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On Maps and Generalized Λ_b-Sets

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Abstract. In this paper we define the concepts of $g.\Lambda_b$-continuous maps, $g.\Lambda_b$-irresolute maps and $g.V_b$-closed maps by using generalized Λ_b-sets and generalized V_b-sets. Also we introduce a new class of topological spaces called T^{V_b}-spaces.

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1. Introduction

The concept of b-open sets in a topological space was introduced by Andrijevic in [1]. One year later, this notion was called γ-open sets by El-Atik [5]. A subset A of a topological space (X, τ) is said to be b-open (=γ-open [5]) if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$, where $\mathrm{Cl}(A)$ denotes the closure of A and $\mathrm{Int}(A)$ denotes the interior of A in (X, τ). The complement A^c of a b-open set A is called b-closed [1] (=γ-closed [5]). The family of all b-open (resp. b-closed) sets in (X, τ) is denoted by $BO(X, \tau)$ (resp. $BC(X, \tau)$). The intersection of all b-closed sets containing A is called the b-closure of A [1] and is denoted by $b\mathrm{Cl}(A)$. Recently, Ekici introduced and studied the concept of b-R_0 [4] in topological spaces. A topological space (X, τ) is called a b-R_0-space if every b-open set contains the b-closure of each of its singletons. Quite recently Caldas et al. [3] used b-open sets to define and investigate the Λ_b-sets (resp. V_b-sets) which are intersections of b-open (resp. union of b-closed) sets. The purpose of the present paper is to introduce and study the concepts of $g.\Lambda_b$-continuous map (which includes the class of b-continuous maps); $g.\Lambda_b$-irresolute maps (defined as an analogy of b-irresolute maps) and $g.V_b$-closed maps by using $g.\Lambda_b$-sets and $g.V_b$-sets. This definition enables us to obtain conditions under which maps and inverse maps preserve $g.\Lambda_b$-sets and $g.V_b$-sets. Moreover, we introduce a new class of topological spaces called T^{V_b}-spaces and as an application, we show that the image of a T^{V_b}-space under a homeomorphism is a T^{V_b}-space.

2. Preliminaries

Throughout this paper, (X,τ), (Y,σ) and (Z,ν) (or simply X, Y and Z) will always denote topological spaces on which no separation axioms are assumed, unless explicitly stated.

Definition 2.1. A map $f:(X,\tau)\to(Y,\sigma)$ is said to be *b-continuous* [5] (=γ-continuous [5]) (resp. *b-irresolute* [5](=γ*-irresolute* [5])) if for every $A\in\sigma$ (resp. $A\in BO(Y,\sigma)$) $f^{-1}(A)\in BO(X,\tau)$, or equivalently, f is b-continuous (resp. b-irresolute) if and only if for every closed set A (resp. b-closed set A) of (Y,σ), $f^{-1}(A)\in BC(X,\tau)$.

Definition 2.2. A space topological (X,τ) is called a b-T_1 [3] if to each pair of distinct points x,y of (X,τ) there corresponds a b-open set A containing x but not y and a b-open set B containing y but not x, or equivalently, (X,τ) is a b-T_1-space if and only if every singleton is b-closed.

Definition 2.3. Let B be a subset of a topological space (X,τ). B is a Λ_b-set (resp. V_b-set) [3], if $B=B^{\Lambda_b}$ (resp. $B=B^{V_b}$), where

$$B^{\Lambda_b}=\bigcap\{O: O\supseteq B, O\in BO(X,\tau)\}$$

and

$$B^{V_b}=\bigcup\{F: F\subseteq B, F^c\in BO(X,\tau)\}.$$

Definition 2.4. In a topological space (X,τ), a subset B is called

(i) *generalized* Λ_b*-set* (or $g.\Lambda_b$*-set*) of (X,τ) [3] if $B^{\Lambda_b}\subseteq F$ and $F\in BC(X,\tau)$.

(ii) *generalized* V_b*-set* (or $g.V_b$*-set*) of (X,τ) [3] if B^c is a $g.\Lambda_b$-set of (X,τ).

By S^{Λ_b} (resp. S^{V_b}) we will denote the family of all $g.\Lambda_b$-sets (resp. $g.V_b$-sets) of (X,τ).

Proposition 2.5. [3] *For some* $\{B_\Lambda:\Lambda\in\Omega\}$*, let* A,B *be subsets of a topological space* (X,τ)*. Then the following properties are valid:*

(a) $B\subseteq B^{\Lambda_b}$;

(b) *If* $A\subseteq B$ *then* $A^{\Lambda_b}\subseteq B^{\Lambda_b}$;

(c) $B^{\Lambda_b\Lambda_b}=B^{\Lambda_b}$;

(d) $[\bigcup_{\Lambda\in\Omega}B_\Lambda]^{\Lambda_b}=\bigcup_{\Lambda\in\Omega}B_\Lambda^{\Lambda_b}$;

(e) *If $A \in BO(X,\tau)$ then $A = A^{\Lambda_b}$ (i.e A is a Λ_b-set);*

(f) *$(B^c)^{\Lambda_b} = (B^{V_b})^c$;*

(g) *$B^{V_b} \subseteq B$;*

(h) *If $B \in BC(X,\tau)$ then $B = B^{V_b}$ (i.e A is a V_b-set);*

(i) *$[\bigcap_{\Lambda\in\Omega} B_\Lambda]^{\Lambda_b} \subseteq \bigcap_{\Lambda\in\Omega} B_\Lambda^{\Lambda_b}$;*

(j) *$[\bigcup_{\Lambda\in\Omega} B_\Lambda]^{V_b} \supseteq \bigcup_{\Lambda\in\Omega} B_\Lambda^{V_b}$;*

Proposition 2.6. [3] *Let (X,τ) be a topological space. Then*

(a) *Every Λ_b-set is a $g.\Lambda_b$-set;*

(b) *Every V_b-set is a $g.V_b$-set;*

(c) *If $B_\Lambda \in S^{\Lambda_b}$ for all $\Lambda \in \Omega$ then $\bigcup_{\Lambda\in\Omega} B_\Lambda \in S^{\Lambda_b}$;*

(d) *If $B_\Lambda \in S^{V_b}$ for all $\Lambda \in \Omega$ then $\bigcup_{\Lambda\in\Omega} B_\Lambda \in S^{V_b}$.*

Proposition 2.7. [3] *A topological space (X,τ) is a b-T_1-space if and only if every subset is a Λ_b-set (or equivalently a V_b-set).*

Corollary 2.8. *Every b-T_1-space is a b-R_0-space.*

3. $G.\Lambda_b$-continuous Maps and $G.\Lambda_b$-irresolute Maps

We introduce the following definition.

Definition 3.1. Let $f : (X,\tau) \to (Y,\sigma)$ be a map from a topological space (X,τ) into a topological space (Y,σ). Then,

(i) f is called generalized Λ_b-continuous map (abbrev. $g.\Lambda_b$-continuous map) if $f^{-1}(A)$ is a $g.\Lambda_b$-set in (X,τ) for every open set A of (Y,σ);

(ii) f is called generalized Λ_b-irresolute map (abbrev. $g.\Lambda_b$-irresolute map) if $f^{-1}(A)$ is a $g.\Lambda_b$-set in (X,τ) for every $g.\Lambda_b$-set of (Y,σ).

Proposition 3.2. *Let $f : (X,\tau) \to (Y,\sigma)$ be b-continuous. Then f is $g.\Lambda_b$-continuous.*

Proof. The proof follows from the fact that every b-open set is $g.\Lambda_b$-set (Proposition 2.5(e) and Proposition 2.6(a)). ■

The converse needs not be true as seen from the following example.

Example 3.3. Let $X = Y = \{a,b,c\}$, $\tau = \{\varnothing, \{a,b\}, X\}$ and $\sigma = \{\varnothing, \{c\}, Y\}$. Then the identity map $f : (X,\tau) \to (Y,\sigma)$ is $g.\Lambda_b$-continuous but not b-continuous.

Proposition 3.4. *Let $f : (X,\tau) \to (Y,\sigma)$ be $g.\Lambda_b$-irresolute. Then f is $g.\Lambda_b$-continuous.*

Proof. Since every open set is b-open and every b-open set is $g.\Lambda_b$-set it proves that f is $g.\Lambda_b$-continuous. ■

The converse needs not be true as seen from the following example.

Example 3.5. Let $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, X\}$ and $\sigma = \{\varnothing, \{a, b\}, Y\}$. The identity map $f : (X, \tau) \to (Y, \sigma)$ is $g.\Lambda_b$-continuous but is not $g.\Lambda_b$-irresolute since for the $g.\Lambda_b$-set $\{b\}$ of (Y, σ), the inverse image $f^{-1}(\{b\}) = \{b\}$ is not a $g.\Lambda_b$-set of (X, τ).

Theorem 3.6. *A map $f : (X, \tau) \to (Y, \sigma)$ is $g.\Lambda_b$-irresolute (resp. $g.\Lambda_b$-continuous) if and only if, for every $g.\Lambda_b$-set A (resp. closed set A) of (Y, σ) the inverse image $f^{-1}(A)$ is a $g.V_b$-set of (X, τ).*

Proof. Necessity. If $f : (X, \tau) \to (Y, \sigma)$ is $g.\Lambda_b$-irresolute, then every $g.\Lambda_b$-set B of (Y, σ), $f^{-1}(B)$ is a $g.\Lambda_b$-set in (X, τ). If A is any $g.V_b$-set of (Y, σ), then A^c is a $g.\Lambda_b$-set. Thus $f^{-1}(A^c)$ is a $g.\Lambda_b$-set, but $f^{-1}(A^c) = (f^{-1}(A))^c$ so $f^{-1}(A)$ is a $g.V_b$-set.

Sufficiency. If, for all $g.V_b$-sets of (Y, σ), $f^{-1}(A)$ is a $g.V_b$-set in (X, τ), and if B is any $g.\Lambda_b$-set of (Y, σ), then B^c is a $g.V_b$-set. Also $f^{-1}(B^c) = (f^{-1}(B))^c$ is a $g.V_b$-set. Thus, $f^{-1}(B)$ is a $g.\Lambda_b$-set. ■

In a similar way we prove the case f is $g.\Lambda_b$-continuous.

Definition 3.7. A map $f : (X, \tau) \to (Y, \sigma)$ is called *pre-b-closed* (resp. *pre-b-open*), if $f(A) \in BC(Y, \sigma)$ (resp. $f(A) \in BO(Y, \sigma)$) for every $A \in BC(X, \tau)$ (resp. $A \in BO(X, \tau)$).

A bijection $f : (X, \tau) \to (Y, \sigma)$ is pre-b-open, if and only if, f is pre-b-closed.

Theorem 3.8. *If a map $f : (X, \tau) \to (Y, \sigma)$ is bijective b-irresolute and pre-b-closed, then*

(i) for every $g.\Lambda_b$-set B of (Y, σ), $f^{-1}(B)$ is a $g.\Lambda_b$-set of (X, τ) (i.e., f is $g.\Lambda_b$-irresolute);

(ii) for every $g.\Lambda_b$-set A of (X, τ), $f(A)$ is a $g.\Lambda_b$-set of (Y, σ) (i.e., f is $g.\Lambda_b$-open).

Proof. (i) Let B be a $g.\Lambda_b$-set of (Y, σ). Suppose that $f^{-1}(B) \subseteq F$ where F is b-closed in (X, τ). Therefore $B \subseteq f(F)$ and $f(F)$ is b-closed, because f is pre-b-closed. Since B is a $g.\Lambda_b$-set, $B^{\Lambda_b} \subseteq f(F)$, hence $f^{-1}(B^{\Lambda_b}) \subseteq F$. Therefore, we have $(f^{-1}(B))^{\Lambda_b} \subseteq f^{-1}(B^{\Lambda_b}) \subseteq F$. Hence $f^{-1}(B)$ is a $g.\Lambda_b$-set in (X, τ).

(ii) Let A be a $g.\Lambda_b$-set of (X, τ). Let $f(A) \subseteq F$ where F is some b-closed set of (Y, σ). Then $A \subseteq f^{-1}(F)$ and $f^{-1}(F)$ is b-closed because f is $g.\Lambda_b$-irresolute. Since f is pre-b-open (f is bijective), $(f(A))^{\Lambda_b} \subseteq f(A^{\Lambda_b}) \subseteq F$. Hence $f(A)$ is a $g.\Lambda_b$-set in (Y, σ). ■

Corollary 3.9. *If a map $f : (X, \tau) \to (Y, \sigma)$ is bijective, b-irresolute and pre-b-closed, then*

(i) *for every $g.V_b$-set B of (Y, σ), $f^{-1}(B)$ is a $g.V_b$-set of (X, τ),*

(ii) *for every $g.V_b$-set A of (X, τ), $f(A)$ is a $g.V_b$-set of (Y, σ).*

Definition 3.10. A topological space (X, τ) is a b-$T_{1/2}$ space if and only if every $g.V_b$-set is a V_b-set.

Proposition 3.11. *Under the same assumption of Theorem 3.8, if (X, τ) is b-$T_{1/2}$ then (Y, σ) is b-$T_{1/2}$.*

Proof. By the above remark, it suffices to prove that every $g.V_b$-set of (Y, σ) is a V_b-set. In fact, let B be a $g.V_b$-set of (Y, σ). Then by Corollary 3.9, $f^{-1}(B)$, say H, is a $g.V_b$-set in (X, τ). But (X, τ) is b-$T_{1/2}$ and so H is a V_b-set. By assumptions and Definition 2.3, $f(H) = f(H^{V_b}) \subseteq (f(H))^{V_b}$ which shows that $B = B^{V_b}$, i.e., B is a V_b-set. ■

Definition 3.12. (i) A map $f : (X, \tau) \to (Y, \sigma)$ is said to be *b-homeomorphism* [5] if f is bijective, b-irresolute and pre-b-closed;
(ii) A map $f : (X, \tau) \to (Y, \sigma)$ is said to be *$g.\Lambda_b$-homeomorphism* if f is bijective, $g.\Lambda_b$-irresolute and $g.\Lambda_b$-open.

Corollary 3.13. *The b-homeomorphic image of a b-$T_{1/2}$ space is b-$T_{1/2}$.*

Corollary 3.14. *Every b-homeomorphism is a $g.\Lambda_b$-homeomorphism.*

Remark 3.15. If $f : (X, \tau) \to (Y, \sigma)$ is a homeomorphism, then f is a b-homeomorphism, hence, we have that every homeomorphism is a $g.\Lambda_b$ -homeomorphism, the converse is not true in general. Let $X = Y = \{a, b, c\}$, and let $\tau = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\varnothing, \{a\}, \{a, b\}, Y\}$. Then the identity map $f : (X, \tau) \to (Y, \sigma)$ is a $g.\Lambda_b$ -homeomorphism, but it is not a homeomorphism.

Proposition 3.16. (i) *If $f : (X, \tau) \to (Y, \sigma)$ is a $g.\Lambda_b$-irresolute map and $h : (Y, \sigma) \to (Z, \nu)$ is a $g.\Lambda_b$-continuous map, then the composition $h \circ f : (X, \tau) \to (Z, \nu)$ is $g.\Lambda_b$-continuous;*

(ii) *If $f : (X, \tau) \to (Y, \sigma)$ and $h : (Y, \sigma) \to (Z, \nu)$ are both $g.\Lambda_b$-irresolute, then the composition $h \circ f : (X, \tau) \to (Z, \nu)$ is a $g.V_b$-irresolute map.*

Proof. The proof follows directly from definitions. ■

4. $G.V_b$-closed Maps and T^{V_b}-spaces

Now, we introduce a new class of map called $g.V_b$-closed map. Let $f : (X, \tau) \to (Y, \sigma)$ be a map from a topological space (X, τ) into a topological space (Y, σ).

Definition 4.1. A map $f : (X, \tau) \to (Y, \sigma)$ is called a generalized V_b-closed map (written as $g.V_b$-closed map), if for each closed set F of X, $f(F)$ is a $g.V_b$-set.

Obviously, every b-closed map (i.e., $f(F) \in BC(Y, \sigma)$ for every closed F in (X, τ)) is a $g.V_b$-closed map and the converse is not always true as the following example shows.

Example 4.2. Let $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, X\}$ and $\sigma = \{\varnothing, \{a, b\}, Y\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then f is $g.V_b$-closed but not b-closed since for a closed set $\{b, c\}$ of (X, τ), the image $f(\{b, c\}) = \{a, b\}$ is not b-closed.

Theorem 4.3. *A map $f : (X, \tau) \to (Y, \sigma)$ is a $g.V_b$-closed if and only if for each subset S of Y and for each open set U containing $f^{-1}(S)$, there is a $g.\Lambda_b$-set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.*

Proof. Necessity. Let S be a subset of Y and let U be an open set of X such that $f^{-1}(S) \subset U$. Then $(f(U^c))^c$, say V, is a $g.\Lambda_b$-set containing S such that $f^{-1}(V) \subset U$.

Sufficiency. Let F be an arbitrary closed set of X. Then, $f^{-1}((f(F))^c) \subset F^c$ and F^c is open. By hypothesis, there is a $g.\Lambda_b$-set V of Y such that $(f(F))^c \subset V$ and hence $V^c \subset f(F) \subset f((f^{-1}(V))^c) \subset V^c$, which implies $f(F) = V^c$. Since V^c is a $g.V_b$-set, $f(F)$ is a $g.V_b$-set and thus f is a $g.V_b$-closed map. ∎

Theorem 4.4. *Let $f : (X, \tau) \to (Y, \sigma)$, $h : (Y, \sigma) \to (Z, \nu)$ be two mapping such that $h \circ f : (X, \tau) \to (Z, \nu)$ is a $g.V_b$-closed map. Then,*

(i) if f is continuous and surjective, then h is $g.V_b$-closed;

(ii) if h is b-irresolute, pre-b-closed and bijective, then f is $g.V_b$-closed.

Proof. (i) Let B be a closed set of Y. Since $f^{-1}(B)$ is closed in X, $(h \circ f)(f^{-1}(B))$ is a $g.V_b$-set in Z and hence $h(B)$ is a $g.V_b$-set in Z. This implies that h is a $g.V_b$-closed map.

(ii) Let F be a closed set of X. Then $(h \circ f)(F)$ is a $g.V_b$-set in Z and $h^{-1}((h \circ f)(F))$ is a $g.V_b$-set in Y using assumptions and Corollary 3.9. Since h is injective, $f(F) = h^{-1}((h \circ f)(F))$ is a $g.V_b$-set in Y. Therefore f is $g.V_b$-closed. ∎

Theorem 4.5. *(i) If $f : (X, \tau) \to (Y, \sigma)$ is a $g.V_b$-closed map and $h : (Y, \sigma) \to (Z, \nu)$ is bijective, b-irresolute and pre-b-closed, then $h \circ f : (X, \tau) \to (Z, \nu)$ is $g.V_b$-closed;*

(ii) If $f : (X, \tau) \to (Y, \sigma)$ is a closed map and $h : (Y, \sigma) \to (Z, \nu)$ is a $g.V_b$-closed map, then $h \circ f : (X, \tau) \to (Z, \nu)$ is $g.V_b$-closed.

Proof. (i) Let F be an arbitrary closed set in (X, τ). Then $f(F)$ is a $g.V_b$-set in (Y, σ). Since h is bijective, b-irresolute and pre-b-closed, $(h \circ f)(F) = h(f(F))$ is a $g.V_b$-set (Corollary 3.9(ii)).

(ii) The proof follows immediately from definitions. ∎

Theorem 4.6. *For a topological space (X,τ), every singleton of X is a $g.\Lambda_b$-set if and only if $A = A^{V_b}$ holds for every $A \in BO(X,\tau)$.*

Proof. Necessity. Let A be a b-open set. Let $y \in A^c$, then $\{y\}^{\Lambda_b} \subset A^c$ by assumption. By using Proposition 2.5(d) we have $A^c \supseteq \bigcup\{\{y\}^{\Lambda_b} : y \in A^c\} = (A^c)^{\Lambda_b}$ and hence $A^c = (A^c)^{\Lambda_b}$. Then it follows from Proposition 2.5(f) that $A = A^{V_b}$.

Sufficiency. Let $x \in X$ and F be a b-closed set such that $\{x\} \subset F$. Since $F^c = (F^c)^{V_b} = (F^{\Lambda_b})^c$, we have $F = F^{\Lambda_b}$. Therefore we have $\{x\}^{\Lambda_b} \subset F^{\Lambda_b} = F$. Hence $\{x\}$ is a $g.\Lambda_b$-set. ■

Recall that τ^{Λ_b} is the topology on X, generated by C^{Λ_b} in the usual manner [4], i.e., $\tau^{\Lambda_b} = \{B : B \subseteq X, C^{\Lambda_b}(B^c) = B^c\}$ where C^{Λ_b} is the Kuratowski closure operator on X, defined by $C^{\Lambda_b}(A) = \bigcap\{U : A \subseteq U, U \in C^{\Lambda_b}\}$ for any subset A of (X,τ).

Lemma 4.7. $\tau^{\Lambda_b} = \{B : B \subseteq X, \mathrm{Int}^{V_b}(B) = B\}$, *where* $\mathrm{Int}^{V_b}(B) = \bigcup\{F : B \supseteq F, F \in S^{V_b}\}$.

In the following proposition we have a further result concerning the transfer of properties from (X,τ) to (X,τ^{Λ_b}).

Theorem 4.8. *For a topological space (X,τ), the following properties are equivalent:*

(*i*) *(X,τ) is a b-R_0-space;*

(*ii*) *For any nonempty set A and $G \in BO(X)$ such that $A \cap G \neq \varnothing$, there exists $F \in BC(X)$ such that $A \cap F \neq \varnothing$ and $F \subset G$;*

(*iii*) *Any $G \in BO(X)$, $G = \bigcup\{F \in BC(X) | F \subset G\}$;*

(*iv*) *Any $F \in BC(X)$, $F = \bigcap\{G \in BO(X) | F \subset G\}$;*

(*v*) *For any $x \in X$, b-$\mathrm{Cl}(\{x\}) \subset b$-$\mathrm{Ker}(\{x\})$.*

Proof. (i)⇒(ii): Let A be a nonempty subset of X and $G \in BO(X)$ such that $A \cap G \neq \varnothing$. Then there exists $x \in A \cap G$. Since $x \in G \in BO(X)$, b-$\mathrm{Cl}(\{x\}) \subset G$. Set $F = b$-$\mathrm{Cl}(\{x\})$, then $F \in BC(X)$, $F \subset G$ and $A \cap F \neq \varnothing$.

(ii)⇒(iii): Let $G \in BO(X)$, then $G \supset \bigcup\{F \in BC(X) | F \subset G\}$. Let x be any point of G. There exists $F \in BC(X)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \bigcup\{F \in BC(X) | F \subset G\}$ and hence $G = \bigcup\{F \in BC(X) | F \subset G\}$.

(iii)⇒(iv): This is obvious.

(iv)⇒(v): Let x be any point of X and $y \in b$-$\mathrm{Ker}(\{x\})$. There exists $V \in BO(X)$ such that $x \in V$ and $y \notin V$; hence b-$\mathrm{Cl}(\{y\}) \cap V = \varnothing$. By (iv), $(\bigcap\{G \in BO(X) | b\text{-}\mathrm{Cl}(\{y\})\} \subset G) \cap V = \varnothing$ and there exists $G \in BO(X)$ such that $x \notin G$ and b-$\mathrm{Cl}(\{y\}) \subset G$. Therefore, b-$\mathrm{Cl}(\{x\}) \cap G = \varnothing$ and $y \notin b$-$\mathrm{Cl}(\{x\})$. Consequently, we obtain b-$\mathrm{Cl}(\{x\}) \subset b$-$\mathrm{Ker}(\{x\})$.

(v)⇒(i): Let $G \in BO(X)$ and $x \in G$. Let $y \in b\text{-Ker}(\{x\})$, then $x \in b\text{-Cl}(\{y\})$ and $y \in G$. This implies that $b\text{-Ker}(\{x\}) \subset G$. Therefore, we obtain $x \in b\text{-Cl}(\{x\}) \subset b\text{-Ker}(\{x\}) \subset G$. This shows that (X, τ) is a b-R_0 space. ■

Proposition 4.9. *If (X, τ) is a b-R_0-space, then (X, τ^{Λ_b}) is a T_1-space (hence b-T_1-space).*

Proof. Since (X, τ) is a $b - R_0$-space, then by Theorem 4.8, any b-open set A in (X, τ) can be expressed as $A = \bigcup\{F : F \subseteq A, F^c \in BO(X, \tau)\}$ i.e., $A = A^{V_b}$. By Theorem 4.6, every singleton $\{x\}$ of X is a $g.\Lambda_b$-set. Then we have $C^{\Lambda_b}(\{x\}) = \{x\}$ and hence $\{x\}$ is a τ^{Λ_b}-closed set. Therefore, every singleton is closed in (X, τ^{Λ_b}). ■

Definition 4.10. A topological space (X, τ) is said to be a T^{V_b}-space, if every τ^{Λ_b}-open set is a $g.V_b$-set.

Theorem 4.11. *(X, τ) is a T^{V_b}-space if and only if $S^{V_b} = \tau^{\Lambda_b}$.*

Proof. Necessity. Since X is a T^{V_b}-space, then $\tau^{\Lambda_b} \subseteq S^{V_b}$. Therefore it is enough to prove that $S^{V_b} \subseteq \tau^{\Lambda_b}$. Let $B \in S^{V_b}$. Then $\text{Int}^{V_b}(B) = B$ since B is a $g.V_b$-set. By Lemma 4.7 $B \in \tau^{\Lambda_b}$. Thus $S^{V_b} = \tau^{\Lambda_b}$.

Sufficiency. It is clear, by assumption and Definition 4.1. ■

Theorem 4.12. *Every b-$T_{1/2}$ space is a T^{V_b}-space.*

Proof. Let B be a τ^{Λ_b}-open set, i.e., $B = \text{Int}^{V_b}(B)$ (Lemma 4.7). Since every V_b-set is a $g.V_b$-set (Proposition 2.6(b)), it is enough to show that, $\text{Int}^{V_b}(B)$ is a V_b-set, i.e., $(\text{Int}^{V_b}(B))^{V_b} = \text{Int}^{V_b}(B)$. Really, let $\Omega_{V_b} = \{B : B \text{ is a } V_b\text{-set}\}$. By Proposition 2.6(b) and assumption we have that $\Omega_{V_b} = S^{V_b}$. Therefore by definition, Proposition 2.5(j) and the fact that $\Omega_{V_b} = S^{V_b}$ we have

$$\begin{aligned}
(\text{Int}^{V_b}(B))^{V_b} &= \left(\bigcup\{F : B \supseteq F, F \in S^{V_b}\}\right)^{V_b} \\
&= \left(\bigcup\{F : B \supseteq F, F \in \Omega_{V_b}\}\right)^{V_b} \\
&\supseteq \bigcup\{F^{V_b} : B \supseteq F, F \in \Omega_{V_b}\} \\
&= \bigcup\{F^{V_b} : B \supseteq F, F \in S^{V_b}\} \\
&= \text{Int}^{V_b}(B).
\end{aligned}$$

By Proposition 2.1(g) we have $(\text{Int}^{V_b}(B))^{V_b} = \text{Int}^{V_b}(B)$ and hence $\text{Int}^{V_b}(B)$ is a V_b-set. ■

Theorem 4.13. *The image of a T^{V_b} space under a $g.\Lambda_b$-homeomorphism is a T^{V_b}-space.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $g.\Lambda_b$-homeomorphism of a T^{V_b}-space (X, τ) onto a topological space (Y, σ). Let B be any σ^{Λ_b}-open set of (Y, σ). We show that B is a $g.V_b$-set of (Y, σ), i.e., $\sigma^{\Lambda_b} = S^{V_b}$ in (Y, σ). We show that B is a $g.V_b$-set of (Y, σ), i.e., $\sigma^{\Lambda_b} = S^{V_b}$ in (Y, σ). It follows from the assumptions that $(f^{-1}(B))^c = f^{-1}(C^{\Lambda_b}(B^c)) \supseteq C^{\Lambda_b}((f^{-1}(B))^c)$, i.e., $(f^{-1}(B))^c = C^{\Lambda_b}((f^{-1}(B))^c)$. Hence $f^{-1}(B)$ is a τ^{Λ_b}-open set of (X, τ). Since (X, τ) is a T^{V_b}-space and f is a $g.\Lambda_b$-homeomorphism we obtain that B is a $g.V_b$-set of (Y, σ). Therefore, (Y, σ) is a T^{V_b}-space. ■

Remark 4.14. In particular we have, that the image of a T^{V_b}-space under a homeomorphism is a T^{V_b}-space (i.e., the property T^{V_b}-space is topological).

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