}$ is a *-isomorphism of $\overline{F(\mathbb{D})}/I_0$ onto $C(\mathbb{T})$. Further we know for $\phi \in \overline{F(\mathbb{D})}$,

$$\|\phi\|_\infty = \text{ ess sup }_{z\in\overline{\mathbb{D}}}|\phi(z)| \geq \|\widetilde{\phi}\|_\infty.$$

Hence we have that

$$\|\phi + I_0\| = \inf_{\psi\in\phi+I_0} \|\psi\|_\infty \geq \inf_{\psi\in\phi+I_0} \|\widetilde{\psi}\|_\infty = \|\widetilde{\phi}\|_\infty.$$

Further, let g be a harmonic function in $\phi + I_0$. Then $\|\phi + I_0\| \leq \|g\|_\infty = \|\widetilde{g}\|_\infty = \|\widetilde{\phi}\|_\infty$. Hence the map $\phi + I_0 \mapsto \widetilde{\phi}$ is an isometrical *-isomorphism of $\overline{F(\mathbb{D})}/I_0$ onto $C(\mathbb{T})$.

Let R be the map from $\overline{F(\mathbb{D})}$ onto $C(\mathbb{T})$ defined as $R(\phi) = \widetilde{\phi}$. We have already seen that there exists an isometrical isomorphism T from $h^\infty(\mathbb{D})$ onto $L^\infty(\mathbb{T})$. In fact, if $\phi \in h^\infty(\mathbb{D})$ then $T\phi = \widetilde{\phi}$ where $\widetilde{\phi}(e^{i\theta}) = \lim_{r\to 1^-} \phi(re^{i\theta})$ and $\widetilde{\phi} \in L^\infty(\mathbb{T})$ (see [4]). Further if $\widetilde{\psi} \in L^\infty(\mathbb{T}),\ T^{-1}\widetilde{\psi} = \widehat{\psi}$ where $\widehat{\psi}$ is the harmonic extension of $\widetilde{\psi}$ into $\mathbb{D}$.

Suppose there exists a bounded projection Q from $h^\infty(\mathbb{D})$ onto $\overline{F(\mathbb{D})}$. Then $M = R \circ Q \circ T^{-1}$ is a map from $L^\infty(\mathbb{T})$ onto $C(\mathbb{T})$ and $M^2 = M$. Hence M is a bounded projection from $L^\infty(\mathbb{T})$ onto $C(\mathbb{T})$. Such map M does not exist by Lemma 2.7. The theorem is proved. ■

Let

$$J_0 = \{\phi \in F(\mathbb{D}) : \phi(e^{i\theta}) = 0 \text{ for all } \theta \in [0, 2\pi]\}.$$

It is easy to check that the closure of J_0 in $L^\infty(\mathbb{D})$ is I_0. For, let Θ be any element in I_0. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that $\text{ess sup}_{1-\delta<|z|<1}|\Theta(z)| < \epsilon$. Let, now, $\phi : \overline{\mathbb{D}} \to \mathbb{C}$ be defined by

$$\phi(z) = \begin{cases} 0 \text{ if } 1 - \delta < |z| \leq 1; \\ \Theta(z) \text{ if } |z| \leq 1 - \delta. \end{cases}$$

Then ϕ is in J_0 and $\|\Theta - \phi\|_\infty < \epsilon$.

We shall now introduce another closed subalgebra of $L^\infty(\mathbb{D})$ which can be complemented in $L^\infty(\mathbb{D})$. Let

$$L^\infty_C(\mathbb{D}) = \{\phi \in L^\infty(\mathbb{D}) : \phi \text{ is continuous and } \lim_{r\to 1^-} \phi(re^{i\theta}) \text{ exists a.e. on } \mathbb{T}\}.$$

Let $\widetilde{\phi}$ represent the radial limit function of ϕ in $L^\infty_C(\mathbb{D})$. A complex valued function ϕ on $\mathbb{D}$ is said to satisfy the condition (**) if there is a compact subset M of $\mathbb{D}$ such that ϕ is essentially bounded on M, bounded and continuous on the complement of M in $\mathbb{D}$ and $\lim_{r\to 1^-} \phi(re^{i\theta})$ exists almost everywhere in $\mathbb{T}$. Then we define $G(\mathbb{D}) = \{\phi \in L^\infty(\mathbb{D}) : \phi \text{ satisfies } (**)\}$. That is, $G(\mathbb{D})$ is the set of those functions in $L^\infty(\mathbb{D})$ which behave like $L^\infty_C(\mathbb{D})$ functions off a compact subset of $\mathbb{D}$. Clearly, the radial limit of functions $\phi \in G(\mathbb{D})$ exists a.e. and let $\widetilde{\phi}(e^{i\theta}) = \lim_{r\to 1^-} \phi(re^{i\theta})$. It is easy to see that $G(\mathbb{D})$ is an algebra but not closed. Let $\overline{G(\mathbb{D})}$ be the closure of $G(\mathbb{D})$ in $L^\infty(\mathbb{D})$. For $\phi \in \overline{G(\mathbb{D})}$, there exists a sequence $\{\phi_n\}$ in $G(\mathbb{D})$ such that $\lim_{n\to\infty} \|\phi_n - \phi\|_\infty = 0$. Hence it follows that $\lim_{m,n\to\infty} \|\phi_n - \phi_m\|_\infty = 0$. Thus, we have that $\lim_{m,n\to\infty} \|\widetilde{\phi}_n - \widetilde{\phi}_m\|_\infty = 0$, which implies that the sequence $\{\widetilde{\phi}_n\}$ is a Cauchy sequence and hence converges in $L^\infty(\mathbb{T})$. Let it converge to $\widetilde{\phi}$ in $L^\infty(\mathbb{T})$. Clearly, $\widetilde{\phi}$ is independent of the choice of the sequence $\{\phi_n\}$, thus is well defined. We, henceforth, call $\widetilde{\phi}$ the boundary value function of ϕ in $\overline{G(\mathbb{D})}$. Now we shall relate $\phi \in \overline{G(\mathbb{D})}$ to $\widetilde{\phi}$ in $L^\infty(\mathbb{T})$. To do this, we define the following subset of $\overline{G(\mathbb{D})}$. Let

$$K_0 = \{\phi \in \overline{G(\mathbb{D})} : \widetilde{\phi} = 0 \textit{ a.e.} \text{ on } \mathbb{T}\}.$$

It is not difficult to check that K_0 is a closed ideal of $\overline{G(\mathbb{D})}$. Let $W(\mathbb{D}) = \overline{G(\mathbb{D})}/K_0$.

Theorem 4.2. *There exists an isometrical *-isomorphism from $W(\mathbb{D})$ onto $L^\infty(\mathbb{T})$ and the space $\overline{G(\mathbb{D})}$ can be complemented in $L^\infty(\mathbb{D})$.*

Proof. We have seen that the map $\phi \to \widetilde{\phi}$ is a $*-$ homomorphism of $\overline{G(\mathbb{D})}$ onto $L^\infty(\mathbb{T})$ with kernel K_0. Therefore, by the first isomorphism theorem the map $\phi + K_0 \mapsto \widetilde{\phi}$ is a $*-$ isomorphism of $\overline{G(\mathbb{D})}/K_0 = W(\mathbb{D})$ onto $L^\infty(\mathbb{T})$. Further, it is easy to see that for $\phi \in \overline{G(\mathbb{D})}$, $\|\phi\|_\infty = \text{ ess sup }_{z\in\overline{\mathbb{D}}} |\phi(z)| \geq \|\widetilde{\phi}\|_\infty$. Hence

$$\|\phi + K_0\| = \inf_{g\in\phi+K_0} \|g\|_\infty \geq \inf_{g\in\phi+K_0} \|\widetilde{g}\|_\infty = \|\widetilde{\phi}\|_\infty.$$

Further, if h is a harmonic function in $\phi + K_0$, then

$$\|\phi + K_0\| \leq \|h\|_\infty = \|\widetilde{h}\|_\infty = \|\widetilde{\phi}\|_\infty.$$

Hence the map $\phi + K_0 \mapsto \widetilde{\phi}$ is an isometrical $*-$ isomorphism of $\overline{G(\mathbb{D})}/K_0 = W(\mathbb{D})$ onto $L^\infty(\mathbb{T})$ and $\|\phi + K_0\| = \|\widetilde{\phi}\|_\infty$.

Let S be the homomorphism from $L^\infty(\mathbb{T})$ onto $\overline{G(\mathbb{D})}$ defined by $S\phi = \widehat{\phi}$ where $\widehat{\phi}$ is the harmonic extension of ϕ into $\mathbb{D}$. Define a map $T : h^\infty(\mathbb{D}) \to L^\infty(\mathbb{T})$ as $Tf = \widetilde{f}$ where $\widetilde{f}(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$. Notice that if $\psi \in L^\infty(\mathbb{T})$ then $T^{-1}\psi = \widehat{\psi}$, the harmonic extension of ψ into $\mathbb{D}$. The map T is bijective, linear and bounded. Let Q be the bounded projection [6, 7] from $L^\infty(\mathbb{D})$ onto $h^\infty(\mathbb{D})$. Then $M = S \circ T \circ Q$ is a map from $L^\infty(\mathbb{D})$ onto $\overline{G(\mathbb{D})}$, M is bounded, linear and $M^2 = M$. Thus M is the required bounded projection from $L^\infty(\mathbb{D})$ onto $\overline{G(\mathbb{D})}$. ■

References

1. J. M. Anderson, J. Clunie, and Ch. Pommerenke, On Bloch functions and normal functions, *J. Reine Angew. Math.* **270** (1974), 12–37.
2. W. Arveson, Interpolation problems in nest algebras, *J. Funct. Anal.* **20** (1975), 208–233.
3. P. L. Duren, *Theory of H^p Spaces*, Academic press, New York, 1970.
4. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, New Jersey, 1962.
5. J. Lindenstrauss and L. Tzafriri, On the complemented subspaces problem, *Israel J. Math.* **9** (1971), 263–269.
6. W. Lusky, On the isomorphic classification of weighted spaces of holomorphic functions, *Acta Univ. Carolinae* **41** (2000), 51–60.
7. W. Lusky, On the isomorphism classes of weighted spaces of harmonic and holomorphic functions, *preprint*, 2004.
8. N. K. Nikolskii, *Treatise on the Shift Operator*, Springer-Verlag, Berlin, 1986.
9. K. Zhu, *Operator Theory in Function Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, Marcell Dekker, Inc. **139**, New York and Basel, 1990.