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Localization of Pseudo Generalized Cohen-Macaulay Modules

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Abstract. We study the pseudo generalized Cohen-Macaulayness for finitely generated modules by using localization when the base ring is commutative Noetherian local. We extend this to the case where the ring is not necessarily local. As consequences, we get similar results for pseudo Cohen-Macaulayness and pseudo Buchsbaumness.

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1. Introduction

Let A be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated A-module with $\dim M = d > 0$. Let $\underline{x} = (x_1, \ldots, x_d)$ be a system of parameters (s.o.p. for short) of M. Consider the difference

$$J_M(\underline{x}) = e(\underline{x}; M) - \ell(M/Q_M(\underline{x})),$$

where $Q_M(\underline{x}) = \bigcup\limits_{t>0}((x_1^{t+1}, \ldots, x_d^{t+1})M :_M x_1^t \ldots x_d^t)$. The submodule $Q_M(\underline{x})$ of M and the number $J_M(\underline{x})$ give some useful information on M. For example, M is Cohen-Macaulay if and only if $Q_M(\underline{x}) = (x_1, \ldots, x_d)M$ for some s.o.p. $\underline{x}$ of M, cf. [8, 2]. In this case, $J_M(\underline{x}) = 0$ for all s.o.p. $\underline{x}$ of M. Further, $\ell(M/Q_M(\underline{x}))$ is just the length of the generalized fraction $1/(x_1, \ldots, x_d, 1)$ defined in [11], therefore if M is generalized Cohen-Macaulay then $\sup\limits_{\underline{x}} J_M(\underline{x}) < \infty$, where $\underline{x}$

runs over the set of all s.o.p. of M. Moreover, if M is Buchsbaum then $J_M(\underline{x})$ takes a constant value not depending on $\underline{x}$ for all s.o.p. $\underline{x}$ of M, [2].

Cuong-Nhan [6] and Cuong-Loan [3] studied the structure of modules called *pseudo Cohen-Macaulay* modules (resp. *pseudo generalized Cohen-Macaulay* modules; *pseudo Buchsbaum* modules) satisfying the condition $J_M(\underline{x}) = 0$ for all s.o.p. $\underline{x}$ of M (resp. $\sup_{\underline{x}} J_M(\underline{x}) < \infty$; $J_M(\underline{x})$ is a constant for all s.o.p. $\underline{x}$ of M). They proved that although the class of pseudo Cohen-Macaulay modules (resp. pseudo Buchsbaum modules, pseudo generalized Cohen-Macaulay modules) contains strictly the class of Cohen-Macaulay modules (resp. Buchsbaum modules, generalized Cohen-Macaulay modules), but these modules still have nice properties, closely related to that of Cohen-Macaulay modules (resp. Buchsbaum modules, generalized Cohen-Macaulay modules). However, the pseudo Cohen-Macaulayness, pseudo Buchsbaumness and pseudo generalized Cohen-Macaulayness are not preserved by localization, cf. [6].

The purpose of this paper is to study the pseudo Cohen-Macaulayness by using localization and apply to define the notions of pseudo Cohen-Macaulay modules, pseudo Buchsbaum modules, pseudo generalized Cohen-Macaulay modules over a commutative Noetherian ring, which are not necessarily local.

The paper is divided into 3 sections. In Sec. 2, we recall some properties of pseudo Cohen-Macaulay modules, pseudo Buchsbaum modules and pseudo generalized Cohen-Macaulay modules over a Noetherian local ring that will be used in the sequel. We study the localization of these modules in Sec. 3. In the last section, we extend the study to the case where the base ring is not necessarily local, and we obtain a theorem of Matijevic-Robert's type [9] for graded pseudo Cohen Macaulay modules.

2. Preliminaries

In this section, let $(A, \mathfrak{m})$ be a commutative Noetherian local ring and let M be a finitely generated R-module with $\dim M = d > 0$. Let $\underline{x} = (x_1, \ldots, x_d)$ be a s.o.p. of M and let $\underline{n} = (n_1, \ldots, n_d)$ be a d-tuple of positive integers. Set $\underline{x}(\underline{n}) = (x_1^{n_1}, \ldots, x_d^{n_d})$. Then the difference between multiplicities and lengths

$$J_M(\underline{x}(\underline{n})) = n_1 ... n_d e(\underline{x}; M) - \ell(M/Q_M(\underline{x}(\underline{n})))$$

can be considered as a function in $\underline{n}$. Note that this function is non-negative ([4, Lemma 3.1]) and ascending, i.e., $J_M(\underline{x}(\underline{n})) \geq J_M(\underline{x}(\underline{m}))$ whenever $n_i \geq m_i$ for all $i = 1, \ldots, d$, cf. [2, Corollary 4.3]. Sharp and Hamieh [11] asked that if the length of generalized fraction $1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1)$ is a polynomial for $\underline{n} \gg 0$, or equivalently, if the function $J_M(\underline{x}(\underline{n}))$ is a polynomial for $\underline{n} \gg 0$? Later on, Cuong-Morales-Nhan in [5] gave an example to show that this function $J_M(\underline{x}(\underline{n}))$ is not a polynomial for $\underline{n} \gg 0$. However, we have the following important result.

Theorem 2.1. (cf. [4, Theorem 3.2]) $J_M(\underline{x}(\underline{n}))$ *is bounded above by polynomials in* $\underline{n}$ *and the least degree of all polynomials bounding above this function is independent of the choice of the s.o.p.* $\underline{x}$ *of* M.

Following Cuong-Minh [4], the least degree in the above theorem is called the *polynomial type of fractions of* M and denoted by $pf(M)$.

Definition 2.2. (cf. [6, 3]) (i) M is called a pseudo Cohen-Macaulay module if $pf(M) = -\infty$, where we stipulate that the degree of the zero-polynomial is $-\infty$.

(ii) M is called a pseudo generalized Cohen-Macaulay module if $pf(M) \leqslant 0$.

(iii) M is called a pseudo Buchsbaum module if there exists a constant K such that $J_M(\underline{x}) = K$ for every s.o.p. $\underline{x}$ of M.

It should be mentioned that every Cohen-Macaulay module (resp. generalized Cohen-Macaulay module, Buchsbaum module) is pseudo Cohen-Macaulay (resp. pseudo generalized Cohen-Macaulay, pseudo Buchsbaum). However, the converse is not true.

In this paper, we use the following notations. Denote by $U_M(0)$ the largest submodule of M of dimension less than d. Set $\overline{M} = M/U_M(0)$ and $\widetilde{M} = \widehat{M}/U_{\widehat{M}}(0)$. It is clear that

$$\text{Ass } \overline{M} = \{\mathfrak{p} \in \text{Ass } M \mid \dim A/\mathfrak{p} = d\}.$$

Therefore $\text{Supp } \overline{M} = \bigcup_{\substack{\mathfrak{p}\in \text{Ass } M \\ \dim A/\mathfrak{p}=d}} V(\mathfrak{p})$. Following [1], $\text{Supp } \overline{M}$ is called the *unmixed support* of M and denoted by Usupp M.

Theorem 2.3. ([6, 3]) *Suppose that* A *admits a dualizing complex. Then the following statements are true.*

(*i*) *M is pseudo Cohen-Macaulay* (*resp. pseudo generalized Cohen-Macaulay*) *if and only if* $\overline{M}$ *is Cohen-Macaulay* (*resp. generalized Cohen-Macaulay*)*;*

(*ii*) *M is pseudo Buchsbaum if and only if* $\overline{M}$ *is Buchsbaum. In this case, for every s.o.p.* $\underline{x}$ *of* M *we have*

$$J_M(\underline{x}) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell(H^i_{\mathfrak{m}}(\overline{M})).$$

In general case (where the ring A does not necessarily admit a dualizing complex), we have $pf(M) = pf(\widehat{M})$, cf. [6]. Therefore M is pseudo Cohen-Macaulay (resp. pseudo generalized Cohen-Macaulay) if and only if so is $\widehat{M}$. So, by Theorem 2.3, M is pseudo Cohen-Macaulay (resp. pseudo generalized Cohen-Macaulay) if and only if $\widetilde{M} = \widehat{M}/U_{\widehat{M}}(0)$ is Cohen-Macaulay (resp. generalized Cohen-Macaulay). The following theorem shows that the same result is true for pseudo Buchsbaumness.

Theorem 2.4. (cf. [3]) *M is pseudo Buchsbaum if and only if $\widetilde{M}$ is Buchsbaum. In this case, for every s.o.p. $\underline{x}$ of M we have*

$$J_M(\underline{x}) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell(H^i_{\hat{\mathfrak{m}}}(\widetilde{M})).$$

3. Localization in Case the Base Ring is Local

In this section we always assume that $(A, \mathfrak{m})$ is a commutative Noetherian local ring and M is a finitely generated A-module with $\dim M = d$.

It is known that the Cohen-Macaulayness, Buchsbaumness and generalized Cohen-Macaulayness are preserved by taking localization with respect to any prime ideal in Supp M. Concretely, if M is generalized Cohen-Macaulay then $M_\mathfrak{p}$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Supp} M \setminus \{\mathfrak{m}\}$. However, this property does not hold for pseudo generalized Cohen-Macaulay modules, i.e. there exists a pseudo generalized Cohen-Macaulay A-module M such that $M_\mathfrak{p}$ is not pseudo generalized Cohen-Macaulay for some $\mathfrak{p} \in \operatorname{Supp} M$, cf. [6]. On the other hand, it is known that $pf(M) = pf(M/N)$ for any submodule N of M of dimension less than d, and hence M is pseudo Cohen-Macaulay (resp. pseudo generalized Cohen-Macaulay) if and only if so is $\overline{M} = M/U_M(0)$. Therefore the pseudo Cohen-Macaulayness and pseudo generalized Cohen-Macaulayness of M depends only on the unmixed support Usupp $M = \operatorname{Supp} \overline{M}$. This suggests studying the pseudo Cohen-Macaulayness and pseudo generalized Cohen-Macaulayness of M by taking localization with respect to prime ideals in the unmixed support M instead of the support of M.

Note that Usupp M is a closed subset of Spec A. We say that Usupp M is catenary if for any prime ideals $\mathfrak{p}, \mathfrak{q} \in \operatorname{Usupp} M$ with $\mathfrak{p} \subset \mathfrak{q}$, all saturated chains of prime ideals starting from $\mathfrak{p}$ and ending at $\mathfrak{q}$ have the same length. It is clear that Usupp M is catenary if and only if the ring $A/\operatorname{Ann} \overline{M}$ is catenary. Note that $\overline{M}$ is equidimensional, i.e. $\dim(A/\mathfrak{p}) = d$ for all $\mathfrak{p} \in \operatorname{Ass} \overline{M}$. So, Usupp M is catenary if and only if $\dim M_\mathfrak{p} + \dim(A/\mathfrak{p}) = d$ for all $\mathfrak{p} \in \operatorname{Usupp} M$.

The main result of this section is the following theorem.

Theorem 3.1. *Suppose that* Usupp M *is catenary and M is pseudo generalized Cohen-Macaulay. Then $M_\mathfrak{p}$ is pseudo Cohen-Macaulay for all prime ideals $\mathfrak{p} \in$* Usupp $M \setminus \{\mathfrak{m}\}$.

From now on we set Assh $M = \{\mathfrak{p} \in \operatorname{Ass} M \mid \dim A/\mathfrak{p} = d\}$. To prove Theorem 3.1, we need the following lemmas.

Lemma 3.2. *Suppose that* Usupp M *is catenary. Then $(U_M(0))_\mathfrak{p} = U_{M_\mathfrak{p}}(0)$ for all $\mathfrak{p} \in$* Usupp M. *In particular, $(U_M(0))_\mathfrak{p}$ is the largest submodule of $M_\mathfrak{p}$ of dimension less than* $\dim M_\mathfrak{p}$.

Proof. Let $0 = \bigcap_{\mathfrak{p}_i \in \text{Ass }(M)} N_i$ be a reduced primary decomposition of the submodule 0 of M, where N_i is $\mathfrak{p}_i$-primary for all i. Let $\mathfrak{p} \in$ Usupp M. Then $0_{M_\mathfrak{p}} = \bigcap_{\mathfrak{p}_i \subseteq \mathfrak{p}} (N_i)_\mathfrak{p}$ is a reduced primary decomposition of the submodule 0 of $M_\mathfrak{p}$. Therefore

$$U_{M_\mathfrak{p}}(0) = \bigcap_{\substack{\mathfrak{p}_i \subseteq \mathfrak{p} \\ \mathfrak{p}_i A_\mathfrak{p} \in \text{Assh } M_\mathfrak{p}}} (N_i)_\mathfrak{p}.$$

Moreover, since $U_M(0) = \bigcap_{\mathfrak{p}_i \in \text{Assh }(M)} N_i$, cf. [6], we can check that

$$(U_M(0))_\mathfrak{p} = \bigcap_{\substack{\mathfrak{p}_i \in \text{Assh } M \\ \mathfrak{p}_i \subseteq \mathfrak{p}}} (N_i)_\mathfrak{p}.$$

Therefore, the lemma will be proved if we can show that $\mathfrak{p}_i \in$ Assh M if and only if $\mathfrak{p}_i A_\mathfrak{p} \in$ Assh $M_\mathfrak{p}$ for every $\mathfrak{p}_i \in$ Ass M with $\mathfrak{p}_i \subseteq \mathfrak{p}$.

Now, let $\mathfrak{p}_i \in$ Assh M such that $\mathfrak{p}_i \subseteq \mathfrak{p}$. Since Usupp M is catenary,

$$\text{ht }(\mathfrak{p}/\mathfrak{p}_i) = \dim A/\mathfrak{p}_i - \dim A/\mathfrak{p} = d - \dim A/\mathfrak{p} = \dim M_\mathfrak{p}.$$

So, $\mathfrak{p}_i A_\mathfrak{p} \in$ Assh $M_\mathfrak{p}$. Conversely, let $\mathfrak{p}_i A_\mathfrak{p} \in$ Assh $M_\mathfrak{p}$. Since $\mathfrak{p} \in$ Usupp M, there exists $\mathfrak{p}_i' \in$ Assh M such that $\mathfrak{p}_i' \subseteq \mathfrak{p}$. Hence $\mathfrak{p}_i' A_\mathfrak{p} \in$ Assh $M_\mathfrak{p}$, and ht $(\mathfrak{p}/\mathfrak{p}_i') =$ ht $(\mathfrak{p}/\mathfrak{p}_i)$. Therefore $\dim A/\mathfrak{p}_i = \dim A/\mathfrak{p}_i' = d$ as Usupp M is catenary. Thus $\mathfrak{p}_i \in$ Assh M. ∎

Lemma 3.3. *Suppose that* Usupp M *is catenary and* $\mathfrak{p} \in$ Usupp M. *Then there exists* $\mathfrak{q} \in$ Assh M *such that* $\mathfrak{q} \subseteq \mathfrak{p}$ *and* $\mathfrak{q}A_\mathfrak{p} \in$ Assh $M_\mathfrak{p}$.

Proof. Let $\mathfrak{p} \in$ Usupp M. Then $M_\mathfrak{p} \neq 0$. Choose $\mathfrak{q}A_\mathfrak{p} \in$ Ass $(M_\mathfrak{p})$ such that $\dim(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}) = \dim(M_\mathfrak{p})$. Hence $\mathfrak{q} \in$ Ass M, $\mathfrak{q} \subseteq \mathfrak{p}$ and ht $(\mathfrak{p}/\mathfrak{q}) = \dim M_\mathfrak{p}$. Since Usupp M is catenary,

$$d \geq \dim A/\mathfrak{q} \geq \text{ht }(\mathfrak{p}/\mathfrak{q}) + \dim A/\mathfrak{p} = \dim M_\mathfrak{p} + \dim A/\mathfrak{p} = d.$$

It follows that $\dim A/\mathfrak{q} = d$, i.e. $\mathfrak{q} \in$ Assh M. ∎

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\mathfrak{p} \in$ Usupp $M \setminus \{\mathfrak{m}\}$. Then, there exists by Lemma 3.3 a prime ideal $\mathfrak{q} \in$ Assh M such that $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q}A_\mathfrak{p} \in$ Assh $M_\mathfrak{p}$. Thus ht $(\mathfrak{p}/\mathfrak{q}) = \dim M_\mathfrak{p}$. Let $\widehat{\mathfrak{p}} \in$ Assh $\widehat{A}/\mathfrak{p}\widehat{A}$. Then $\dim(\widehat{A}/\widehat{\mathfrak{p}}) = \dim(A/\mathfrak{p})$ and $\widehat{\mathfrak{p}} \cap A = \mathfrak{p}$. Since the natural map $A \longrightarrow \widehat{A}$ is flat, the going down theorem holds (cf. [10]), so there exists $\widehat{\mathfrak{q}} \in$ Spec $\widehat{A}$ contained in $\widehat{\mathfrak{p}}$ such that $\widehat{\mathfrak{q}} \cap A = \mathfrak{q}$ and ht $(\widehat{\mathfrak{p}}/\widehat{\mathfrak{q}}) \geq$ ht $\mathfrak{p}/\mathfrak{q}$. Because $M_\mathfrak{q} \neq 0$ and the induced map $A_\mathfrak{q} \longrightarrow \widehat{A}_{\widehat{\mathfrak{q}}}$ is faithful by flat, it follows that

$$\widehat{M}_{\widehat{\mathfrak{q}}} \cong M_{\mathfrak{q}} \otimes_{A_{\mathfrak{q}}} \widehat{A}_{\widehat{\mathfrak{q}}} \neq 0.$$

Therefore $\widehat{\mathfrak{q}} \in \operatorname{Supp} \widehat{M}$. Note that $\widehat{A}$ is a catenary ring. Moreover Usupp M is catenary by the hypothesis. So by the above fact we have

$$\begin{aligned}\dim(\widehat{A}/\widehat{\mathfrak{q}}) &= \dim(\widehat{A}/\widehat{\mathfrak{p}}) + \operatorname{ht}(\widehat{\mathfrak{p}}/\widehat{\mathfrak{q}}) \\ &\geq \dim(A/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) \\ &= \dim(A/\mathfrak{p}) + \dim(M_{\mathfrak{p}}) \\ &= d.\end{aligned}$$

It follows that $\widehat{\mathfrak{q}} \in \operatorname{Assh} \widehat{M}$. Therefore $\widehat{\mathfrak{p}} \in \operatorname{Usupp}\widehat{M}$.

Since M is pseudo generalized Cohen-Macaulay, so is $\widehat{M}$. Therefore $\widehat{M}/U_{\widehat{M}}(0)$ is generalized Cohen-Macaulay by Theorem 2.3, (i). It follows that $(\widehat{M}/U_{\widehat{M}}(0))_{\widehat{\mathfrak{p}}}$ is Cohen-Macaulay, i.e. $\widehat{M}_{\widehat{\mathfrak{p}}}/(U_{\widehat{M}}(0))_{\widehat{\mathfrak{p}}}$ is Cohen-Macaulay. Since $\widehat{A}$ is a catenary ring, $(U_{\widehat{M}}(0))_{\widehat{\mathfrak{p}}}$ is the largest submodules of $\widehat{M}_{\widehat{\mathfrak{p}}}$ of dimension less than $\dim(\widehat{M}_{\widehat{\mathfrak{p}}})$ by Lemma 3.2. Therefore $\widehat{M}_{\widehat{\mathfrak{p}}}$ is pseudo Cohen-Macaulay by Theorem 2.3, (i). Let $f : A_{\mathfrak{p}} \to \widehat{A}_{\widehat{\mathfrak{p}}}$ be the natural homomorphism. Then f is faithfully flat and $\dim(M_{\mathfrak{p}}) = \dim(\widehat{M}_{\widehat{\mathfrak{p}}})$. Therefore we can check that

$$pf(M_{\mathfrak{p}}) = pf(\widehat{M}_{\widehat{\mathfrak{p}}}) = -\infty.$$

Thus $M_{\mathfrak{p}}$ is pseudo Cohen-Macaulay. ∎

Theorem 3.1 leads immediately to the following consequence.

Corollary 3.4. *Assume that* Usupp M *is catenary. Then M is a pseudo Cohen-Macaulay* (*resp. pseudo Buchsbaum, pseudo generalized Cohen-Macaulay*) *A-module if and only if $M_{\mathfrak{p}}$ is a pseudo Cohen-Macaulay* (*resp. pseudo Buchsbaum, pseudo generalized Cohen-Macaulay*) *$A_{\mathfrak{p}}$-module for all* $\mathfrak{p} \in \operatorname{Usupp}M$.

4. Extension to General Case

In this section, let A be a commutative Noetherian (not necessarily local) ring and let M be a finitely generated A-module of dimension d. Corollary 3.4 suggests the following notions of pseudo Cohen-Macaulay modules, pseudo Buchsbaum modules and pseudo generalized Cohen-Macaulay modules.

Definition 4.1. M is called a pseudo Cohen-Macaulay module (respectively, pseudo Buchsbaum module, pseudo generalized Cohen-Macaulay module) if $M_{\mathfrak{p}}$ is pseudo Cohen-Macaulay (respectively, pseudo Buchsbaum, pseudo generalized Cohen-Macaulay) for every prime ideal $\mathfrak{p} \in \operatorname{Usupp}M$.

Firstly, we give a characterization for the pseudo Cohen-Macaulayness, pseudo Buchsbaumness and pseudo generalized Cohen-Macaulayness in non local case.

Theorem 4.2. *Suppose that A admits a dualizing complex. Let $U_M(0)$ be the largest submodule of M of dimension less than d. The following statements are equivalent:*

(i) *M is pseudo Cohen-Macaulay (resp. pseudo Buchsbaum, pseudo generalized Cohen-Macaulay).*

(ii) *$M/U_M(0)$ is Cohen-Macaulay (resp. Buchsbaum, generalized Cohen-Macaulay).*

Proof. We first prove that M is pseudo Cohen-Macaulay if and only if $M/U_M(0)$ is Cohen-Macaulay. Assume that M is pseudo Cohen-Macaulay. Let $\mathfrak{p} \in \mathrm{Usupp} M$. Then $M_{\mathfrak{p}}$ is pseudo Cohen-Macaulay. Since A admits a dualizing complex, $\mathrm{Usupp} M$ is catenary. Therefore by the same arguments as in the proof of Lemma 3.2 we can show that $(U_M(0))_{\mathfrak{p}}$ is the largest submodule of $M_{\mathfrak{p}}$ of dimension less than $\dim M_{\mathfrak{p}}$. Note that $A_{\mathfrak{p}}$ also admits a dualizing complex. Therefore by Theorem 2.3, $M_{\mathfrak{p}}/(U_M(0))_{\mathfrak{p}}$ is Cohen-Macaulay. Thus $(M/U_M(0))_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \in \mathrm{Usupp} M = \mathrm{Supp}(M/U_M(0))$, i.e. $M/U_M(0)$ is Cohen-Macaulay. Conversely, assume that $M/U_M(0)$ is Cohen-Macaulay. Then $(M/U_M(0))_{\mathfrak{p}} = M_{\mathfrak{p}}/(U_M(0))_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \in \mathrm{Supp}(M/U_M(0)) = \mathrm{Usupp} M$. Note that $(U_M(0))_{\mathfrak{p}} = U_{M_{\mathfrak{p}}}(0)$. Hence $M_{\mathfrak{p}}/U_{M_{\mathfrak{p}}}(0)$ is Cohen-Macaulay, and hence $M_{\mathfrak{p}}$ is pseudo Cohen-Macaulay by Theorem 2.3 for all $\mathfrak{p} \in \mathrm{Usupp} M$. Thus M is pseudo Cohen-Macaulay.

The rest follows similarly. ■

For the rest of this paper, we investigate the pseudo Cohen-Macaulayness, pseudo Buchsbaumness and pseudo generalized Cohen-Macaulayness in the graded case. Assume that R_0 is a commutative Noetherian ring admitting a dualizing complex, $R = \bigoplus_{n \geq 0} R_n$ is a finitely generated R_0-algebra and G is a finitely generated graded R-module. Let $\wp$ be a prime ideal of R. We denote by $\wp_h$ the largest graded ideal of R contained in $\wp$. Then $\wp_h$ is again a prime ideal of R. We have the following lemma.

Lemma 4.3. *If $\wp \in \mathrm{Usupp}\, G$ then $\wp_h \in \mathrm{Usupp}\, G$. Furthermore, $G_{\wp}$ is a pseudo Cohen-Macaulay $R_{\wp}$-module if and only if $G_{\wp_h}$ is a pseudo Cohen-Macaulay $R_{\wp_h}$-module.*

Proof. Assume that $\mathrm{Ass}\, G = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_n\}$. Then $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ are homogeneous ideals. Moreover there exists by [10, Proposition 10B, (ii)] a reduced primary decomposition $0_G = \bigcap_{i=1}^{n} N_i$ of the submodule 0 in G, where N_i is $\mathfrak{P}_i$-primary and homogeneous. Then we have

$$U_G(0) = \bigcap_{\dim R/\mathfrak{P}_j = \dim\ G} N_j.$$

Therefore $U_G(0)$ is homogeneous, and hence $G/U_G(0)$ is a graded module. So any associated prime of $G/U_G(0)$ is homogeneous. Since $\wp \in \mathrm{Usupp}\, G$, it contains an associated prime of $G/U_G(0)$. As this associated prime is homogeneous, it must be contained in $\wp_h$. Therefore $\wp_h \in \operatorname{Usupp} G$.

Now, $G_\wp$ is pseudo Cohen-Macaulay if and only if $(G/U_G(0))_\wp$ is Cohen-Macaulay, if and only if $(G/U_G(0))_{\wp_h}$ is Cohen-Macaulay by [7, Corollary 1.1.3]. Because R is catenary, by similar arguments as in the proof of Lemma 3.2, we can show that $(U_G(0))_\wp$ (resp. $(U_G(0))_{\wp_h}$) is the largest submodule of $G_\wp$ (resp. $G_{\wp_h}$) of dimension less than $\dim G_\wp$ (resp. $\dim G_{\wp_h}$). Thus, by Theorem 2.3, $G_\wp$ is pseudo Cohen-Macaulay if and only if $G_{\wp_h}$ is pseudo Cohen-Macaulay. ■

The following result is similar to the main result of J. Matijevic and P. Roberts in [9].

Theorem 4.4. *The following statements are equivalent:*

(i) *G is pseudo Cohen-Macaulay* (*resp. pseudo Buchsbaum, pseudo generalized Cohen-Macaulay*)*;*

(ii) *$G_{\mathfrak{P}}$ is pseudo Cohen-Macaulay* (*resp. pseudo Buchsbaum, pseudo generalized Cohen-Macaulay*) *for all homogeneous ideals* $\mathfrak{P} \in \operatorname{Usupp} G$;

(iii) *$G_{\mathfrak{M}}$ is pseudo Cohen-Macaulay* (*resp. pseudo Buchsbaum, pseudo generalized Cohen-Macaulay*) *for all homogeneous maximal ideals* $\mathfrak{M} \in \operatorname{Usupp} G$.

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial.

$(iii) \Rightarrow (i)$. Assume that $G_{\mathfrak{M}}$ is pseudo Cohen-Macaulay for all homogeneous maximal ideals $\mathfrak{M} \in \operatorname{Usupp} G$. Let $\wp \in \operatorname{Usupp} M$. Denote by $\wp_h$ the largest homogeneous ideal of R contained in $\wp$. Then $\wp_h \in \operatorname{Usupp} G$ by Lemma 4.3. There exists a maximal homogeneous ideal $\mathfrak{M} \in \operatorname{Usupp} G$ such that $\wp_h \subseteq \mathfrak{M}$. It is clear that $G_{\wp_h} \cong (G_{\mathfrak{M}})_{\wp_h}$. Since $G_{\mathfrak{M}}$ is pseudo Cohen-Macaulay, so is $G_{\wp_h}$ by Theorem 3.1. Hence $G_\wp$ is pseudo Cohen-Macaulay by Lemma 4.3. Thus, $G_\wp$ is pseudo Cohen-Macaulay. The rest of the theorem follows similarly. ■

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