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New Lacunary Strongly Summable Difference Sequences and $\triangle_v^m$- Lacunary Almost Statistical Convergence

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Abstract. The purpose of this paper is to introduce the concept of $\triangle_v^m$-lacunary strongly almost convergence with respect to a sequence of moduli and $\triangle_v^m$- lacunary almost statistical convergence and give some relations between these two kinds of convergence. The results proved here are analogous to those by R. Colak, B. C. Tripathy and M. Et (Vietnam Journal of Mathematics **34**:2(2006) 129 -138).

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1. Introduction

Let ω denote the set of all real sequences $x = (x_k)$. Let l_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ normed as usual by $||x||_\infty = \sup_k |x_k|$. Kizmaz [20] defined the sequence spaces:

$$l_\infty(\triangle) = \{x = (x_k) : (\triangle x_k) \in l_\infty\},$$

$$c(\triangle) = \{x = (x_k) : (\triangle x_k) \in c\},$$

and

$$c_0(\triangle) = \{x = (x_k) : (\triangle x_k) \in c_0\},$$

where $\triangle x_k = (x_k - x_{k+1})$. These are Banach spaces with the norm

$$||x||_{\triangle} = |x_1| + ||\triangle x||_{\infty}.$$

Difference sequence spaces have been studied by Colak and Et [4], Et [10], Et and Esi [11], Vakeel A. Khan [17, 18, 19] and many others.

A linear functional $\pounds$ on l_∞ is said to be a Banach limit [1] if it has the properties:

(i) $\pounds(x) \geq 0$ if $x \geq 0$ (i.e. $x_n \geq 0$ for all n);
(ii) $\pounds(e) = 1$, where $e = (1, 1, \ldots)$;
(iii) $\pounds(Dx) = \pounds(x)$, where D is the shift operator defined by $(Dx_n) = (x_{n+1})$.

Let ß be the set of all Banach limits on l_∞. A sequence x is said to be almost convergent to a number L if $\pounds(x) = L$ for all $\pounds \in$ ß. Lorentz [23] has shown that x is almost convergent to L if and only if

$$t_{kj} = t_{kj}(x) = \frac{x_j + x_{j+1} + \cdots + x_{j+k}}{k+1} \to L, \text{ as } k \to \infty, \text{ uniformly in } j.$$

Let f denote the set of all almost convergent sequences. We write f-$\lim x = L$ if x is almost convergent to L. Maddox [24] and Freedman et al. [13] have defined x to be strongly almost convergent to a number L if

$$\frac{1}{k+1} \sum_{i=0}^{k} |x_{i+j} - L| \to 0 \;\; k \to \infty, \text{ uniformly in } j.$$

Let $[f]$ denote the set of all strongly almost convergent sequences. If x is strongly almost convergent to L, we write $[f]$-$\lim x = L$. It is easy to see that $[f] \subset f \subset l_\infty$. Das and Sahoo [8] defined the sequence space

$$[w(p)] = \left\{ x \in w : \frac{1}{n+1} \sum_{k=0}^{n} |t_{kj}(x) - L|^{p_k} \to 0 \;\; n \to \infty, \text{ uniformly in } j \right\}.$$

The concept of statistical convergence was first introduced by Fast [12] and also independently by Buck [3] and Schoenberg [29] for real and complex sequences. Further this concept was studied by Šalát [27], Fridy [14], Connor [5], Connor, Fridy, and Kline [6], and many others.

Let $\mathbb{N}$ and $\mathbb{C}$ be the set of natural numbers and complex numbers, respectively. If $E \subseteq \mathbb{N}$, then the natural density of E (see Freedman et al. [13]) is defined by

$$\delta(E) = \lim_n \frac{1}{n} |\{k \leq n : k \in E\}|,$$

where the vertical bars denote the cardinality of the enclosed set. The sequence x is said to be statistically convergent to L, denoted by $stat$-$\lim x = L$, if for every $\epsilon > 0$, the set

$$\{k : |x_k - L| \geq \epsilon\}$$

has natural density zero. In this case we write $stat\text{-}\lim x_k = L$.

A sequence of positive integers $\theta = (k_r)$ is called "lacunary" if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r)$ and $\eta_r = \dfrac{k_r}{k_{r-1}}$. Lacunary sequences have been studied in [2, 9, 14, 16, 28].

A function $f : [0, \infty) \to [0, \infty)$ is called a modulus if

1. $f(t) = 0$ if and only if $t = 0$,
2. $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
3. f is increasing, and
4. f is continuous from the right of 0.

Let X be a sequence space. Then the sequence space $X(f)$ is defined as

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}$$

for a modulus f ([25, 26]). Kolk [21, 22] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ i.e.

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.$$

Let $F = (f_k)$ be a sequence of moduli, $p = (p_k)$ a sequence of positive real numbers and let $v = (v_k)$ be any fixed sequence of non zero complex numbers and let $m \in \mathbb{N}$ be fixed (see [10]). This assumption is made throughout the rest of this paper.

Now we define the following sequence spaces:

$$L_\theta(\triangle_v^m, F, p) = \left\{ x \in \omega : \lim_r \frac{1}{h_r} \sum_{k \in I_r} [f_k(|\triangle_v^m t_{kj}(x) - L|)]^{p_k} = 0, \text{ uniformly in } j \text{ for some } L \right\},$$

$$L_\theta^0(\triangle_v^m, F, p) = \left\{ x \in \omega : \lim_r \frac{1}{h_r} \sum_{k \in I_r} [f_k(|\triangle_v^m t_{kj}(x)|)]^{p_k} = 0, \text{ uniformly in } j \right\},$$

$$L_\theta^\infty(\triangle_v^m, F, p) = \left\{ x \in \omega : \sup_{r,j} \frac{1}{h_r} \sum_{k \in I_r} [f_k(|\triangle_v^m t_{kj}(x)|)]^{p_k} < \infty \right\}.$$

where $\triangle_v^0 x_k = (v_k x_k)$, $\triangle_v x_k = (v_k x_k - v_{k+1} x_{k+1})$, and $\triangle_v^m x_k = (\triangle_v^{m-1} x_k - \triangle_v^{m-1} x_{k+1})$, and so that

$$\triangle_v^m x_k = \sum_{i=0}^{m} (-1)^i \begin{bmatrix} m \\ i \end{bmatrix} \ v_{k+i} x_{k+i}.$$

2. $\triangle_v^m$ - Lacunary Almost Statistical Convergence

In this section we define the following definitions.

Definition 2.1. Let θ be a lacunary sequence. A sequence $x = (x_k)$ is said to be $\triangle_v^m$ - lacunary almost statistically convergent to the number L provided that for every $\epsilon > 0$,

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |\triangle_v^m t_{kj}(x) - L| \geq \epsilon\}| = 0, \text{ uniformly in} j.$$

In this case we write $[S_\theta]_{\triangle_v^m}$-$\lim x = L$ or $x_k \to L([S_\theta]_{\triangle_v^m})$ and we define

$$[S_\theta]_{\triangle_v^m} = \{x \in w(X) : [S_\theta]_{\triangle_v^m}\text{-}\lim x = L, \text{ for some } L\}.$$

If $\theta = (2^r)$, then we shall write $[S]_{\triangle_v^m}$ instead of $[S_\theta]_{\triangle_v^m}$.

Definition 2.2. Let θ be a lacunary sequence and $0 < q < \infty$. Then a sequence $x = (x_k)$ is said to be $\triangle_v^m$ - lacunary strongly almost convergent to the number L provided that,

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} |\triangle_v^m t_{kj}(x) - L|^q = 0, \text{ uniformly in } j.$$

In this case we write $[w_\theta]_{\triangle_v^m}$-$\lim x = L$ or $x_k \to L([w_\theta]_{\triangle_v^m})$ and we define

$$[w_\theta]_{\triangle_v^m} = \{x \in w : [w_\theta]_{\triangle_v^m}\text{-}\lim x = L, \text{ for some } L\}.$$

3. Main Results

Theorem 3.1. *Let $F = (f_k)$ be a sequence of moduli, then*

$$L_\theta^0(\triangle_v^m, F, p) \subset L_\theta(\triangle_v^m, F, p) \subset L_\theta^\infty(\triangle_v^m, F, p).$$

Proof. Let $x \in L_\theta(\triangle_v^m, F, p)$. Then we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}[f_k(|\triangle_v^m t_{kj}(x)|)]^{p_k}\\
&\le \frac{D}{h_r}\sum_{k\in I_r}[f_k(|\triangle_v^m t_{kj}(x)-L|)]^{p_k}+\frac{D}{h_r}\sum_{k\in I_r}[f_k(|\triangle_v^m L|)]^{p_k}\\
&\le \frac{D}{h_r}\sum_{k\in I_r}[f_k(|\triangle_v^m t_{kj}(x)-L|)]^{p_k}+D\max\{1,\sup[f_k(|\triangle_v^m L|)]^H,
\end{aligned}$$

where $\sup_k p_k = G$, $H=\max(1,G)$ and $D=\max(1,2^{G-1})$.

Thus we get $x\in L_\theta^\infty(\triangle_v^m,F,p)$. The inclusion $L_\theta^0(\triangle_v^m,F,p)\subset L_\theta(\triangle_v^m,F,p)$ is obvious. ■

Theorem 3.2. *Let $F=(f_k)$ be a sequence of moduli, and let the sequence (p_k) be bounded. Then $L_\theta^0(\triangle_v^m,F,p)$, $L_\theta(\triangle_v^m,F,p)$ and $L_\theta^\infty(\triangle_v^m,F,p)$ are linear spaces over the set of complex numbers.*

Theorem 3.3. *Let θ be a lacunary sequence.*

(i) If a sequence $x=(x_k)$ is $\triangle_v^m$ - lacunary strongly almost convergent to L, then it is $\triangle_v^m$ - lacunary almost statistically convergent to L.

(ii) If $x\in l_\infty(\triangle_v^m)$ is $\triangle_v^m$ - lacunary almost statistically convergent to L, then it is $\triangle_v^m$ - lacunary strongly almost convergent to L.

(iii) $l_\infty(\triangle_v^m)\cap[S_\theta]_{\triangle_v^m}=l_\infty(\triangle_v^m)\cap[w_\theta]_{\triangle_v^m}$,

where $l_\infty(\triangle_v^m)=\{x\in w:\sup_k|\triangle_v^m x|<\infty\}$.

Proof. (i) Let $\epsilon>0$ and $x_k\to L([w_\theta]_{\triangle_v^m})$. Then we can write

$$\begin{aligned}
\sum_{k\in I_r}|\triangle_v^m t_{kj}(x)-L|^p &\ge \sum_{\substack{k\in I_r\\ |\triangle_v^m t_{kj}(x)-L|\ge\epsilon}}|\triangle_v^m t_{kj}(x)-L|^p\\
&\ge \epsilon^p|\{k\in I_r:|\triangle_v^m t_{kj}(x)-L|\ge\epsilon\}|.
\end{aligned}$$

Hence $x_k\to L([S_\theta]_{\triangle_v^m})$.

(ii) Suppose that $x_k\to L([S_\theta]_{\triangle_v^m})$ and let $x\in l_\infty(\triangle_v^m)$. Let $\epsilon>0$ be given and choose N_ϵ such that

$$\frac{1}{h_r}\left|\left\{k\in I_r:|\triangle_v^m t_{kj}(x)-L|\ge\left(\frac{\epsilon}{2}\right)^{1/p}\right\}\right|\le\frac{\epsilon}{2K^p}\quad\text{for all } m \quad\text{and } r>N_\epsilon$$

and set

$$L_{rm}=\left\{k\in I_r:|\triangle_v^m t_{kj}(x)-L|\ge\left(\frac{\epsilon}{2}\right)^{1/p}\right\},$$

where $K=\sup_{k,j}|\triangle_v^m t_{kj}(x)-L|^p$.

Now for all m and $r > N_\epsilon$ we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}|\triangle_v^m t_{kj}(x)-L|^p\\
&=\frac{1}{h_r}\sum_{\substack{k\in I_r\\ k\in L_{rj}}}|\triangle_v^m t_{kj}(x)-L|^p+\sum_{\substack{\in I_r\\ k\notin L_{rj}}}|\triangle_v^m t_{kj}(x)-L|^p\\
&\le\frac{1}{h_r}(\frac{h_r\epsilon}{2K^p})K^p+\frac{\epsilon}{2h_r}h_r=\epsilon.
\end{aligned}$$

Thus $(x_k)\in[w_\theta]_{\triangle_v^m}$.

The proof of (iii) follows from (i) and (ii). ∎

Theorem 3.4. *For any lacunary sequence θ, if $\lim\inf_{r\to\infty}\eta_r > 1$, then $[S]_{\triangle_v^m}\subset [S_\theta]_{\triangle_v^m}$.*

Proof. If $\lim\inf_{r\to\infty}\eta_r > 1$, then there exists a $\delta > 0$ such that $1+\delta\le\eta_r$ for sufficiently large r. Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r}\le\frac{1+\delta}{\delta}$. Let $x_k\to L([S_\theta]_{\triangle_v^m})$. Then for every $\epsilon > 0$ and for all m we have

$$\begin{aligned}
\frac{1}{k_r}\left|\{k\le k_r:|\triangle_v^m t_{kj}(x)-L|\ge\epsilon\}\right| &\ge\frac{1}{k_r}\left|\{k\in I_r:|\triangle_v^m t_{kj}(x)-L|\ge\epsilon\}\right|\\
&\ge\frac{\delta}{1+\delta}\frac{1}{h_r}\left|\{k\in I_r:|\triangle_v^m t_{kj}(x)-L|\ge\epsilon\}\right|.
\end{aligned}$$

Hence $[S]_{\triangle_v^m}\subset[S_\theta]_{\triangle_v^m}$. ∎

Theorem 3.5. *For any lacunary sequence θ, if $\limsup_r\rho_r<\infty$, then $[S_\theta]_{\triangle_v^m}\subset [S]_{\triangle_v^m}$.*

Proof. If $\limsup_r\rho_r<\infty$ then there exists a $\beta>0$ such that $\eta_r<\beta$ for all r. Let $x_k\to L([S_\theta]_{\triangle_v^m})$, and for each $m\ge 1$ set

$$E_{rj}=|\{k\in I_r:|\triangle_v^m t_{kj}(x)-L|\ge\epsilon\}|.$$

Then there exists an $r_0\in\mathbb{N}$ such that $\frac{E_{rm}}{h_r}<\epsilon$ for all $r>r_0$ and for each $m\ge 1$. Let $K=\max\{E_{rj}:1\le r\le r_0\}$ and choose n such that $k_{r-1}<n\le k_r$. Then for each $m\ge 1$ we have

$$
\begin{aligned}
&\frac{1}{n}|\{k \leq n : |\triangle_v^m t_{kj}(x) - L| \geq \epsilon\}| \\
&\qquad \leq \frac{1}{k_{r-1}}\,|\{k \leq k_r : |\triangle_v^m t_{kj}(x) - L| \geq \epsilon\}| \\
&\qquad \leq \frac{1}{k_{r-1}}\left\{E_{1j} + E_{2j} + \cdots + E_{r_0 j} + E_{(r_0+1)j} + \cdots + E_{rj}\right\} \\
&\qquad \leq \frac{K}{k_{r-1}} r_0 + \frac{1}{k_{r-1}}\left\{\frac{E_{(r_0+1)j}}{h_{r_0+1}} h_{r_0+1} + \cdots + \frac{E_{rj}}{h_r} h_r\right\} \\
&\qquad \leq \frac{K}{k_{r-1}} r_0 + \frac{1}{k_{r-1}}\left(\sup_{r > r_0} \frac{E_{rj}}{h_r}\right)\{h_{r_0+1} + \cdots + h_r\} \\
&\qquad \leq \frac{K}{k_{r-1}} r_0 + \epsilon \frac{k_r - k_{r_0}}{k_{r-1}} \\
&\qquad \leq \frac{K}{k_{r-1}} r_0 + \epsilon \rho_r \\
&\qquad \leq \frac{K}{k_{r-1}} r_0 + \epsilon \beta.
\end{aligned}
$$

Hence $[S_\theta]_{\triangle_v^m} \subset [S]_{\triangle_v^m}$. ■

Theorem 3.6. *Let $F = (f_k)$ be a sequence of moduli. Then $L_\theta(\triangle_v^m, F, p) \subset [S_\theta]_{\triangle_v^m}$.*

Proof. Let $x \in L_\theta(\triangle_v^m, F, p)$. Then

$$\frac{1}{h_r}\sum_{k \in I_r} [f_k(|\triangle_v^m t_{kj}(x) - L|)]^{p_k} \to 0, \quad \text{as} \quad r \to \infty.$$

For given $\epsilon > 0$ we have

$$
\begin{aligned}
&\frac{1}{h_r}\sum_{k \in I_r} [f_k(|\triangle_v^m t_{kj}(x) - L|)]^{p_k} \geq \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |\triangle_v^m t_{kj}(x) - L| \geq \epsilon}} [f_k(|\triangle_v^m t_{kj}(x) - L|)]^{p_k} \\
&\geq \frac{1}{h_r}\sum_{k \in I_r} [f_k(|\triangle_v^m t_{kj}(x) - L|)]^{p_k} \geq \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |\triangle_v^m t_{kj}(x) - L| \geq \epsilon}} [f_k(\epsilon)]^{p_k} \\
&\geq \frac{1}{h_r}\sum \min\{[f_k(\epsilon)]^{\inf p_k}, [f_k(\epsilon)]^G\} \\
&\geq \frac{1}{h_r}|\{k \leq I_r : |\triangle_v^m t_{kj}(x) - L| \geq \epsilon\}| \min\{[f_k(\epsilon)]^{\inf p_k}, [f_k(\epsilon)]^G\}.
\end{aligned}
$$

Hence $x \in [S_\theta]_{\triangle_v^m}$. ■

Theorem 3.7. $[S_\theta]_{\triangle_v^m} \cap \triangle_v^m(l_\infty) = L_\theta(\triangle_v^m, F, p) \cap l_\infty(\triangle_v^m)$.

Proof. By Theorem 3.6, we need to show that

$$[S_\theta]_{\triangle_v^m} \cap l_\infty(\triangle_v^m) \subset L_\theta(\triangle_v^m, F, p) \cap l_\infty(\triangle_v^m).$$

For each $m \geq 1$, let $y_{km} = t_{km}(x) - L(S_\theta)$. Since $(x_k) \in l_\infty(\triangle_v^m)$, for each $m \geq 1$, there exists $K > 0$ such that

$$f_k[\triangle_v^m(y_{kj})] \leq K, \text{ for all } y_{km}.$$

Then, for a given $\epsilon > 0$ and for each $m \in \mathbb{N}$, we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}[f_k(|\triangle_v^m y_{kj}|)] \\
&= \frac{1}{h_r}\sum_{\substack{k\in I_r \\ |\triangle_v^m t_{km}(x)-L|\geq\epsilon}}[f_k(|\triangle_v^m y_{kj}|)] + \frac{1}{h_r}\sum_{\substack{k\in I_r \\ |\triangle_v^m t_{kj}(x)-L|<\epsilon}}[f_k(|\triangle_v^m y_{kj}|)] \\
&\leq \frac{K}{h_r}|\{k \in I_r : |\triangle_v^m y_{kj}| \geq \epsilon\}| + f_k(\epsilon).
\end{aligned}$$

Hence $x \in L_\theta(\triangle_v^m, F, p) \cap l_\infty(\triangle_v^m)$. ■

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