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Some Generalized Quasi-Ky Fan Inequalities in Topological Ordered Spaces

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Abstract. In this paper, we obtain some existence results for solutions of some generalized quasi-Ky Fan inequalities and Nash equilibrium points for a game system in topological semilattices. Our results are similar to the ones obtained by Luo under some other conditions about continuity and convexity of vector set-valued mappings.

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1. Introduction

To start with let us recall the following well-known theorem of Ky Fan (1972, [4]):

Theorem 1.1. *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and let $f : K \times K \to \mathbb{R}$ be such that*

(1) $f(x, x) \leq 0,\ \forall x \in K$;
(2) $\forall y \in K,\ f(., y)$ *is quasiconcave;*
(3) $\forall x \in K,\ f(x, .)$ *is lower semi-continuous.*

Then there exists $y^* \in K$ such that $f(x, y^*) \leq 0,\ \forall x \in K$.

Ky Fan's minimax inequality has become a versatile tool in nonlinear and convex analysis, for instance, optimization problems, variational inequalities, problems of Nash equilibria, etc. In the framework of topological semilattices, Horvath and Llinares Ciscar (1996, [5]) were the first to establish an order theoretical version of the classical result of Knaster-Kuratowski-Mazurkiewicz, as well as fixed point theorems for multivalued mappings.

In 2001, by using Horvath and Llinares Ciscar's results, Luo [8] has proved a result similar to Theorem 1.1 for topological semilattices under some more general conditions. In 2006, Luo [9] studied Ky Fan's minimax inequalities for vector set-valued mappings in topological semilattices.

Based on these facts, in the first part of this note, we will give some results which are similar to the ones of Luo [9] under some other conditions about continuity and convexity of vector set-valued mappings. Next, by applying the vector Ky Fan minimax inequality, we give an existence theorem of the Nash equilibrium point for finitely many players.

2. Preliminaries

Definition 2.1. ([5]) A partially ordered set $(X, \leqslant)$ is called a *sup-semilattice* if any two elements x, y of X have a least upper bound, denoted by $x \vee y$ or $\sup\{x, y\}$. $(X, \leqslant)$ is a *topological semilattice* if X is a sup-semilattice equipped with a topology such that the mapping $X \times X \to X$, $(x, y) \mapsto x \vee y$ is continuous.

Once we have the notion of a sup-semilattice, we could obviously also consider the notion of an inf-semilattice. Should no confusion arise, we will simply use the word semilattice. It is also evident that each nonempty finite set A of X will have a least upper bound, denoted by $\sup A$.

In a partially ordered set $(X, \leq)$, two arbitrary elements x and x' do not have to be comparable but, in the case where $x \leq x'$, the set $[x, x'] = \{y \in X : x \leq y \leq x'\}$ is called an order interval. Now assume that $(X, \leq)$ is a semilattice and A is a nonempty finite subset; then the set $\Delta(A) = \bigcup_{a \in A}[a, \sup A]$ is well defined and it has the following properties:

(a) $A \subseteq \Delta(A)$;
(b) If $A \subset A'$, then $\Delta(A) \subseteq \Delta(A')$.

We say that a subset $E \subseteq X$ is Δ-convex if for any nonempty finite subset $A \subseteq E$ we have $\Delta(A) \subseteq E$.

Example 2.2. ([8]) Let

$$X = \{(x, 1) : 0 \leq x < 1\} \cup \{(x, y) : 0 \leq y \leq 1, x \geq 1, y \geq x - 1\} \subset \mathbb{R}^2;$$

A partial ordering of $\mathbb{R}^2$ is defined by

$$(a, b), (c, d) \in \mathbb{R}^2, (a, b) \leq (c, d) \Leftrightarrow c - a \geq 0, d - b \geq 0 \text{ and } d - b \leq c - a.$$

Then X is Δ-convex.

For any $D \subset X, \mathcal{F}(D)$ denotes the family of all finite subsets of D, $\Delta(D) = \bigcup_{A \in \mathcal{F}(D)} \Delta(A)$.

Lemma 2.3. ([*15*, Lemma 1.1]) *Let Y be a topological vector space and let C be a closed, convex and pointed cone of Y with* int$C \neq \emptyset$, *where* int$C$ *denotes the interior of C. Then* int$C + C \subset$ int$C$.

Definition 2.4. Let X be a topological semilattice or a Δ-convex subset of a topological semilattice, Y a topological vector space, $C \subset Y$ a closed, pointed and convex cone with $\operatorname{int} C \neq \emptyset$. A mapping $F : X \to 2^Y \setminus \{\emptyset\}$ is said to be a

(1) *lower C-Δ-quasiconvex mapping* if, for any pair $x_1, x_2 \in X$ and for any $x \in \Delta(\{x_1, x_2\})$, we have either

$$F(x) \subset F(x_1) - C$$

or

$$F(x) \subset F(x_2) - C;$$

(2) *upper C-Δ-quasiconvex* if, for any pair $x_1, x_2 \in X$ and for any $x \in \Delta(\{x_1, x_2\})$, we have either

$$F(x_1) \subset F(x) + C$$

or

$$F(x_2) \subset F(x) + C.$$

If F is single-valued, then the lower C-Δ-quasiconvexity and the upper C-Δ-quasiconvexity of F coincide and we say that F *is C-Δ-quasiconvex.*

Remark 2.5. If $Y = \mathbb{R} = (-\infty, +\infty)$ and $C = [0, +\infty)$, and $F = \varphi$ is a real function, then C-Δ-quasiconvexity of φ is equivalent to Δ-quasiconvexity of φ.

Definition 2.6. ([7, Definition 2.2]) Let X be a topological space, Y a topological vector space with a cone C. Given a subset $D \subset X$, we consider a multi-valued mapping $F : D \to 2^Y$. The domain of F is defined to be the set dom$F = \{x \in D : F(x) \neq \emptyset\}$.

1. F is said to be *upper (lower) C-continuous* at $\bar{x} \in$ dom$F$ if for any neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that

$$F(x) \subset F(\bar{x}) + V + C \quad (F(\bar{x}) \subset F(x) + V - C, \text{ respectively})$$

holds for all $x \in$ dom$F \cap U$.

2. If F is upper C-continuous and lower C-continuous at $\bar{x}$ simultaneously, we say that it is *C-continuous at $\bar{x}$*; and F is *upper* (respectively, *lower*) *C-continuous* on D if it is upper (respectively, lower) C-continuous at every point of D.

3. If F is single-valued, then the upper C-continuity and the lower C-continuity of F at $\bar{x}$ coincide and we say that F is *C-continuous at $\bar{x}$*.

Remark 2.7. If $Y = \mathbb{R}$ and $C = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ (or $C = \mathbb{R}_- = \{x \in \mathbb{R} : x \leq 0\}$) and F is C-continuous at $\bar{x}$, then F is lower semicontinuous (upper semicontinuous, respectively) at $\bar{x}$ in the usual sense.

Definition 2.8. Let X, Y be two topological spaces; $F : X \to 2^Y$ is said to *have open lower sections* if $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open for any $y \in Y$.

Lemma 2.9. (Browder-Fan fixed point theorem, [5, Corollary 1]) *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals M, $F : K \to 2^K$ with nonempty Δ-convex values, and let $F^{-1}(y) \subset K$ be open, for any $y \in K$. Then F has a fixed point.*

3. Generalized Quasi-Ky Fan Inequalities

Let X be a topological semilattice, $K \subset X$ a nonempty Δ-convex subset, Y a topological vector space with a cone C.

We consider the following quasi-Ky Fan inequalities:

I. **Upper ideal quasi-Ky Fan inequality:** Find $x \in K$ such that

$$x \in A(x), \quad f(x, y) \subset C, \quad \forall y \in A(x).$$

II. **Lower ideal quasi-Ky Fan inequality:** Find $x \in K$ such that

$$x \in A(x), \quad f(x, y) \cap C \neq \emptyset, \quad \forall y \in A(x).$$

III. **Upper Pareto quasi-Ky Fan inequality:** Find $x \in K$ such that

$$x \in A(x), \quad f(x, y) \not\subset -(C \setminus \{0\}), \quad \forall y \in A(x).$$

IV. **Lower Pareto quasi-Ky Fan inequality:** Find $x \in K$ such that

$$x \in A(x), \quad f(x, y) \cap -(C \setminus \{0\}) = \emptyset, \quad \forall y \in A(x).$$

V. **Upper weak quasi-Ky Fan inequality:** Find $x \in K$ such that

$$x \in A(x), \quad f(x, y) \not\subset -\text{int}C, \quad \forall y \in A(x).$$

VI. **Lower weak quasi-Ky Fan inequality:** Find $x \in K$ such that

$$x \in A(x), \quad f(x, y) \cap -\text{int}C = \emptyset, \quad \forall y \in A(x).$$

Some existence results for solutions of the problems (I, V, VI) are studied by Luo in [9]. In this paper, we give some results which are similar to the ones of Luo under some other conditions about continuity and convexity of vector set-valued mappings.

Theorem 3.1. *Let K be a nonempty compact Δ-convex subset of a Hausdorff topological semilattice with path-connected intervals M, Y a topological vector space, $A : K \to 2^K$ with nonempty Δ-convex values, $f : K \times K \to 2^Y$, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$. Assume that*

(1) *A has open lower sections and $B := \{x \in K : x \in A(x)\}$ is closed;*
(2) *$f(x,x) \cap -\operatorname{int} C = \emptyset,\ \forall x \in K$;*
(3) *For all $x \in K$, $f(x, .)$ is upper C-Δ-quasiconvex;*
(4) *For all $y \in K$, $f(., y)$ is lower $-C$-continuous.*

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$, and $f(x^*, y) \cap -\operatorname{int} C = \emptyset$, for all $y \in A(x^*)$.

Proof. Define $P : K \to K$ by

$$P(x) = \{y \in K : f(x, y) \cap -\operatorname{int} C \neq \emptyset\},\ \forall x \in K.$$

Suppose that there exists an $x' \in K$ such that $P(x')$ is not Δ-convex; then there exist $y_1, y_2 \in P(x')$ such that $\Delta(\{y_1, y_2\}) \not\subset P(x')$, i.e., there exists a $z \in \Delta(\{y_1, y_2\})$ and $z \notin P(x')$; hence $f(x', z) \cap -\operatorname{int} C = \emptyset$. By (3), we have either

$$f(x', y_1) \subset f(x', z) + C$$

or

$$f(x', y_2) \subset f(x', z) + C.$$

Since $f(x', y_i) \cap -\operatorname{int} C \neq \emptyset$, take $u_i \in f(x', y_i) \cap -\operatorname{int} C,\ i = 1, 2$. Then there exist $v_i \in f(x', z)$ and $w_i \in C$ such that either $u_1 = v_1 + w_1$ or $u_2 = v_2 + w_2$. By Lemma 2.3, we have either $v_1 = u_1 - w_1 \in -\operatorname{int} C$ or $v_2 = u_2 - w_2 \in -\operatorname{int} C$ which contradicts $f(x', z) \cap -\operatorname{int} C = \emptyset$. Therefore, for any $x \in X$, $P(x)$ is Δ-convex.

Next, we prove that $P^{-1}(y)$ is open for each $y \in K$. We have

$$\begin{aligned} K \setminus P^{-1}(y) &= \{x \in K : x \notin P^{-1}(y)\} = \{x \in K : y \notin P(x)\} \\ &= \{x \in K : f(x, y) \cap -\operatorname{int} C = \emptyset\}. \end{aligned}$$

In order to prove that for any $y \in K$, $P^{-1}(y)$ is open, it suffices to show that for any $y \in K$, $D := K \setminus P^{-1}(y)$ is closed. Letting $\overline{D}$ denote the closure of D and taking $\bar{x} \in \overline{D}$, we shall deduce that $\bar{x} \in D$. Assuming on the contrary that $\bar{x} \notin D$, we get $f(\bar{x}, y) \cap -\operatorname{int} C \neq \emptyset$. Therefore, there exist an $\bar{y} \in f(\bar{x}, y)$ and a neighborhood V of 0 in Y such that $\bar{y} + V \subset -\operatorname{int} C$. By assumption (4), there exists a neighborhood U of $\bar{x}$ such that

$$f(\bar{x}, y) \subset f(x, y) + V + C,\quad \forall x \in U.$$

Consequently, we have $\bar{y} \in f(x,y) + V + C, \ \ \forall x \in U$ and then

$$0 \in f(x,y) - \bar{y} + V + C \subset f(x,y) + \operatorname{int} C + C$$
$$\subset f(x,y) + \operatorname{int} C = f(x,y) - (-\operatorname{int} C), \ \ \forall x \in U.$$

We get $f(x,y) \cap -\operatorname{int} C \neq \emptyset, \ \ \forall x \in U$. Let $\{x_\alpha\}$ be any net in D converging to $\bar{x}$. Then there exists β such that $x_\alpha \in U$, $\forall \alpha \geq \beta$ and then $f(x_\alpha, y) \cap -\operatorname{int} C \neq \emptyset$, which contradicts $x_\alpha \in D$. Therefore, $\bar{x} \in D$ and D is closed. Consequently, we infer that $P^{-1}(y)$ is open for each $y \in K$.

By Lemma 2.9, B is a nonempty set. Define $S : K \to 2^K$ by

$$S(x) = \begin{cases} A(x) \cap P(x), & \text{if } x \in B, \\ A(x), & \text{if } x \in K \setminus B. \end{cases}$$

Then for any $x \in K$, $S(x)$ is Δ-convex. And for any $y \in K$,

$$S^{-1}(y) = (A^{-1}(y) \cap P^{-1}(y)) \cup ((K \setminus B) \cap A^{-1}(y))$$

is open.

Suppose that for all $x \in K, \ S(x)$ is nonempty; by Lemma 2.9, S has a fixed point, i.e., there exists an $x_0 \in K$ such that $x_0 \in S(x_0)$. If $x_0 \in B$, then $x_0 \in S(x_0) = A(x_0) \cap P(x_0)$; hence $x_0 \in P(x_0), \ f(x_0, x_0) \cap -\operatorname{int} C \neq \emptyset$ which contradicts our assumption (2); if $x_0 \in K \setminus B$, then $x_0 \in S(x_0) = A(x_0) \subset A(x_0)$, and hence $x_0 \in B$ which contradicts $x_0 \in K \setminus B$. Therefore, there exists an $x^* \in K$ such that $S(x^*) = \emptyset$. Since $A(x)$ is nonempty for all $x \in K$, hence $x^* \in B$, $S(x^*) = A(x^*) \cap P(x^*) = \emptyset$, i.e., $x^* \in A(x^*)$ and for any $y \in A(x^*), \ y \notin P(x^*)$, we have

$$x^* \in A(x^*), \ \ f(x^*, y) \cap -\operatorname{int} C = \emptyset, \ \ \forall y \in A(x^*).$$

Therefore, the assertion of Theorem 3.1 is true. ■

Corollary 3.2. *Let K be a nonempty compact Δ-convex subset of a Hausdorff topological semilattice with path-connected intervals M, Y a topological vector space, $A : K \to 2^K$ with nonempty Δ-convex values, $f : K \times K \to Y$, C a closed, pointed and convex cone in Y with* $\operatorname{int} C \neq \emptyset$. *Assume that*

(1) *A has open lower sections and $B := \{x \in K : x \in A(x)\}$ is closed;*
(2) *$f(x,x) \notin -\operatorname{int} C, \ \forall x \in K$;*
(3) *$\forall x \in K, \ f(x,.)$ is C-Δ-quasiconvex;*
(4) *$\forall y \in K, \ f(.,y)$ is $-C$-continuous.*

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$, and $f(x^*, y) \notin -\operatorname{int} C, \ \forall y \in A(x^*)$.

When $Y = (-\infty, +\infty), \ C = (-\infty, 0]$ and $A(x) = K$, for any $x \in K$, we get Ky Fan inequality for real-valued fucntions (see, for instance, [8, 13]).

Corollary 3.3. *Let K be a nonempty compact Δ-convex subset of a Hausdorff topological semilattice with path-connected intervals M and let $f : K \times K \to \mathbb{R}$ be such that*

(1) $f(x,x) \leq 0,\ \forall x \in K$;
(2) $\forall x \in K,\ f(x,.)$ *is Δ-quasiconcave;*
(3) $\forall y \in K,\ f(.,y)$ *is lower semi-continuous.*

Then there exists $x^* \in K$ such that $f(x^*, y) \leq 0,\ \forall y \in K$.

Theorem 3.4. *Let K be a nonempty compact Δ-convex subset of a Hausdorff topological semilattice with path-connected intervals M, Y a topological vector space, $A : K \to 2^K$ with nonempty Δ-convex values, $f : K \times K \to 2^Y$, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$. Assume that*

(1) *A has open lower sections and $B := \{x \in K : x \in A(x)\}$ is closed;*
(2) $f(x,x) \not\subset -\mathrm{int}C,\ \forall x \in K$;
(3) $\forall x \in K,\ f(x,.)$ *is lower C-Δ-quasiconvex;*
(4) $\forall y \in K,\ f(.,y)$ *is upper $-C$-continuous.*

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$, and $f(x^*, y) \not\subset -\operatorname{int} C$, for all $y \in A(x^*)$.

Proof. Define $P : K \to 2^K$ by

$$P(x) = \{y \in K : f(x,y) \subset -\operatorname{int} C\}, \quad \forall x \in K.$$

Suppose that there exists an $x' \in K$ such that $P(x')$ is not Δ-convex; then there exist $y_1, y_2 \in P(x')$ such that $\Delta(\{y_1, y_2\}) \not\subset P(x')$, i.e., there exists a $z \in \Delta(\{y_1, y_2\})$ and $z \notin P(x')$; hence $f(x', z) \not\subset -\operatorname{int} C$. By (3), we have either

$$f(x', z) \subset f(x', y_1) - C$$

or

$$f(x', z) \subset f(x', y_2) - C.$$

By Lemma 2.3, we have either

$$f(x', z) \subset f(x', y_1) - C \subset -\operatorname{int} C - C \subset -\operatorname{int} C$$

or

$$f(x', z) \subset f(x', y_2) - C \subset -\operatorname{int} C - C \subset -\operatorname{int} C$$

which is a contradiction. Therefore, for any $x \in X, P(x)$ is Δ-convex.

Next, we prove that $P^{-1}(y)$ is open for each $y \in K$. We have

$$P^{-1}(y) = \{x \in K : f(x,y) \subset -\operatorname{int} C\}$$

For each $y \in K$ and each $x \in P^{-1}(y)$, we have $f(x,y) \subset -\operatorname{int} C$, which implies that there exists a neighborhood V of the origin in Y such that $f(x,y) + V \subset -\operatorname{int} C$. By (4), there exists a neighborhood $U(x)$ of x such that $f(x',y) \subset f(x,y) + V - C \subset -\operatorname{int} C - C \subset -\operatorname{int} C$ whenever $x' \in U(x)$, which implies that $U(x) \subset P^{-1}(y)$, i.e., $P^{-1}(y)$ is open. The rest of the proof is similar to that of Theorem 3.1. ■

Theorem 3.5. *Let K be a nonempty compact Δ-convex subset of a Hausdorff topological semilattice with path-connected intervals M, Y a topological vector space, $A : K \to 2^K$ with nonempty Δ-convex values, $f : K \times K \to 2^Y$, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$. Assume that*

(1) *A has open lower sections and $B := \{x \in K : x \in A(x)\}$ is closed;*
(2) *$f(x,x) \subset C$, $\forall x \in K$;*
(3) *For all $x \in K$, $f(x,.)$ is upper C-Δ-quasiconvex;*
(4) *For all $y \in K$, $f(.,y)$ is lower $-C$-continuous.*

Then there exists $x^* \in K$ such that $x^* \in A(x^*)$, and $f(x^*,y) \subset C$, for all $y \in A(x^*)$.

Proof. Define $P : K \to 2^K$ by

$$P(x) = \{y \in K : f(x,y) \not\subset C\}, \quad \forall x \in K.$$

Suppose that there exists an $x' \in K$ such that $P(x')$ is not Δ-convex; then there exist $y_1, y_2 \in P(x')$ such that $\Delta(\{y_1,y_2\}) \not\subset P(x')$, i.e., there exists a $z \in \Delta(\{y_1,y_2\})$ and $z \notin P(x')$; hence $f(x',z) \subset C$. By (3), we have either

$$f(x',y_1) \subset f(x',z) + C$$

or

$$f(x',y_2) \subset f(x',z) + C.$$

Consequently, we have either

$$f(x',y_1) \subset f(x',z) + C \subset C + C \subset C$$

or

$$f(x',y_2) \subset f(x',z) + C \subset C + C \subset C,$$

which is a contradiction. Therefore, for any $x \in X, P(x)$ is Δ-convex.

Next, we prove that $P^{-1}(y)$ is open for each $y \in K$. We have

$$\begin{aligned} K \setminus P^{-1}(y) &= \{x \in K : x \notin P^{-1}(y)\} = \{x \in K : y \notin P(x)\} \\ &= \{x \in K : f(x,y) \subset C\}. \end{aligned}$$

In order to prove that for any $y \in K$, $P^{-1}(y)$ is open, it suffices to show that for any $y \in K$, $D := K \setminus P^{-1}(y)$ is closed. Taking $\bar{x} \in \overline{D}$, we shall deduce that $\bar{x} \in D$.

By (4), the lower C-continuity of $f(.,y)$ implies that for any neighborhood V of the origin in Y there is a neighborhood $U(\bar{x})$ of $\bar{x}$ such that

$$f(\bar{x},y) \subset f(x,y) + V + C, \quad \text{for all } x \in U(\bar{x}).$$

Let $\{x_\alpha\}$ be any net in D converging to $\bar{x}$, hence there exists β such that $x_\alpha \in U(\bar{x})$, for all $\alpha \geq \beta$ and then

$$f(\bar{x},y) \subset f(x_\alpha,y) + V + C, \ \forall \alpha \geq \beta$$

and so

$$f(\bar{x},y) \subset f(x_\alpha,y) + V + C \subset C + V + C \subset C + V \ \text{ for all } V.$$

Since C is closed, the last inclusion shows $f(\bar{x},y) \subset C$. Therefore, $\bar{x} \in D$ and D is closed. Consequently, we infer that $P^{-1}(y)$ is open for each $y \in K$.

The rest of the proof is similar to that of Theorem 3.1. ■

4. Game System

Let $(X_i, \leqslant_i), i \in I$, be a family of topological semilattices, and let $X := \prod_{i\in I} X_i$, $X_{-i} := \prod_{j\in I\setminus\{i\}} X_j$, be the product spaces with the product topology.

For $x, x' \in X := \prod_{i\in I} X_i$, define $x \leq x'$ if and only if $x_i \leq_i x'_i$. Then $(X, \leqslant)$ is a topological semilattice with $(x \bigvee x')_i = x_i \vee_i x'_i$ for each $i \in I$. For any $x \in X$, $x = (x_i, x_{-i})$, where $x_i \in X_i, \ x_{-i} \in X_{-i}$.

Let Y be a Hausdorff topological vector space with a cone C; for each $i \in I$, let $A_i : X \to 2^{X_i}$ be the ith constraint correspondence and $f_i : X \to Y$ the ith pay-off mapping.

Definition 4.1. A point $x^* \in X$ is called a *generalized Nash equilibrium point* of the game system $\Gamma = (X_i, A_i, f_i)_{i\in I}$, if for each $i \in I$,

$$x_i^* \in A_i(x^*), \quad f_i(u_i, x_{-i}^*) - f_i(x_i^*, x_{-i}^*) \notin -\mathrm{int}\, C, \quad \forall u_i \in A_i(x^*). \qquad (1)$$

Remark 4.2. When $Y = (-\infty, +\infty),\ C = (-\infty, 0]$, (1) reduces to: Find $x^* \in X$ such that for each $i \in I$,

$$x_i^* \in A_i(x^*), \quad f_i(x_i^*, x_{-i}^*) \geq f_i(u_i, x_{-i}^*), \quad \forall u_i \in A_i(x^*). \qquad (2)$$

The special cases of (2) are studied in [6, 14].

Lemma 4.3. *Let $I = \{1, 2, ..., n\}$, and for each $i \in I$, let X_i be a nonempty Δ-convex subset of a topological semilattice with path-connected intervals and set*

$$X := \prod_{i\in I} X_i, \quad X_{-i} := \prod_{j\in I\setminus\{i\}} X_j.$$

Let Y be a topological vector space, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$. For each $i \in I$, let $f_i : X \to Y$ be such that for any $x \in X$, $f_i(., x_{-i})$ is a C-Δ-quasiconvex mapping. Define

$$\varphi(x, y) = \sum_{i=1}^{n} (f_i(y_i, x_{-i}) - f_i(x_i, x_{-i})), \quad \forall x, y \in X.$$

Then for any $x \in X$, $\varphi(x, .)$ is also a C-Δ-quasiconvex mapping.

Proof. For any $x \in X$, $\{y^1, y^2\} \subset X$, and $y \in \Delta(\{y^1, y^2\})$, since $f_i(., x_{-i})$ is a C-Δ-quasiconvex mapping, for each $i \in I$, we have either

$$f_i(y_i^1, x_{-i}) \in f_i(y_i, x_{-i}) + C$$

or

$$f_i(y_i^2, x_{-i}) \in f_i(y_i, x_{-i}) + C.$$

We get

$$\varphi(x, y^1) = \sum_{i=1}^{n} (f_i(y_i^1, x_{-i}) - f_i(x_i, x_{-i})) \in \sum_{i=1}^{n} (f_i(y_i, x_{-i}) + C - f_i(x_i, x_{-i}))$$

or

$$\varphi(x, y^2) = \sum_{i=1}^{n} (f_i(y_i^2, x_{-i}) - f_i(x_i, x_{-i})) \in \sum_{i=1}^{n} (f_i(y_i, x_{-i}) + C - f_i(x_i, x_{-i}))$$

Since C is a convex cone, we have either

$$\begin{aligned}\varphi(x, y^1) \in \sum_{i=1}^{n} (f_i(y_i, x_{-i}) + C - f_i(x_i, x_{-i})) &\in \sum_{i=1}^{n} (f_i(y_i, x_{-i}) - f_i(x_i, x_{-i})) + C \\ &= \varphi(x, y) + C\end{aligned}$$

or

$$\begin{aligned}\varphi(x, y^2) \in \sum_{i=1}^{n} (f_i(y_i, x_{-i}) + C - f_i(x_i, x_{-i})) &\in \sum_{i=1}^{n} (f_i(y_i, x_{-i}) - f_i(x_i, x_{-i})) + C \\ &= \varphi(x, y) + C\end{aligned}$$

Hence $\varphi(x, .)$ is a C-Δ-quasiconvex mapping. ∎

The following lemma is a particular form of [11, Theorem 3.1].

Lemma 4.4. *Let $I = \{1, 2, ..., n\}$ and for each $i \in I$, let X_i be a nonempty, compact and Δ-convex subset of a Hausdorff topological semilattice with path-connected intervals and $X = \prod_{i \in I} X_i$. For each $i \in I$, let $T_i : X \to 2^{X_i}$ be a set-valued map such that*

(1) *T_i has nonempty Δ-convex values;*

(2) *T_i has open lower sections.*

Then there exists a point $x \in X$ such that $x \in T(x) := \prod_{i\in I} T_i(x)$; that is, $x_i \in T_i(x)$ for each $i \in I$, where $x_i = \pi_i(x)$ is the projection of x onto X_i for each $i \in I$.

Theorem 4.5. *Let $I = \{1, 2, ..., n\}$ and for each $i \in I$, let X_i be a nonempty compact Δ-convex subset of a Hausdorff topological semilattice with path-connected intervals,*

$$X := \prod_{i\in I} X_i, \quad X_{-i} := \prod_{j\in I\setminus\{i\}} X_j.$$

Let Y be a locally convex topological vector space. For each $i \in I$, let $A_i : X \to 2^{X_i}$ be a set-valued map, $f_i : X \to Y$ a map, C a closed, pointed and convex cone in Y with $\operatorname{int} C \neq \emptyset$, such that

(1) *$\forall i \in I$, A_i has open lower sections and nonempty Δ-convex values;*

(2) *The set $\{x \in X : x \in \prod_{i\in I} A_i(x)\}$ is closed;*

(3) *For all $i \in I$, f_i is C-continuous;*

(4) *For all $i \in I$, $f_i(u_i, x_{-i})$ is $-C$-continuous in x_{-i};*

(5) *For all $i \in I$, for any $x_{-i} \in X_{-i}$, the function $f_i(., x_{-i})$ is C-Δ-quasiconvex.*

Then there exists $x^* \in X$ such that for each $i \in I$,

$$x_i^* \in A_i(x^*), \quad f_i(u_i, x_{-i}^*) - f_i(x_i^*, x_{-i}^*) \notin -\operatorname{int} C, \quad \forall u_i \in A_i(x^*).$$

Proof. For any $x \in X$, let $A(x) = \prod_{i\in I} A_i(x)$. By Lemma 4.4 and (2), the set $B = \{x \in X : x \in A(x)\} = \{x \in X : x \in \prod_{i\in I} A_i(x)\}$ is nonempty and closed.

For any $x, y \in X$, let

$$\varphi(x, y) = \sum_{i=1}^{n} (f_i(y_i, x_{-i}) - f_i(x_i, x_{-i})).$$

By (5) and Lemma 4.3, for any $x \in X, \varphi(x, .)$ is a C-Δ-quasiconvex mapping. Next, we prove that for any $y \in X$, $\varphi(., y)$ is a $-C$-continuous mapping. Indeed, by assumption (3), for any neighborhood V of 0 in Y, there exists a neighborhood $U_1(x)$ such that

$$f_i(x') \in f_i(x) + \frac{V}{2n} + C, \quad \forall x' \in U_1(x),$$

or

$$-f_i(x'_i, x'_{-i}) \in -f_i(x_i, x_{-i}) + \frac{V}{2n} - C, \quad \forall x' \in U_1(x). \tag{3}$$

From (4), there exists a neighborhood $U_2(x_{-i})$ such that

$$f_i(y_i, x'_{-i}) \in f_i(y_i, x_{-i}) + \frac{V}{2n} - C, \quad \forall x'_{-i} \in U_2(x_{-i}). \tag{4}$$

Take $U(x) = U_1(x) \cap (U_3(x_i) \times U_2(x_{-i}))$, where $U_3(x_i)$ is a neighborhood of x_i in X_i. By (3) and (4), for any $x' \in U(x)$, we have

$$f_i(y_i, x'_{-i}) - f_i(x'_i, x'_{-i}) \in f_i(y_i, x_{-i}) - f_i(x_i, x_{-i}) + \frac{V}{n} - C.$$

Therefore, for any $x' \in U(x)$, we get

$$\begin{aligned}\varphi(x', y) = \sum_{i=1}^{n}(f_i(y_i, x'_{-i}) - f_i(x'_i, x'_{-i})) &\in \sum_{i=1}^{n}(f_i(y_i, x_{-i}) - f_i(x_i, x_{-i})) + V - C \\ &= \varphi(x, y) + V - C.\end{aligned}$$

Hence $\varphi(., y)$ is a $-C$-continuous mapping. By Corollary 3.2, there exists an $x^* \in X$ such that

$$x^* \in A(x^*), \quad \varphi(x^*, y) \notin -\mathrm{int}C, \quad \forall y \in A(x^*).$$

For each $i \in I$, for any $u_i \in A_i(x^*)$, taking $\bar{y} = (u_i, x^*_{-i}) \in A(x^*)$; then

$$\varphi(x^*, \bar{y}) = \sum_{k=1}^{n}(f_k(u_k, x^*_{-k}) - f_k(x^*_k, x^*_{-k})) = f_i(u_i, x^*_{-i}) - f_i(x^*_i, x^*_{-i}).$$

Therefore, for each $i \in I$,

$$x^*_i \in A_i(x^*), \quad f_i(u_i, x^*_{-i}) - f_i(x^*_i, x^*_{-i}) \notin -\mathrm{int}C, \quad \forall u_i \in A_i(x^*).$$

Hence our proof is finished. ■

Remark 4.6. Other interesting results on semilattices and equilibria can be found in [5, 8, 9, 11, 13].

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