Vietnam Journal of Mathematics **36**:4(2008) 415-426

Vietnam Journal
of
MATHEMATICS

Prime Ideals of Group Graded Semirings and Their Smash Products

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Received February 20, 2008
Revised September 24, 2008

Abstract. From an algebraic point of view, semirings provide the most natural generalization of group theory and ring theory. In the absence of additive inverses, one has to impose some weaker versions of additive inverses on semirings. This paper generalizes some ring theoretic results of M. Cohen and S. Montgomery [1] for the additively cancellative and yoked semirings.

1991 Mathematics Subject Classification: Primary 16Y60; Secondary 16Y99.

Key words: Graded prime ideals, incomparability, semirings, smash products.

1. Introduction

M. Cohen and S. Montgomery [1] obtained some results (e.g. Incomparability and orbit problem) about graded rings using the fact that grading and group actions are dual concepts, even when G is not abelian. For a K-algebra R graded by G, they formed an algebra $R \# K[G]^*$ which plays the role for graded rings that the skew group algebra $R * G$ plays for a ring with finite group action on it. The identity component in the graded ring corresponds to the fixed ring of the group action. M. Cohen and S. Montgomery proved results similar to those of M. Lorenz and D. S. Passman [3] for R and $R * G$ where a finite group G acts on R. The results of M. Lorenz and D. S. Passman were generalized for a semiring with finite group action on it and its skew group semiring $R * G$ by Sharma and Sharma [4], under some weaker conditions in the absence of additive inverses. Thus we start this article with the aim to settle Incomparability and Orbit problem for a K-semialgebra R graded by a finite group G and its smash product $R \# K[G]^*$. For the definitions of a semiring, semimodule and tensor product of semimodules one can see [2].

In the absence of additive inverses, throughout this article we also have to impose a weaker condition on the semiring R and the commutative semiring K, that is, R and K are assumed to be additively cancellative (that is, $a + b = a + c$ implies $b = c$). If R and K are additively cancellative semirings, then the rings of differences R^{Δ} and K^{Δ} exist. Thus if the semiring R is graded by G, then R^{Δ} becomes a ring graded by G and therefore we have the smash product $R^{\Delta} \# K^{\Delta}[G]^* \cong (R \# K[G]^*)^{\Delta}$. The semiring $R \# K[G]^*$ embeds in $R^{\Delta} \# K^{\Delta}[G]^*$ as R and K embed in R^{Δ} and K^{Δ} respectively. These embeddings become useful as the results of [1] are valid for the ring R^{Δ} and the smash product $R^{\Delta} \# K^{\Delta}[G]^*$, thereby providing us with an incisive technique for analysing these results for R and $R \# K[G]^*$.

We also have to impose another weak version of the condition of having additive inverses on a G-graded semiring R, that is, R is assumed to be yoked semiring whenever it is required (that is, for $a, b \in R$ there exists $r \in R$ such that $a+r = b$ or $b + r = a$). Before proceeding further, we give an example of an additively cancellative and yoked graded semiring. Let $R = \{(x, y) \mid x \in \mathbb{N}, y = 0, 1\}$, where $\mathbb{N}$ is the set of nonnegative integers. Define operations of $'+'$ and $'.'$ in R as follows:

$$(x,\ y) + (x', y') = (x + x', r),\ \text{where } r \equiv y + y' (\text{mod } 2)$$

and

$$(x, y).(x', y') = (xx', s),\ \text{where } s \equiv xy' + yx' (\text{mod } 2).$$

Then $(R, +, .)$ becomes a semiring with additive identity $(0, 0)$ and multiplicative identity $(1, 0)$. Using the additively cancellative and yoked characters of the semiring $\mathbb{N}$ with respect to usual addition and multiplication, one can easily see that $(R, +, .)$ is an additively cancellative and yoked semiring. Let $G = \{1, g\}$ be a group of order 2. Define $R_1 = \{(x, 0) \mid x \in \mathbb{N}\}$ and $R_g = \{(0, y) \mid y = 0, 1\}$, then R_1 and R_g are additive submonoids of R, $R = R_1 \oplus R_g$ and $R_1R_g \subseteq R_g, R_gR_1 \subseteq R_g, R_1R_1 \subseteq R_1, R_gR_g = \{(0, 0)\} \subseteq R_1$. Hence R is graded by G (c.f. Definitions 3.3 and 3.4).

Before mentioning the main results proved in this paper, we recall from [2] that a subset (ideal) A of a semiring R is subtractive if $a,\ a + b \in A$ implies $b \in A$. The graded ideal $A = \{(2n, y) \mid n \in \mathbb{N}, y = 0, 1\}$ of the semiring R considered above is subtractive. The subtractive ideals play an important role in this paper (c.f. Lemma 3.2 and Note thereafter).

We now state the main results:

1. (Incomparability) Let R be a yoked semiring graded by a finite group G. If $P \subsetneq Q$ are subtractive prime ideals of R then $P \cap R_1 \subsetneq Q \cap R_1$, where R_1 is the identity component of R.

2. (Orbit Problem) Let R be a K-semialgebra graded by a finite group G. (i) If R is yoked and P is a subtractive prime ideal of R, then there exists a

prime Q of $R\#K[G]^*$ such that $Q \cap R = P_G$, Q is unique up to its G-orbit $\{Q^g\}$ and $P_G\#K[G]^* = \cap_{g\in G} Q^g$, a G-prime ideal of $R\#K[G]^*$;
(ii) If I is any subtractive prime ideal of $R\#K[G]^*$, then $I \cap R = P_G$ for some subtractive prime P of R.

2. Semialgebras Over Commutative Semirings

First, we recall two definitions from [2] about semimodules.

Definition 2.1. Let M be a left R-semimodule and A be a nonempty subset of M. Then the *subsemimodule* generated by A is

$$RA = \{r_1a_1 + r_2a_2 + \ldots + r_na_n \mid r_i \in R, a_i \in A\}.$$

If $RA = M$, then A is a set of generators for M.

Definition 2.2. Let R be a semiring and M be a left R-semimodule. A subset $A = \{a_1, a_2, \ldots, a_n\}$ of M is linearly independent if and only if for $r_i, r'_i \in R, r_1a_1 + r_2a_2 + \ldots + r_na_n = r^{'}_1a_1 + r^{'}_2a_2 + \ldots + r^{'}_na_n$ implies that $r_i = r'_i$, $i = 1, 2, \ldots, n$. If A is not linearly independent, then it is linearly dependent. A linearly independent set of generators of M is a basis for M over R.

Definition 2.3. Let K be a commutative semiring. A K-*semialgebra* is a K-semimodule R on which is defined a bilinear map $R \times R \longrightarrow R$, defined by $(x, y) \mapsto xy$, that is associative $x(yz) = (xy)z$ for all $x, y, z \in R$ and there is a unity element 1_R in R which satisfies $1_Rx = x1_R = x$ for all $x \in R$.

Definition 2.4. A semicoalgebra (C, Δ, ϵ), where C is a K-semimodule, $\Delta : C \to C \otimes C$ a map called comultiplication and $\epsilon : C \to K$ a map called the counit, is defined in the analogous manner as a coalgebra.

Definition 2.5. Any system $(H, M, u, \Delta, \epsilon)$, where H has a K-semialgebra structure with multiplication M and unit u and H has a K-semicoalgebra structure with comultiplication Δ and counit ϵ satisfying:

(i) M and u are semicoalgebra maps;
(ii) Δ and ϵ are semialgebra maps;
(iii) (a) $\Delta(1) = 1 \otimes 1$;
(b) $\Delta(gh) = \sum_{(g)(h)} g_{(1)}h_{(1)} \otimes g_{(2)}h_{(2)}$, where $\Delta(g) = \sum_{(g)} g_{(1)} \otimes g_{(2)}$ and $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$;
(c) $\epsilon(1) = 1$;
(d) $\epsilon(gh) = \epsilon(g)\epsilon(h)$,

is called a semibialgebra or K-semibialgebra.

Definition 2.6. Let R be a semialgebra over a commutative semiring K, and H a K-semibialgebra with comultiplication Δ and counit ϵ. Then R is an H-semimodule semialgebra if there exists a map $\Psi : H \otimes R \to R$ satisfying:

(i) R is an H-semimodule under Ψ;

(ii) $\Psi(h \otimes ab) = \sum_{(h)} \Psi(h_{(1)} \otimes a)\Psi(h_{(2)} \otimes b)$, for $a, b \in R,\ h \in H,\ \Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$;

(iii) $\Psi(h \otimes 1) = \epsilon(h)1_R$.

For simplification, we also write $h.a$ for $\Psi(h \otimes a)$. Let R be an H-semimodule semialgebra. Then we can define the smash product $R \# H$ as follows: as a semimodule, $R\#H$ is $R \otimes_K H$, whose elements are of the form $a \otimes h$ written as $a \# h$. Multiplication in $R \# H$ is defined by

$$(a\#g)(b\#h) = \sum_{(g)} a(g_{(1)}.b)\#(g_{(2)}h),$$

where $\Delta(g) = \sum_{(g)} g_{(1)} \otimes g_{(2)}$. This makes $R\#H$ a K-semialgebra with unit element $1 = 1_R \# 1_H$.

Let G be a finite group and K a commutative semiring. Then $K[G]$ is a semibialgebra over K with multiplication $(\sum_i r_i g_i)(\sum_j s_j h_j) = \sum r_i s_j g_i h_j$, the unit map $u(k) = k1$, the comultiplication $\Delta(g) = g \otimes g$ and counit given by $\epsilon(g) = 1$ for any $g \in G$.

Let R be an additively cancellative K-semialgebra, where K is an additively cancellative, commutative semiring. Since R and K are additively cancellative, the ring of differences R^Δ and K^Δ exist. Following [4], we write $R^\Delta = \{a - b \mid a, b \in R\}$. For any $a \in R$, the element $a - 0 \in R^\Delta$ will be simply written as a. The zero element of R^Δ is $a - a$, denoted by 0 and multiplicative identity is 1. For any nonempty subset A of R, we have $A^\Delta = \{a - b \mid a, b \in A\}$. If A is an additive submonoid (ideal) of R, then A^Δ is an additive subgroup (ideal) of R^Δ. Let M be a left K-semimodule, then its module of differences is denoted by M^Δ.

Let

$$K^\Delta[G] = \left\{ \sum_i x_i\ g_i \mid x_i = r_i - r'_i,\ r_i, r'_i \in K,\ g_i \in G \right\}.$$

Since the addition in $K[G]$ is defined pointwise, it is clear that $K[G]$ is additively cancellative and hence $(K[G])^\Delta$ exists. It is also easy to see that $K^\Delta[G] \cong (K[G])^\Delta$. Then $K^\Delta[G]$ becomes a bialgebra over K^Δ under the operations defined above. Let $K[G]^*$ be the dual semialgebra and $K^\Delta[G]^*$ be the dual algebra.

A K^Δ-basis of $K^\Delta[G]^*$ is the set of projections $\{p_g \mid g \in G\}$; that is, for any $g \in G$ and $x = \sum_{g_i \in G} (\alpha_i - \beta_i)g_i \in K^\Delta[G]$, $p_{g_j}(x) = \alpha_j - \beta_j \in K^\Delta$. The set

$\{p_g\}$ consists of orthogonal idempotents whose sum is 1. The comultiplication on $K^{\Delta}[G]^*$ is given by $\Delta(p_g) = \sum_{h\in G} p_{gh^{-1}} \otimes p_h$ and the counit is $\epsilon(p_g) = \delta_{1.g}$. The set $\{p_g \mid p_{g/K[G]}{:}K[G]\to K\}$ forms a K-basis for $K[G]^*$.

3. Graded Prime Ideals of R and R^{Δ}

We, first, recall some properties of the ideals of R and R^{Δ} from Sharma and Sharma [4].

Lemma 3.1. *Let R be an additively cancellative semiring and R^{Δ} its ring of differences. Let A, B be two ideals of R and I, J be two ideals of R^{Δ}. Then*

(i) $A^{\Delta}B^{\Delta} = (AB)^{\Delta}$;

(ii) *If* $A \subseteq B$, *then* $A^{\Delta} \subseteq B^{\Delta}$. *Further, if A is subtractive and* $A \subsetneq B$, *then* $A^{\Delta} \subsetneq B^{\Delta}$;

(iii) $(I \cap R)(J \cap R) \subseteq (IJ) \cap R$;

(iv) $(I \cap R)^{\Delta} \subseteq I$. *Equality holds if R is a yoked semiring.*

We prove somewhat similar results which will be used in this paper.

Lemma 3.2. *Let R be a semiring and R^{Δ} its ring of differences. Let A, B be two nonempty subsets of R and I, J two subgroups of $(R^{\Delta}, +)$. Then*

(i) (a) $A \subseteq A^{\Delta} \cap R$. *Equality holds if A is subtractive;*

(b) $A \subseteq B$ *implies* $A^{\Delta} \subseteq B^{\Delta}$;

(ii) $(A \cap B)^{\Delta} \subseteq A^{\Delta} \cap B^{\Delta}$. *Equality holds if A and B are subtractive submonoids of $(R, +)$ and R is yoked;*

(iii) $(A + B)^{\Delta} \subseteq A^{\Delta} + B^{\Delta}$. *Equality holds if A and B are submonoids of $(R, +)$;*

(iv) *If A and B are subtractive, then* $A^{\Delta} \subseteq B^{\Delta}$ *implies* $A \subseteq B$ *and* $A^{\Delta} \subsetneq B^{\Delta}$ *implies* $A \subsetneq B$;

(v) $I \cap R$ *is subtractive;*

(vi) $(I \cap R) + (J \cap R) \subseteq (I + J) \cap R$.

Proof. $(i)\,(a)$ and $(i)\,(b)$ follow in a similar way as in the case of ideals (c.f. [4]).

(ii) Clearly $(A \cap B)^{\Delta} \subseteq A^{\Delta} \cap B^{\Delta}$. Conversely, assume that A and B are subtractive and R is yoked. Let $a - b \in A^{\Delta} \cap B^{\Delta}$. Then $a - b \in A^{\Delta} \subseteq R^{\Delta}$. We note that in an additively cancellative yoked semiring, for all $a - b \in R^{\Delta}$, either $a - b \in R$ or $b - a \in R$. If $a - b \in R$, then $a - b \in A^{\Delta} \cap R = A$ by $(i)(a)$ and similarly, $a - b \in B^{\Delta} \cap R = B$. So $a - b \in A \cap B \subseteq (A \cap B)^{\Delta}$. If $b - a \in R$, then $b - a \in A \cap B \subseteq (A \cap B)^{\Delta}$. Since $(A \cap B)^{\Delta}$ is a subgroup of the ring R^{Δ}, $a - b \in (A \cap B)^{\Delta}$.

(iii) Clearly $(A + B)^{\Delta} \subseteq A^{\Delta} + B^{\Delta}$. The converse follows from $(i)(b)$ and the fact that $A \subseteq A + B$ and $B \subseteq A + B$.

(iv) Assume that $A^{\Delta} \subseteq B^{\Delta}$. Then by $(i)(a)$, $A = A^{\Delta} \cap R \subseteq B^{\Delta} \cap R = B$, as A and B are subtractive. Since $A = B$ implies $A^{\Delta} = B^{\Delta}$, $A^{\Delta} \subsetneq B^{\Delta}$ implies $A \subsetneq B$.

(v) Let $x, y \in R$ such that $x + y, y \in I \cap R$. Then $x + y, y \in I$ and I is a subgroup of $(R^{\Delta}, +)$. Hence $x = (x - 0) + (y - y) = (x + y) - y \in I$. So, $x \in I \cap R$ proving that $I \cap R$ is subtractive.

The result (vi) is obious to prove. ∎

Note. Let $R = \{(x, y) \mid x \in \mathbb{N}, y = 0, 1\}$ be the semiring considered in Sec. 1 and $A = \{(2n, 0) \mid n \in \mathbb{N}\}, B = \{(2n, 0) \cup (0, 0) \mid n$ is integer greater than or equal to $2\}$. Then A is a subtractive ideal of R, whereas B is an ideal of R which is not subtractive. Clearly $A^{\Delta} = B^{\Delta} = \{(2x, 0) \mid x \in \mathbb{Z}\}$, but $A \nsubseteq B$. Thus (iv) in the above lemma does not hold if B is not subtractive.

Definition 3.3. A semiring (semialgebra) R is graded by a finite group G if $R = \sum_{g \in G} \oplus R_g$, where the R_g are additive submonoids (K-subsemimodules) of R and if $R_g R_h \subseteq R_{gh}$ for all $g,\ h \in G$.

Definition 3.4. For any subset (ideal) A of R, define $A_G = \sum_{g \in G} \oplus (A \cap R_g)$. A is graded if $A = A_G$. Moreover A_G is the largest graded subset (ideal) of R contained in A.

The presence of the following result makes R^{Δ} a G-graded ring.

Proposition 3.5. *If R is a K-semialgebra graded by G, then R^{Δ} is a $K^{\Delta}[G]^*$-module algebra.*

Proof. Assume that R is graded by G. Then $R = \sum_{g \in G} \oplus R_g$ and $R_g R_h \subseteq R_{gh}$. Define the action of $K^{\Delta}[G]^*$ on R^{Δ} by $p_g(a - b) = a_g - b_g$, where a_g and b_g are the g^{th} components of a and b in R_g. Then obviously R^{Δ} becomes a $K^{\Delta}[G]^*$-module.

Moreover, $p_g(xy) = \sum_{h \in G} p_{gh^{-1}}(x) p_h(y)$, where $x, y \in R^{\Delta}$. This is compatible with $\Delta(p_g) = \sum_{h \in G} p_{gh^{-1}} \otimes p_h$. Also, $p_g(1) = 1_g - 0_g = \delta_{1,g}(1 - 0) = \delta_{1,g} = \epsilon(p_g).1_{R^{\Delta}}$. Therefore, R^{Δ} is a $K^{\Delta}[G]^*$-module algebra. ∎

Remark 3.6. Since R^{Δ} is a $K^{\Delta}[G]^*$-module algebra, R^{Δ} is graded by G (c.f. [1], Proposition 1.3). Therefore for $x \in R^{\Delta}$,

$$x = a - b \ \left(a = \sum_{g \in G} a_g, b = \sum_{g \in G} b_g \in R \right),$$

we have the unique representation $a - b = \sum_{g \in G} (a - b)_g$, where $(a - b)_g = a_g - b_g$.

In the following lemma, we make some observations regarding the g^{th} components of R and R^{Δ}, for $g \in G$.

Lemma 3.7. *Let R be a semiring graded by a finite group G and A any subset of R. Then, for $g \in G$*

(i) *Each R_g is subtractive;*

(ii) (a) *Each A_g is subtractive, if A is a subtractive subset of R;*

(b) *A_G is subtractive, if A is a subtractive submonoid of $(R, +)$;*

(iii) $(R_g)^{\Delta} = (R^{\Delta})_g$;

(iv) $R_g = (R^{\Delta})_g \cap R$;

(v) $(A_g\ )^{\Delta} \subseteq (A^{\Delta})_g$. *Equality holds if R is yoked and A is a subtractive submonoid of R;*

(vi) $(A_G\ )^{\Delta} \subseteq (A^{\Delta})_G$. *Equality holds if R is yoked and A is a subtractive submonoid of R;*

(vii) *Let I be an ideal of R^{Δ}. Then*

(a) $(I \cap R)_g = I_g \cap R$;

(b) $(I \cap R)_G = I_G \cap R$;

$(viii)$ *If R is a yoked semiring, then R_1 is a yoked subsemiring of R.*

Proof. (i) Let $a \in R_g$ and $a + b \in R_g$. This implies $a + b = (a + b)_g$ and $a = a_g$. So $a + b = a + b_g$. Since R is additively cancellative, we get $b = b_g \in R_g$.

(ii) (a) Assume that a and $a + b \in A_g = A \cap R_g$. Then, as in (i), $b \in R_g$. If A is subtractive, then we also get $b \in A$. Hence, $b \in A_g$.
(b) Assume that $r + s \in A_G$ and $s \in A_G$. Then by the definition of A_G, s_g and $(r + s)_g \in A$ for all $g \in G$. Since A is subtractive, we get each $r_g \in A$. Moreover since $(r_g)_h \in A$ for all $h \in G$ as $r_g, 0 \in A$. So again by the definition of A_G, we get $r_g \in A_G$ for all $g \in G$. Since A_G is a submonoid of $R, r = \sum\limits_{g \in G} r_g \in A_G$.

(iii) Since R is additively cancellative, R_g is additively cancellative and hence $(R_g)^{\Delta}$ exists. Now the result follows from the fact that $(a - b)_g = a_g - b_g$.

(iv) Using subtractive character of R_g and the fact that $(R_g)^{\Delta} = (R^{\Delta})_g$ together with Lemma 3.2 $(i)(a)$, we have $R_g = (R_g)^{\Delta} \cap R = (R^{\Delta})_g \cap R$.

(v) $(A_g)^{\Delta} = (A \cap R_g)^{\Delta} \subseteq A^{\Delta} \cap (R_g)^{\Delta} = A^{\Delta} \cap (R^{\Delta})_g = (A^{\Delta})_g$ (c.f.Lemma 3.2 (ii)). Further if A is a subtractive submonoid of R, then by using $(i), (iii)$ and Lemma 3.2 (ii),

$$(A^{\Delta})_g = A^{\Delta} \cap (R^{\Delta})_g = A^{\Delta} \cap (R_g)^{\Delta} = (A \cap R_g)^{\Delta} = (A_g)^{\Delta}.$$

(vi) We have $(A_G)^{\Delta} = (A^{\Delta})_G$ (using Lemma 3.2 $((iii), (ii))$ and (iii)). Now assume that R is yoked and A is a subtractive submonoid of R. Then the above inclusions become equalities using (iii) and Lemma 3.2 $((ii)$ and $(iii))$.

(*vii*) (*a*) follows from (*iv*), (*b*) The inclusion $(I \cap R)_G \subseteq I_G \cap R$ follows by using (*iv*) and Lemma 3.2 (*vi*).

For the other inclusion, note that $I_G \cap R$ is a graded ideal of R and $I_G \cap R \subseteq I \cap R$. Since $(I \cap R)_G$ is the largest graded ideal of R contained in $I \cap R$, we have $I_G \cap R \subseteq (I \cap R)_G$.

(*viii*) Let $a, b \in R_1 \subseteq R$. Since R is yoked there exists an element $r \in R$ such that $a + r = b$ or $b + r = a$. If $a + r = b$, then $(a + r)_1 = a + r_1 = b \in R_1$. Therefore, we have $a+r = a+r_1$. Since R is additively cancellative, this implies $r = r_1 \in R_1$. Similarly, the result follows if $b + r = a$. Hence R_1 is yoked. ∎

Lemma 3.8. *Let R be a semiring graded by G.*

(*i*) *If A is a graded ideal of R, then A^Δ is a graded ideal of R^Δ. The converse also holds if A is subtractive.*

(*ii*) *If I is a graded ideal of R^Δ, then $I \cap R$ is a graded ideal of R. The converse also holds if R is yoked.*

Proof. (*i*) Let A be a graded ideal of R. Then by using Lemma 3.2 ((*iii*) and (*ii*)) and Lemma 3.7 (*iii*), we get $A^\Delta = (A^\Delta)_G$. So A^Δ is a graded ideal of R^Δ. Conversely, assume that A^Δ is a graded ideal of R^Δ, where A is subtractive. Then using Lemmas 3.2 (*i*)(*a*) and 3.7 (*vii*)(*b*) upon $(A^\Delta)_G = A^\Delta$, we have

$$A_G = (A^\Delta \cap R)_G = (A^\Delta)_G \cap R = A^\Delta \cap R = A.$$

Hence A is graded.

(*ii*) Assume that I is a graded ideal of R^Δ. Then $(I \cap R)_G = I_G \cap R = I \cap R$, using Lemma 3.7 (*vii*)(*b*). Hence $I \cap R$ is a graded ideal of R. Conversely, assume that $I \cap R$ is a graded ideal of R. Then by (*i*), $(I \cap R)^\Delta$ is a graded ideal of R^Δ. But $(I \cap R)^\Delta = I$, as R is yoked (c.f. Lemma 3.1 (*iv*)). ∎

Definition 3.9. Let P be a graded ideal of R. Then P is graded prime if whenever $AB \subseteq P$, where A, B are graded ideals of R, then $A \subseteq P$ or $B \subseteq P$.

Lemma 3.10. *Let R be a graded semiring and R^Δ its ring of differences.*

(*i*) *If I is a graded prime ideal of R^Δ, then $I \cap R$ is a graded prime ideal of R;*

(*ii*) *Let R be yoked and P be a subtractive ideal of R. Then P is a graded prime ideal of R if and only if P^Δ is a graded prime ideal of R^Δ.*

Proof. (*i*) Assume that I is a graded prime ideal of R^Δ. Then $I \cap R$ is a graded ideal of R (c.f. Lemma 3.8 (*ii*)). Let A and B be graded ideals of R such that $AB \subseteq I \cap R$. Then $A^\Delta B^\Delta = (A\ B)^\Delta \subseteq (I \cap R)^\Delta \subseteq I$ (c.f. Lemma 3.1 ((*i*) and (*iv*)) and Lemma 3.2 (*i*)(*b*)). Now the graded primeness of I implies that either $A^\Delta \subseteq I$ or $B^\Delta \subseteq I$, as A^Δ and B^Δ are graded ideals of R^Δ. Hence $A \subseteq A^\Delta \cap R \subseteq I \cap R$ or $B \subseteq B^\Delta \cap R \subseteq I \cap R$. So $I \cap R$ is a graded prime ideal of R.

(ii) Let I and J be any two graded ideals of R^{Δ} such that $IJ \subseteq P^{\Delta}$. Then $(I \cap R)(J \cap R) \subseteq (IJ) \cap R \subseteq P^{\Delta} \cap R = P$, as P is subtractive (c.f. Lemma 3.1 (iii) and Lemma 3.2 (i)(a)). This implies that either $(I \cap R) \subseteq P$ or $(J \cap R) \subseteq P$, as $(I \cap R), (J \cap R)$ are graded ideals of R and P is a graded prime ideal of R. Hence either $I = (I \cap R)^{\Delta} \subseteq P^{\Delta}$ or $J = (J \cap R)^{\Delta} \subseteq P^{\Delta}$ (c.f. Lemma 3.2 (i)(b)). So P^{Δ} is a graded prime ideal of R.

On the other hand if P^{Δ} is a graded prime ideal of R^{Δ}, then by using (i) we get $P = P^{\Delta} \cap R$ is graded prime, as P is subtractive. ■

To prove a result analogous to ([1, Lemma 5.1]), first we recall the following lemma from [4].

Lemma 3.11. *Let R be an additively cancellative semiring and R^{Δ} its ring of differences.*

(i) *Let R be a yoked semiring and P a subtractive ideal of R. If P is a prime ideal of R, then P^{Δ} is a prime ideal of R^{Δ};*

(ii) *If P is a prime ideal of R^{Δ}, then $P \cap R$ is a prime ideal of R.*

Lemma 3.12. *Let R be a yoked semiring graded by a finite group G. Then a subtractive graded ideal A of R is graded prime if and only if there exists a subtractive, prime ideal P of R such that $P_G = A$.*

Proof. Since A is a subtractive graded prime ideal of R, A^{Δ} is a graded prime ideal of R^{Δ} (c.f. Lemma 3.10 (ii)). Hence by ([1, Lemma 5.1]), there exists a prime ideal Q of R^{Δ} such that $Q_G = A^{\Delta}$. Let $P = Q \cap R$. Then by Lemma 3.11 (ii), P is prime. Moreover by Lemma 3.2 (v), P is a subtractive ideal of R. So by using Lemma 3.7 (vii)(b) and Lemma 3.2 (i), we have

$$P_G = (Q \cap R)_G = Q_G \cap R = A^{\Delta} \cap R = A.$$

The converse follows from the facts that P_G is the largest graded ideal of R contained in P and P_G is subtractive if P is subtractive (c.f. Lemma 3.7 (ii)(b)). ■

We now prove the first main result of this paper.

Theorem 3.13. [*Incomparability*] *Let R be a yoked semiring graded by a finite group G. If $P \subsetneq Q$ are subtractive prime ideals of R, then $P \cap R_1 \subsetneq Q \cap R_1$, where R_1 is the identity component of R.*

Proof. Since R is yoked and P and Q are subtractive prime ideals of R, P^{Δ} and Q^{Δ} are prime ideals of R^{Δ} (c.f. Lemma 3.11 (i)). Further since P is subtractive and $P \subsetneq Q$, we have $P^{\Delta} \subsetneq Q^{\Delta}$ (c.f. Lemma 3.1 (ii)). Now by Theorem 7.1 of [1], we have $P^{\Delta} \cap (R^{\Delta})_1 \subsetneq Q^{\Delta} \cap (R^{\Delta})_1$. Hence by using Lemma 3.2 (ii) and Lemma 3.7 ((i) iii)), we get

$$(P \cap R_1)^{\Delta} = P^{\Delta} \cap (R_1)^{\Delta} = P^{\Delta} \cap (R^{\Delta})_1$$
$$\subsetneq Q^{\Delta} \cap (R^{\Delta})_1 = Q^{\Delta} \cap (R_1)^{\Delta} = (Q \cap R_1)^{\Delta}.$$

So by Lemma 3.2 (*iv*), we finally get $P \cap R_1 \subsetneq Q \cap R_1$. ∎

4. Prime Ideals of R and $R\#K[G]^*$

Let R be a K-semialgebra graded by a finite group G. Then as in the case of K-algebra graded by a finite group G (c.f. [1, Proposition 1.3]), it is easy to see that R is a $K[G]^*$-semimodule semialgebra and hence the smash product $R\#K[G]^*$ can be formed. For $a, b \in R$ and $p_g, p_h \in K[G]^*$, the product is defined by $(ap_g)(bp_h) = ab_{gh^{-1}}p_h$. Moreover, if R is a K-semialgebra graded by a finite group G, then R^{Δ} is a $K^{\Delta}[G]^*$-module algebra (c.f. Proposition 3.5). Hence we can also construct the smash product $R^{\Delta}\#K^{\Delta}[G]^*$, wherein

$$[(a-b)p_g][(c-d)p_h] = (a-b)(c-d)_{gh^{-1}}p_h,$$

for $a-b, c-d \in R^{\Delta}$ and basis elements $p_g, p_h \in K^{\Delta}[G]^*$.

Since the addition in $R\#K[G]^*$ is componentwise and R is additively cancellative, this implies $R\#K[G]^*$ is additively cancellative. So its ring of differences $(R\#K[G]^*)^{\Delta}$ exists. We note that $(R\#K[G]^*)^{\Delta} \cong R^{\Delta}\#K^{\Delta}[G]^*$.

Proposition 4.1. *Let R be a K-semialgebra graded by a finite group G. Then $R^{\Delta}\#K^{\Delta}[G]^* \cong (R\#K[G]^*)^{\Delta}$.*

Proof. The proof follows as that of $R^{\Delta} * G \cong (R * G)^{\Delta}$ (c.f. [4, Proposition 3.7]). ∎

The following lemma will be utilized to prove the main result of this section.

Lemma 4.2. *Let R be a K-subtractive ideal of R and Q' be an ideal of $R^{\Delta}\#K^{\Delta}[G]^*$ with $Q' \cap R^{\Delta} = A^{\Delta}$. Then $Q \cap R = A$, where $Q = Q' \cap (R\#K[G]^*)$ and Q' is identified as an ideal of $(R\#K[G]^*)^{\Delta}$ under the isomorphism given in Proposition 4.1.*

Proof. Let $r \in A \subseteq A^{\Delta}$. Then $r \in Q' \cap R^{\Delta}$ as $A^{\Delta} = Q' \cap R^{\Delta}$. This implies that $r \in Q \cap R$ where $Q = Q' \cap (R\#K[G]^*)$. Hence $A \subseteq Q \cap R$. Conversely if $r \in Q \cap R \subseteq Q'$, then $r \in Q' \cap R^{\Delta} = A^{\Delta}$. So $r \in A^{\Delta} \cap R = A$, as A is subtractive.

Let R be a semiring graded by the finite group G. Then an action of G on $R\#K[G]^*$ is given by $(rp_h)^g = rp_{hg}$, for $r \in R$, $p_h \in K[G]^*$, $g \in G$. Similarly the action of G on $R^{\Delta}\#K^{\Delta}[G]^*$ is given by $[(r-s)p_h]^g = (r-s)p_{hg}$ for $r, s \in R, p_h \in K^{\Delta}[G]^*, g \in G$. ∎

The following results follow exactly as that of a ring graded by a finite group G (c.f. [1], Lemma 6.1).

Lemma 4.3. *Let R be a K-semialgebra graded by a finite group G and I be an ideal of $R\#K[G]^*$. Then*

(i) $I \cap R = (\cap_{g\in G} I^g) \cap R$;

(ii) $I \cap R$ *is a graded ideal of* R;

(iii) *If* I *is* G*-invariant, then* $I = (I \cap R)\#K[G]^*$.

Lemma 4.4. *Let R be a K-semialgebra graded by a finite group G and A a graded ideal of R. Then*

(i) $A\#K[G]^*$ *is a* $G-$ *invariant ideal of* $R\#K[G]^*$;

(ii) A *is subtractive if and only if* $A\#K[G]^*$ *is subtractive;*

(iii) A *is graded prime if and only if* $A\#K[G]^*$ *is* G*-prime.*

Proof. (i) This is obvious.

(ii) Since every element of $A\#K[G]^*$ is of the type $\sum_i a_i p_{g_i}$, where $a_i \in A$ and A is subtractive, the proof is obvious.

(iii) This follows using Lemma 4.3 ((ii) and (iii)). ■

Finally we can prove the second main result of this paper.

Theorem 4.5. *Let R be a K-semialgebra graded by a finite group G.*

(i) *Let R be yoked and P a subtractive prime ideal of R. Then there exists a prime Q of $R\#K[G]^*$ such that $Q \cap R = P_G$, Q is unique up to its G-orbit $\{Q^g\}$ and $P_G\#K[G]^* = \cap_{g\in G} Q^g$, a G-prime ideal of $R\#K[G]^*$;*

(ii) *If I is any subtractive prime ideal of $R\#K[G]^*$, then $I \cap R = P_G$ for some subtractive prime P of R.*

Proof. (i) Suppose P is a subtractive prime ideal of R. Then P^Δ is a prime ideal of R^Δ. Hence there exists a prime Q' of $R^\Delta\#K^\Delta[G]^* \cong (R\#K[G]^*)^\Delta$ such that $Q' \cap R^\Delta = (P^\Delta)_G = (P_G)^\Delta$ and Q' is unique up to its G-orbit $\{Q'^g\}$ and $(P^\Delta)_G\#K^\Delta[G]^* = \cap_{g\in G} Q'^g$, a G-prime ideal of $R^\Delta\#K^\Delta[G]^*$ (c.f. [1], Theorem 6.2 (1)). Let $S = R\#K[G]^*$. Then Q' is a prime ideal of S^Δ. Hence $Q' \cap S$ is a prime ideal of S (c.f. Lemma 3.11 (ii)). Put $Q = Q' \cap S$. Then by taking $A = P_G$ in Lemma 4.2, we get $Q \cap R = P_G$. The uniqueness of Q follows from the uniqueness of Q'. We have $(P_G)^\Delta\#K^\Delta[G]^* = \cap_{g\in G} Q'^g$. This implies $(P_G\#K[G]^*)^\Delta = \cap_{g\in G} Q'^g$. By Lemma 4.4 (ii), $P_G\#K[G]^*$ is subtractive and so $P_G\#K[G]^* = (P_G\#K[G]^*)^\Delta \cap S$. Now by using ([4], Lemma 3.5 (iv)), we get

$$P_G\#K[G]^* = (P_G\#K[G]^*)^\Delta \cap S = (\cap_{g\in G} Q'^g) \cap S = \cap_{g\in G}(Q' \cap S)^g = \cap_{g\in G} Q^g.$$

Also, by Lemma 4.4 (iii), $P_G\#K[G]^*$ is G-prime.

(ii) Since I is a prime ideal of $R\#K[G]^*, I^G$ is a G-prime ideal of $R\#K[G]^*$. Then by using Lemma 4.3, we get $I^G \cap R = I \cap R$ and $I \cap R$ is a graded ideal of R. Also we have

$$I^G = (I^G \cap R) \# K[G]^* = (I \cap R) \# K[G]^*.$$

This implies $(I \cap R) \# K[G]^*$ is G-prime and hence by Lemma 4.4 (*iii*), $I \cap R$ is a graded prime ideal of R. Since I is subtractive, $I \cap R$ is subtractive. Thus $I \cap R$ is a subtractive graded prime ideal of R. Hence by Lemma 3.12, there exists a subtractive prime ideal P of R such that $I \cap R = P_G$. ■

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