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# On a Problem by A. Hajnal and I. Juhász

**Nuno C. Freire**[1] **and M. F. Veiga**[2]

[1] *DMAT Universidade de Évora Col. Verney,*
*R. Romão Ramalho 59, 7000 Évora, Portugal*
[2] *Faculdade de Ciências e Tecnologia Universidade Nova de Lisboa,*
*Quinta da Torre 2829-516 Monte de Caparica, Portugal*

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**Abstract.** Assuming (GCH), we obtain in ZFC the answer to a Problem by A. Hajnal and I. Juhász on the cardinality of the class of the sets determining the topology of an infinite $T_2$ space in the negative.

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*Key words:* Space, separated, cardinality.

## 1. Introduction

We can find in [2] Problem 2, by A. Hajnal and I. Juhász in the form of the following question: Is it true that for $\lambda$ being the cardinality of the class of all open sets in an infinite $T_2$ space it holds that, $\lambda^\omega = \lambda$? We obtain in ZFC, assuming (GCH) a counterexample showing that the answer is in the negative. In Sec. 2., we state definitions and properties concerning cardinal arithmetic that we apply in the proof of Theorem 3.1 in Sec. 3., which gives the Counterexample.

## 2. On Ordinals and Cardinals

Recall that a topological space is Hausdorff or a $T_2$ space if each two different points have disjoint neighborhoods.

Also recall that a partially ordered set $(\mathfrak{M}, \leq)$ is well-ordered if for each nonempty set $\mathfrak{B} \subset \mathfrak{M}$ there is some $b_0 \in \mathfrak{B}$ such that $b_0 \leq b$ for all $b \in \mathfrak{B}$.

**Definition 2.1.** ([1]) An ordinal number is a set $\alpha$ with the properties that, for each $x, y \in \alpha$ such that $x \neq y$ it holds that either $x \in y$ or $y \in x$ and, $(x \in y) \wedge (y \in \alpha) \Rightarrow x \in \alpha$.

Following [1] we say that a bijection $f : (\mathfrak{M}, \leq) \to (\mathfrak{N}, \preceq)$ between two well-ordered sets is an isomorphism if $a \leq b$ implies $f(a) \preceq f(b)$. If $a \in \mathfrak{M}$, the set $\mathfrak{M}_a = \{x \in \mathfrak{M} : (x \leq a) \wedge (x \neq a)\}$ is the initial interval determined by $a$. Also if $\alpha$ is an ordinal number $\mathfrak{C}_\alpha = \alpha$, $\mathfrak{C}$ is the class of all ordinal numbers, well-ordered by putting $\alpha \leq \beta$ if and only if $\alpha \subset \beta$; we write $\alpha < \beta$ if $\alpha \leq \beta, \alpha \neq \beta$ and we say that $\alpha$ is strictly less than $\beta$. If $\alpha$ is an ordinal number, $A \subset \alpha$, we write $\cup A$ for the supremum of $A$. Each well-ordered set is isomorphic to a suitable $\mathfrak{C}(\alpha)$ (see [1, p. 36, 42, 43 and Theorem 6.4, on the same page]). We say that two sets $A, B$ are equipotent if there is a bijection on $A$ onto $B$. It follows from Zermelo's theorem (in [1, Theorem 2.1 (3) , p. 32]) that in the class of all sets equipotent to $A$, there exists an ordinal number and, also, the smallest such ordinal number, which is called the initial ordinal number of the equipotence class ([1, p. 46]).

**Definition 2.2.** (Following [1, 5]) Let $A$ be a set. The cardinal number of $A$ is the initial ordinal number $\alpha$ in the class of all sets equipotent to $A$ .We denote by $\mid A \mid= \alpha$ the cardinality or cardinal number of $A$.

If $\alpha =\mid A \mid$ and $\beta =\mid B \mid$ are cardinal numbers such that there is an injection from $A$ to $B$ and the sets $A, B$ are not equipotent, we say that $\beta$ is greater than $\alpha$ and we write $\beta > \alpha$.

**Definition 2.3.** The generalized continuum hypothesis (GCH) consists of the assumption that, for any infinite set $X$, there is no cardinal number between the cardinal number of $X$ and the cardinal number of the power set $\mathcal{P}(X)$.

The first ordinal number is $\phi$. For each ordinal number $\alpha$, its successor is $\alpha \cup \{\alpha\} = \alpha + 1$. We denote $\phi = 0, \{\phi\} = 1, \{\phi\} \cup \{\{\phi\}\} = 2, ...$ These are the finite ordinal numbers, which can be viewed as the natural numbers. The first infinite ordinal number is $\omega$, which is the cardinal number of the set of natural numbers $\mathbf{N}$. $\omega$ is usually denoted by $\chi_0$ when viewed as the cardinal number $\mid \mathbf{N} \mid$. Each ordinal number $\beta$ that is of the form $\beta = \alpha + 1$ is said to have the immediate predecessor $\alpha$. We denote by the same symbol $\omega$ the ordinal number as well as the first infinite cardinal number $\mid \mathbf{N} \mid$. Its successor in the class of all cardinal numbers is the first cardinal number that is greater than $\omega$, which we denote by $\chi_1$. Writing $2^\omega$ for the cardinality of the power set $\mathcal{P}(\omega)$ we thus have $\chi_1 = 2^\omega$; we may consider $\chi_2 = 2^{\chi_1}$ and so forth, writing the alephs $\chi$ with an ordinal subscript. Hence we denote $\chi_\omega = \cup\{\chi_0, \chi_1, \chi_2, \ldots\}$.

**Definition 2.4.** ([5]) $\alpha, \beta$ being ordinal numbers, $\alpha + 0 = \alpha$, $\alpha + (\beta + 1) = (\alpha + \beta) + 1$. If $\beta$ has no immediate predecessor, then $\alpha + \beta = \bigcup_{\gamma<\beta} (\alpha + \gamma)$.

**Remark 2.5.** A cardinal number is either a natural number or an aleph ([4, p. 132]).

Recall that $\alpha, \beta$ being cardinal numbers, $\alpha =\mid A \mid, \beta =\mid B \mid$ where $A, B$ are disjoint sets, the sum $\gamma = \alpha + \beta$ is the cardinal number $\gamma =\mid A \cup B \mid$ and the product $\alpha\beta =\mid A \times B \mid$. Concerning an infinite sum, if the sets $A_i$ $(i \in I)$ are pairwise disjoint and $k_i =\mid A_i \mid$ then $\sum\limits_{i \in I} k_i =\mid \bigcup\limits_{i \in I} A_i \mid$ ([5, p. 92]) if $k_i = k$ for all $i$ we denote the infinite sum by $\sum\limits_{i \in I} k$. We denote by $\lambda^\mu$ the cardinal number of the set of all functions on $X$ to $Y$, where $\lambda =\mid Y \mid$ and $\mu =\mid X \mid$.

**Remark 2.6.** If $k, \tau$ are infinite cardinal numbers then $k + \tau = \max\{k, \tau\}$, $k.\tau = \max\{k, \tau\}$.

*Proof.* This follows from [5, Theorem 35. (b), (c), p. 88/9]. ■

**Remark 2.7.** If each $k_i \geq \omega$ $(i \in I)$ then $\sum_{i \in I} k_i = \sup\{k_i : i \in I\} + \mid I \mid$.

*Proof.* See [4, 26 (a), p. 98]. ■

**Lemma 2.8.** *It holds that* $(\chi_\omega)^\omega > \chi_\omega$.

*Proof.* This follows from [3, 2.*, p. 66]. ■

## 3. The Counterexample

Consider the intervals $[0, \chi_1[= I_1$, $[\chi_1, \chi_2[= I_2, ..., [\chi_n, \chi_{n+1}[= I_{n+1}, \ldots$ of ordinal numbers equipped with the discrete topologies. Let $E = \prod\limits_{n=1}^{\infty} I_n$ equipped with the product topology $\mathcal{T}$.

**Theorem 3.1.** *The topology* $\mathcal{T}$ *on* $E$ *has cardinality* $\mid \mathcal{T} \mid = \chi_\omega$.

*Proof.* We have that a base of $\mathcal{T}$ is determined by the open rectangles

$$
\begin{aligned}
&< U_{n(1)}, ..., U_{n(m)} > \\
&= \{(\lambda_n) \in E : \lambda_n \in U_n, U_n \subset I_n, n = n(k), \lambda_n \in I_n, n \neq n(k), 1 \leq k \leq m, m \in \mathbf{N}\}.
\end{aligned}
$$

Consequently, the class of all open sets in $\mathcal{T}$ has cardinality

$$
\lambda = \sum_{A \in \mathcal{F}} \chi_{\max A + 1} = \chi_\omega + \omega = \chi_\omega,
$$

where $A$ ranges over the class $\mathcal{F}$ of nonempty finite subsets of $\mathbf{N}$. In fact, this follows from Remark 2.7, since the cardinality of the discrete topology of $I_n$ is $\chi_{n+1}$ and $\chi_{n(1)} \cdots \chi_{n(k)} \cdots \chi_{n(p)} = \chi_{n(p)}$ $(n(1) \leq n(k) \leq n(p))$. The theorem is proved. ■

**Corollary 3.2.** *There exists an infinite Hausdorff topological space whose topology has a cardinal number* $\lambda$ *such that* $\lambda^\omega > \lambda$.

*Proof.* Since the topology $\mathcal{T}$ on $E$ is Hausdorff, the corollary follows from Theorem 3.1 and Lemma 2.8. ■

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