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Ore Extensions over 2-primal Rings

V. K. Bhat and Ravi Raina

School of Applied Physics and Mathematics, SMVD University,
P/o Kakryal, Udhampur, J and K, India- 182121

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Abstract. Let R be a ring, σ an automorphism of R and let δ be a σ-derivation of R. Recall that a ring R is said to be a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, where $P(R)$ denotes the prime radical of R.

It is known that if R is a δ-Noetherian Q-algebra, σ and δ are as usual such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and $\sigma(P) = P$, for all minimal prime ideals P of R, then $R[x, \sigma, \delta]$ is a 2-primal Noetherian ring. In this article it is proved that in the case δ is the zero map, R is a 2-primal Noetherian ring implies that $R[x, \sigma]$ is a 2-primal Noetherian ring. In the case σ is the identity map, a similar result is proved for the differential operator ring $R[x, \delta]$ (R in this case is moreover a Noetherian Q-algebra).

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1. Introduction

In this article a ring R always means an associative ring, Q denotes the field of rational numbers. Spec(R) denotes the set of prime ideals of R. MinSpec(R) denotes the sets of minimal prime ideals of R, $P(R)$ and $N(R)$ denote the prime radical and the set of nilpotent elements of R respectively. Let I and J be any two ideals of a ring R. Then $I \subset J$ means that I is strictly contained in J.

This article concerns 2-primal rings and their extensions. 2-primal rings have been studied in recent years and the 2-primal property is being studied for various types of rings. In [14], Marks discusses the 2-primal property of $R[x, \sigma, \delta]$, where R is a local ring, σ is an automorphism of R and δ is a σ-derivation of R. He has proved that when R is a local ring with a nilpotent maximal ideal, the

Ore extension $R[x,\sigma,\delta]$ will or will not be 2-primal depending on the δ-stability of the maximal ideal of R.

Recall that a σ-derivation of R is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$. In the case σ is the identity map, δ is called just a derivation of R. For example let σ be an automorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be defined by $\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}$, for all $r \in R$. Then δ is a σ-derivation of R.

Also let $R = K[x]$, where K is a field. Then the formal derivative d/dx is a derivation of R. Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [11]. 2-primal near rings have been discussed by Argac and Groenewald in [2]. Recall that a ring R is 2-primal if and only if the set of nilpotent elements of R and prime radical of R are the same if and only if the prime radical is a completely semiprime ideal. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$, where $a \in R$. We also note that a reduced is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [8, 10, 11, 16].

Recall that $R[x,\sigma,\delta]$ is the usual polynomial ring with coefficients in R in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x,\sigma,\delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$. We denote $R[x,\sigma,\delta]$ by $O(R)$. An ideal I of a ring R is called σ-invariant if $\sigma(I) = I$ and is called δ-invariant if $\delta(I) \subseteq I$. If an ideal I of R is σ-invariant and δ-invariant, then $I[x,\sigma,\delta]$ is an ideal of $R[x,\sigma,\delta]$ and as usual we denote it by $O(I)$.

In the case δ is the zero map, we denote the skew polynomial ring $R[x,\sigma]$ by $S(R)$. If J is an ideal of R such that $\sigma(J) = J$, we denote $J[x,\sigma]$ by $S(J)$.

In the case σ is the identity map, we denote the differential operator ring $R[x,\delta]$ by $D(R)$. If K is an ideal of R such that $\delta(K) \subseteq K$, we denote $K[x,\delta]$ by $D(K)$.

Now we study skew polynomial rings over 2-primal rings. There arises a natural question:

Question 1.1. Let R be a 2-primal ring. Is $R[x,\sigma,\delta]$ also a 2-primal ring?

For the time being we are not able to answer this question, but towards this we recall that in [12], a ring R is called σ-rigid if there exists an endomorphism σ of R with the property that $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. In [13], Kwak defines a $\sigma(*)$-ring R to be a ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ and establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. The property is also extended to the skew-polynomial ring $R[x,\sigma]$.

Recall also that the notion of a δ-ring was introduced in [4] in the following way.

Let R be a ring, σ an automorphism of R and δ be a σ-derivation of R. R is said to be a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, where $P(R)$ denotes the prime radical of R. We note that a ring with identity is not a δ-ring. Regarding δ-rings, the following result was proved.

Theorem 2.8 of [4]. *Let R be a δ-Noetherian Q-algebra such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$; $\sigma(P) = P$ for all $P \in \text{MinSpec}(R)$ and $\delta(P(R)) \subseteq P(R)$, where σ and δ are as usual. Then $O(R)$ is 2-primal.*

We now have a partial answer to the above question, namely Question 1.1 in the following way:

1. Let R be a 2-primal Noetherian ring and σ be an automorphism of R. Then $R[x, \sigma]$ is 2-primal Noetherian. This is proved in Theorem 2.4.
2. Let R be a 2-primal Noetherian Q-algebra and let δ be a derivation of R such that $N(D(R)) = D(N(R))$. Then $D(R)$ is 2-primal Noetherian. This is proved in Theorem 3.3.

Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. For example see [1,3–5,9,12,13].

In order to make the manuscript self contained, we state some results of [4]:

Theorem 2.2 of [4]. *Let R be a Noetherian Q-algebra. Let σ be an automorphism of R and δ be a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for $a \in R$. Then $P \in \text{MinSpec}(O(R))$ such that $\sigma(P \cap R) = P \cap R$ implies $P \cap R \in \text{MinSpec}(R)$ and $P_1 \in \text{MinSpec}(R)$ such that $\sigma(P_1) = P_1$ implies $O(P_1) \in \text{MinSpec}(O(R))$.*

Gabriel proved in [6, Lemma 3.4] that if R is a Noetherian Q-algebra and δ is a derivation of R, then $\delta(P) \subseteq P$, for all $P \in \text{MinSpec}(R)$. Towards this the following has been proved:

Proposition 1.1 (Proposition 2.3 of [4]) *Let R be a 2-primal ring. Let σ and δ be as usual such that $\delta(P(R)) \subseteq P(R)$. If $P \in \text{MinSpec}(R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.*

Theorem 2.4 of [4]. *Let R be a δ-ring. Let σ and δ be as above such that $\delta(P(R)) \subseteq P(R)$. Then R is 2-primal.*

Recall that an ideal I of a ring R is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$.

Proposition 1.2 (Proposition 2.5 of [4]) *Let R be a ring. Let σ and δ be as usual. Then*

1. *For any completely prime ideal P of R with $\delta(P) \subseteq P$, $O(P)$ is a completely prime ideal of $O(R)$.*
2. *For any completely prime ideal U of $O(R)$, $U \cap R$ is a completely prime ideal of R.*

Theorem 2.7 of [4]. *Let R be a δ-ring, where σ and δ as usual such that $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in \text{MinSpec}(R)$. Then $O(R)$ is 2-primal if and only if $O(P(R)) = P(O(R))$.*

2. Skew Polynomial Ring $S(R) = R[x, \sigma]$

Throughout this section R is an associative ring with identity. Recall that an ideal I of a ring R is called σ invariant if $\sigma(I) = I$. Also I is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$. We also note that in a right Noetherian ring R, MinSpec(R) is finite (Theorem 2.4 of Goodearl and Warfield [7]), and for any $P \in$ MinSpec(R), $\sigma^i(P) \in$ MinSpec(R) for all integers $i \geq 1$. Therefore there exists an integer $u \geq 1$, such that $\sigma^u(P) = P$ for all $P \in$ MinSpec(R). We use the same u henceforth, and as mentioned above, we denote $\cap_{i=1}^{u}\sigma^i(P)$ by P^0.

Proposition 2.1. *Let R be a Noetherian ring. Let σ be an automorphism of R. Then $\sigma(N(R)) = N(R)$.*

Proof. See [3, Proposition 2.2]. ■

Proposition 2.2. *Let R be a Noetherian ring. Let σ be as usual. Then $S(N(R)) = N(S(R))$.*

Proof. It is easy to see that $S(N(R)) \subseteq N(S(R))$. We will show that $N(S(R)) \subseteq S(N(R))$. Let $f = \sum_{i=0}^{m} x^i a_i \in N(S(R))$. Then $(f)(S(R)) \subseteq N(S(R))$, and $(f)(R) \subseteq N(S(R))$. Let $((f)(R))^k = 0$, $k > 0$. Then equating leading term to zero, we get $(x^m a_m R)^k = 0$. This implies on simplification that

$$x^{km}\sigma^{(k-1)m}(a_m R).\sigma^{(k-2)m}(a_m R).\sigma^{(k-3)m}(a_m R)\ldots a_m R = 0.$$

Therefore $\sigma^{(k-1)m}(a_m R).\sigma^{(k-2)m}(a_m R).\sigma^{(k-3)m}(a_m R)\ldots a_m R = 0 \subseteq P$, for all $P \in$ MinSpec(R). Now there are two cases: $u \geq m$, or $m \geq u$. If $u \geq m$, then we have

$$\sigma^{(k-1)u}(a_m R).\sigma^{(k-2)u}(a_m R).\sigma^{(k-3)u}(a_m R)...a_m R \subseteq P.$$

This implies that $\sigma^{(k-j)u}(a_m R) \subseteq P$, for some j, $1 \leq j \leq k$, i.e., $a_m R \subseteq \sigma^{-(k-j)u}(P) = P$. So we have $a_m R \subseteq P$, for all $P \in$ MinSpec(R). Therefore $a_m \in P(R) = N(R)$. Now $x^m a_m \in S(N(R)) \subseteq N(S(R))$ implies that $\sum_{i=0}^{m-1} x^i a_i \in N(S(R))$, and with the same process, in a finite number of steps, it can be seen that $a_i \in P(R) = N(R)$, $0 \leq i \leq m-1$. Therefore $f \in S(N(R))$. Hence $N(S(R)) \subseteq S(N(R))$ and the result. ■

We now establish a relation between the minimal prime ideals of R and those of $S(R)$ in the following theorem.

Theorem 2.3. *Let R be a Noetherian ring and σ be an automorphism of R. Then $P \in$ MinSpec$(S(R))$ if and only if there exists $L \in$ MinSpec(R), such that $S(P \cap R) = P$ and $P \cap R = L^0$.*

Proof. Let $L \in$ MinSpec(R). Then $\sigma^u(L) = L$ for some integer $u \geq 1$. Let $L_1 = L^0$. Then by [15, (10.6.12)] and by [7, Theorem 7.27], $Q_2 = S(L_1) \in$ MinSpec$(S(R))$.

Conversely, suppose that $P \in \text{MinSpec}(S(R))$. Then $P \cap R = U^0$ for some $U \in \text{Spec}(R)$ and U contains a minimal prime U_1. Now $P \supseteq S(R)U_1^0$, which is a prime ideal of $S(R)$. Hence $P = S(R)U_1^0$. ∎

We are now in a position to prove the main result of this section in the form of the following theorem.

Theorem 2.4. *Let R be a 2-primal Noetherian ring. Then $S(R)$ is 2-primal Noetherian.*

Proof. The fact that R is Noetherian implies $S(R)$ is Noetherian follows from Hilbert Basis Theorem, namely Theorem 1.12 of [7]. Now R is 2-primal implies $N(R) = P(R)$ and Proposition 2.1 implies that $\sigma(N(R)) = N(R)$. Therefore $S(N(R)) = S(P(R))$. Now by Proposition 2.2 $S(N(R)) = N(S(R))$.

We now show that $S(P(R)) = P(S(R))$. It is easy to see that $S(P(R)) \subseteq P(S(R))$. Now let $g = \sum_{i=0}^{t} x^i b_i \in P(S(R))$. Then $g \in P_i$, for all distinct $P_i \in \text{MinSpec}(S(R))$. Now Theorem 2.3 implies that there exists $U_i \in \text{MinSpec}(R)$ such that $P_i = S((U_i)^0)$. Now it can be seen that P_i are distinct implies that U_i are distinct. Therefore $g \in S((U_i)^0)$. This implies that $b_i \in (U_i)^0 \subseteq U_i$. Thus we have $b_i \in U_i$, for all $U_i \in \text{MinSpec}(R)$. Therefore $b_i \in P(R)$, which implies that $g \in S(P(R))$. So we have $P(S(R)) \subseteq S(P(R))$, and hence $S(P(R)) = P(S(R))$. Thus we have $P(S(R)) = S(P(R)) = S(N(R)) = N(S(R))$. Hence $S(R)$ is 2-primal. ∎

Question 2.5. Let R be a 2-primal ring. Is $S(R)$ 2-primal?

The main difficulty is that Proposition 2.2 and Theorem 2.3 do not hold.

3. Differential Operator Ring $D(R) = R[x, \delta]$

Throughout this section R is a Noetherian Q-algebra and δ is a derivation of R.

Proposition 3.1. *Let R be a Noetherian Q-algebra and δ be a derivation of R. Then $\delta(P) \subseteq P$, for all $P \in \text{MinSpec}(R)$ and $\delta(P(R)) \subseteq P(R)$.*

Proof. See [7, Lemma 2.20]. ∎

Theorem 3.2. *Let R be a Noetherian Q-algebra and δ be a derivation of R. Then $P \in \text{MinSpec}(D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in \text{MinSpec}(R)$.*

Proof. Let $P_1 \in \text{MinSpec}(R)$. Then $\delta(P_1) \subseteq P_1$ by Proposition 3.1. Therefore by (14.2.5) (ii) of [15], $D(P_1) \in \text{Spec}(D(R))$. Suppose $P_2 \subset D(P_1)$ is a minimal prime ideal of $D(R)$. Then $P_2 = D(P_2 \cap R) \subset D(P_1) \in \text{MinSpec}(D(R))$. So $P_2 \cap R \subset P_1$ which is not possible.

Conversely suppose that $P \in \text{MinSpec}(D(R))$. Then $P \cap R \in \text{Spec}(R)$ by Lemma 2.21 of [7]. Let $P_3 \subset P \cap R$ be a minimal prime ideal of R. Then $D(P_3) \subset D(P \cap R)$ and as in the first paragraph $D(P_3) \in \text{Spec}(D(R))$, which is a contradiction. Hence $P \cap R \in \text{MinSpec}(R)$. ■

Theorem 3.3. *Let R be a 2-primal Noetherian Q-algebra and let δ be a derivation of R such that $N(D(R)) = D(N(R))$. Then $D(R)$ is 2-primal Noetherian.*

Proof. First of all we note that $D(P(R))$ is well defined by Proposition 3.1. Also $D(R)$ is Noetherian by Theorem 1.12 of [7]. Now R is 2-primal implies that $N(R) = P(R)$. We will show that $P(D(R)) = D(P(R))$. Now let $g = \sum_{i=0}^{t} x^i b_i \in P(D(R))$. Then $g \in P_i$, for all distinct $P_i \in \text{MinSpec}(S(R))$. Now Theorem 3.2 implies that $P_i \cap R \in \text{MinSpec}(R)$ and that $P_i = D(P_i \cap R)$. Denote $P_i \cap R$ by U_i. Now it can be seen that U_i are distinct. Therefore $g \in D(U_i)$. This implies that $b_i \in U_i$. Thus we have $b_i \in U_i$, for all $U_i \in \text{MinSpec}(R)$. Therefore $b_i \in P(R)$, which implies that $g \in D(P(R))$. So we have $P(D(R)) \subseteq D(P(R))$. Now let $h = \sum_{i=0}^{m} x^i c_i \in D(P(R))$. Then $c_i \in D(P(R)) \subseteq T_i$, for all distinct $T_i \in \text{MinSpec}(R)$. Now Theorem 3.2 implies that $D(T_i) \in \text{MinSpec}(D(R))$. Denote $D(T_i)$ by L_i. Now it can be seen that L_i are distinct and therefore $h \in L_i$ for all $L_i \in \text{MinSpec}(D(R))$. Thus $h \in P(D(R))$ and therefore $D(P(R)) \subseteq P(D(R))$.

So we have $P(D(R)) = D(P(R))$. Now it is given that $N(D(R)) = D(N(R))$ and thus we have $P(D(R)) = D(P(R)) = D(N(R)) = N(D(R))$. Hence $D(R)$ is 2-primal. ■

We terminate the article with the following question.

Question 3.4. Let R be a 2-primal Noetherian Q-algebra. Is $D(R)$ 2-primal?

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