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Existence and Uniqueness of Periodic Solutions for a Class of Nonlinear n-th Order Differential Equations with Delays *

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Abstract. In this paper, we use the coincidence degree theory to establish new results on the existence of T-periodic solutions for a class of nonlinear n-th order differential equations with delays of the form

$$x^{(n)}(t) + f(x^{(n-1)}(t)) + g(t, x(t - \tau(t))) = p(t).$$

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1. Introduction

Consider nonlinear n-th order differential equations with delays of the form

$$x^{(n)}(t) + f(x^{(n-1)}(t)) + g(t, x(t - \tau(t))) = p(t), \tag{1}$$

where f, τ, $p : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $f(0) = 0$, τ and p are T-periodic, g is T-periodic in its first argument, $n \geq 2$ is an integer, and $T > 0$ is a constant.

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Clearly, when $n = 2$ and $g(t, x(t - \tau(t))) = g(x(t - \tau(t)))$, Equation (1) can be reduced to

$$x'' + f(x'(t)) + g(x(t - \tau(t))) = p(t), \tag{2}$$

which has been known as the delayed Rayleigh equation. Therefore, we can consider Equation (1) as a high-order delayed Rayleigh equation. Arising from problems in applied sciences, it is well known that the existence of periodic solutions of Equation (1) has been extensively studied over the past twenty years. We refer the reader to [1]-[4], [6]-[18] and the references cited therein. However, to the best of our knowledge, there exist no results for the existence and uniqueness of periodic solutions of Equation (1). Thus, it is worth while to continue to investigate the existence and uniqueness of periodic solutions of Equation (1).

The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of T-periodic solutions of Equation (1). The results of this paper are new and they complement previously known results. An illustrative example is given in Sec. 4.

For ease of exposition, throughout this paper we will adopt the following notations:

$$|x|_k = \left(\int_0^T |x(t)|^k dt \right)^{1/k}, \qquad |x|_\infty = \max_{t \in [0,T]} |x(t)|,$$

$$||x|| = \sum_{j=0}^{n-1} |x^{(j)}|_\infty, \qquad x^{(0)} = x.$$

Let

$$X = \{x | x \in C^{n-1}(\mathbb{R}, \mathbb{R}),\ x(t+T) = x(t), \ \text{ for all } \ t \in \mathbb{R}\}$$

and

$$Y = \{x | x \in C(\mathbb{R}, \mathbb{R}),\ x(t+T) = x(t), \ \text{for all } \ t \in \mathbb{R}\}$$

be two Banach spaces with the norms

$$\|x\|_X = ||x|| = \sum_{j=0}^{n-1} |x^{(j)}|_\infty, \ \text{and} \quad \|x\|_Y = |x|_\infty = \max_{t \in [0,T]} |x(t)|.$$

Define a linear operator $L : D(L) \subset X \longrightarrow Y$ by setting

$$D(L) = \{x | x \in X, x^{(n)} \in C(\mathbb{R}, \mathbb{R})\}$$

and for $x \in D(L)$,

$$Lx = x^{(n)}. \tag{3}$$

We also define a nonlinear operator $N : X \to Y$ by setting

$$Nx = -\left[f(x^{(n-1)}(t)) + g(t, x(t - \tau(t))) \right] + p(t). \tag{4}$$

It is easy to see that

$$\text{Ker}L = \mathbb{R}, \quad \text{and Im}L = \left\{ x | x \in Y, \int_0^T x(s)ds = 0 \right\}.$$

Thus the operator L is a Fredholm operator with index zero.

Define the continuous projectors $P : X \to \text{Ker}L$ and $Q : Y \to Y/\text{Im}L$ by setting

$$Px(t) = \frac{1}{T}\int_0^T x(s)ds,$$

and

$$Qx(t) = \frac{1}{T}\int_0^T x(s)ds.$$

Hence, $\text{Im}P = \text{Ker}L$ and $\text{Ker}Q = \text{Im}L$. Denoting by $L_P^{-1} : \text{Im}L \to D(L) \cap \text{Ker}P$ the inverse of $L|_{D(L)\cap\text{Ker}P}$, one observes that L_P^{-1} is a compact operator. Therefore, N is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X.

It is convenient to introduce the following assumptions.

(A_0) There exists a nonnegative constant C_1 such that

$$|f(x_1) - f(x_2)| \le C_1|x_1 - x_2|, \text{ for all } x_1, x_2 \in \mathbb{R}.$$

$(\widetilde{A_0})$ There exists a nonnegative constant C_2 such that

$$C_2|x_1 - x_2|^2 \le (x_1 - x_2)(f(x_1) - f(x_2)) \text{ for all } x_1, x_2 \in \mathbb{R}.$$

2. Preliminary Results

In view of (3) and (4), the operator equation

$$Lx = \lambda Nx$$

is equivalent to the following equation

$$x^{(n)}(t) + \lambda\left[f(x^{(n-1)}(t)) + g(t, x(t - \tau(t)))\right] = \lambda p(t), \tag{5}$$

where $\lambda \in (0, 1)$.

For convenience of use, we introduce the following Continuation theorem [5].

Lemma 2.1. *Let X and Y be two Banach spaces. Suppose that $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero and $N : X \to Y$ is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied.*

(a) *$Lx \ne \lambda Nx$, for all $x \in \partial\Omega \cap D(L), \lambda \in (0,\ 1)$;*

(b) $Nx \notin \mathrm{Im}L$, *for all* $x \in \partial\Omega \cap \mathrm{Ker}L$;

(c) *The Brouwer degree*

$$\deg\{QN,\ \Omega \cap \mathrm{Ker}L, 0\} \neq 0.$$

Then equation $Lx = Nx$ *has at least one* T*-periodic solution in* $\overline{\Omega}$.

The following lemmas will be useful to prove our main results in Sec. 3.

Lemma 2.2. (Wirtinger inequality, see [17]) *If* $x \in C^2(\mathbb{R}, \mathbb{R})$, $x(t+T) = x(t)$, *then*

$$|x'(t)|_2 \leq \frac{T}{2\pi}|x''(t)|_2. \tag{6}$$

Lemma 2.3. *Suppose that there exists a constant* $d > 0$ *such that one of the following conditions holds:*

(A_1) $x(g(t,x) - p(t)) < 0$, *for all* $t \in \mathbb{R}$, $|x| \geq d$,

(A_2) $x(g(t,x) - p(t)) > 0$, *for all* $t \in \mathbb{R}, |x| \geq d$.

If $x(t)$ *is a* T*-periodic solution of* (5), *then*

$$|x|_\infty \leq d + \sqrt{T}|x'|_2. \tag{7}$$

Proof. Let $x(t)$ be a T-periodic solution of (5). Set

$$x^{(n-2)}(t_1) = \max_{t\in R} x^{(n-2)}(t),$$
$$x^{(n-2)}(t_2) = \min_{t\in R} x^{(n-2)}(t),$$

where $t_1,\ t_2 \in \mathbb{R}$.
Then we have

$$x^{(n-1)}(t_1) = 0,\quad x^{(n)}(t_1) \leq 0, \quad \text{and} \quad x^{(n-1)}(t_2) = 0,\quad x^{(n)}(t_2) \geq 0. \tag{8}$$

In view of (A_1) and (A_2), we shall consider two cases as follows.

Case (i). If (A_1) holds, it follows from $f(0) = 0$, (5) and (8) that

$$g(t_1, x(t_1 - \tau(t_1))) - p(t_1) = -\frac{x^{(n)}(t_1)}{\lambda} \geq 0$$

and

$$g(t_2, x(t_2 - \tau(t_2))) - p(t_2) = -\frac{x^{(n)}(t_2)}{\lambda} \leq 0.$$

Then, in view of (A_1), we obtain

$$x(t_1 - \tau(t_1)) < d \quad \text{and} \quad x(t_2 - \tau(t_2)) > -d.$$

Since $x(t - \tau(t))$ is a continuous function on $\mathbb{R}$, it follows that there exists a constant $\xi \in \mathbb{R}$ such that $|x(\xi - \tau(\xi))| \leq d$.

Let $\xi - \tau(\xi) = mT + t_0$, where $t_0 \in [0, T]$, and m is an integer. Then, using Schwarz inequality and the following relation

$$|x(t)| = \left| x(t_0) + \int_{t_0}^{t} x'(s)ds \right| \leq d + \int_0^T |x'(s)|ds, t \in [0, T],$$

we have

$$|x|_\infty = \max_{t \in [0,T]} |x(t)| \leq d + \sqrt{T}|x'|_2.$$

Case (ii). If (A_2) holds, then using a similar argument as that in the proof of case (i), we see that (7) holds. This completes the proof of Lemma 2.3. ■

Lemma 2.4. *Assume that one of the following conditions is satisfied:*

(A_3) *Suppose that* (A_0) *holds,* $g(t, x)$ *is a strictly monotone function in* x*, and there exists a nonnegative constant* b *such that*

$$C_1 \frac{T}{2\pi} + b \frac{T^n}{(2\pi)^{n-1}} < 1, \quad \text{and } |g(t, x_1) - g(t, x_2)| \leq b|x_1 - x_2|,$$

for all $t, x_1, x_2 \in \mathbb{R}$;

(A_4) *Suppose that* $(\widetilde{A_0})$ *hold,* $g(t, x)$ *is a strictly monotone function in* x*, and there exists a constant* b *such that*

$$0 \leq b < \frac{C_2(2\pi)^{n-2}}{T^{n-1}}, \text{ and } |g(t, x_1) - g(t, x_2)| \leq b|x_1 - x_2|, \text{ for all } t, x_1, x_2 \in \mathbb{R}.$$

Then Equation (1) *has at most one* T*-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of Equation (1). Then, we have

$$x_1^{(n)}(t) + f(x_1^{(n-1)}(t)) + g(t, x_1(t - \tau(t))) = p(t)$$

and

$$x_2^{(n)}(t) + f(x_2^{(n-1)}(t)) + g(t, x_2(t - \tau(t))) = p(t).$$

This implies that

$$\begin{aligned}(x_1(t) - x_2(t))^{(n)} &+ (f(x_1^{(n-1)}(t)) - f(x_2^{(n-1)}(t))) \\ &+ (g(t, x_1(t - \tau(t))) - g(t, x_2(t - \tau(t)))) = 0. \qquad (9)\end{aligned}$$

Set $Z(t) = x_1(t) - x_2(t)$. Then, from (9), we obtain

$$\begin{aligned}Z^{(n)}(t) &+ (f(x_1^{(n-1)}(t)) - f(x_2^{(n-1)}(t))) \\ &+ (g(t, x_1(t - \tau(t))) - g(t, x_2(t - \tau(t)))) = 0. \qquad (10)\end{aligned}$$

Set

$$Z^{(n-2)}(\bar{t}_1) = \max_{t\in R} Z^{(n-2)}(t), \qquad Z^{(n-2)}(\bar{t}_2) = \min_{t\in R} Z^{(n-2)}(t), \quad \text{where} \ \ \bar{t}_1, \ \bar{t}_2 \in \mathbb{R}.$$

Then, we have

$$Z^{(n-1)}(\bar{t}_1) = x_1^{(n-1)}(\bar{t}_1) - x_2^{(n-1)}(\bar{t}_1) = 0, \quad Z^{(n)}(\bar{t}_1) \le 0,$$

and

$$Z^{(n-1)}(\bar{t}_2) = x_1^{(n-1)}(\bar{t}_2) - x_2^{(n-1)}(\bar{t}_2) = 0, \quad Z^{(n)}(\bar{t}_2) \ge 0.$$

In view of (10), we obtain

$$g(\bar{t}_1, x_1(\bar{t}_1 - \tau(\bar{t}_1))) - g(\bar{t}_1, x_2(\bar{t}_1 - \tau(\bar{t}_1))) \ge 0$$

and

$$g(\bar{t}_2, x_1(\bar{t}_2 - \tau(\bar{t}_2))) - g(\bar{t}_2, x_2(\bar{t}_2 - \tau(\bar{t}_2))) \le 0.$$

Since $g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t)))$ is a continuous function on $\mathbb{R}$, it follows that there exists a constant $\bar{\xi} \in \mathbb{R}$ such that

$$g(\bar{\xi}, x_1(\bar{\xi} - \tau(\bar{\xi}))) - g(\bar{\xi}, x_2(\bar{\xi} - \tau(\bar{\xi}))) = 0. \tag{11}$$

Let $\bar{\xi} - \tau(\bar{\xi}) = nT + \widetilde{\gamma}$, where $\widetilde{\gamma} \in [0, \ T]$ and n is an integer. Then, (11), together with (A_3) (or (A_4)), implies that

$$Z(\widetilde{\gamma}) = x_1(\widetilde{\gamma}) - x_2(\widetilde{\gamma}) = x_1(\bar{\xi} - \tau(\bar{\xi})) - x_2(\bar{\xi} - \tau(\bar{\xi})) = 0. \tag{12}$$

Hence,

$$|Z(t)| = |Z(\widetilde{\gamma}) + \int_{\widetilde{\gamma}}^{t} Z'(s)ds| \le \int_0^T |Z'(s)|ds, \ t \in [0, \ T],$$

and

$$|Z|_\infty \le \sqrt{T}|Z'|_2. \tag{13}$$

Now suppose that (A_3) (or (A_4)) holds, we shall consider two cases as follows.

Case (i). If (A_3) holds, multiplying $Z^{(n)}(t)$ and (10) and then integrating it from 0 to T, we have

$$\begin{aligned}
|Z^{(n)}|_2^2 =& \int_0^T |Z^{(n)}(t)|^2 dt \\
=& -\int_0^T (f(x_1^{(n-1)}(t)) - f(x_2^{(n-1)}(t)))Z^{(n)}(t)dt - \int_0^T (g(t, x_1(t-\tau(t))) \\
&- g(t, x_2(t-\tau(t))))Z^{(n)}(t)dt \\
\le& C_1 \int_0^T |x_1^{(n-1)}(t) - x_2^{(n-1)}(t)||Z^{(n)}(t)|dt \\
&+ b\int_0^T |x_1(t-\tau(t)) - x_2(t-\tau(t))|\,|Z^{(n)}(t)|dt.
\end{aligned} \tag{14}$$

From (6), (13) and Schwarz inequality, (14) implies that

$$\begin{aligned}|Z^{(n)}|_2^2 \leq & C_1|Z^{(n-1)}|_2|Z^{(n)}|_2 + b|Z|_\infty\sqrt{T}|Z^{(n)}|_2\\ \leq & C_1\frac{T}{2\pi}|Z^{(n)}|_2^2 + b\sqrt{T}|Z'|_2\sqrt{T}|Z^{(n)}|_2\\ \leq & \left(C_1\frac{T}{2\pi} + b\frac{T^n}{(2\pi)^{n-1}}\right)|Z^{(n)}|_2^2. \end{aligned} \tag{15}$$

Since $Z(t), Z'(t), \ldots, Z^{(n)}(t)$ are T-periodic and continuous functions, in view of (A_3), (12) and (15), we have

$$Z(t) \equiv Z'(t) \equiv \ldots \equiv Z^{(n)}(t) \equiv 0, \quad \text{for all} \quad t \in \mathbb{R}.$$

Thus, $x_1(t) \equiv x_2(t)$, for all $t \in \mathbb{R}$. Therefore, Equation (1) has at most one T-periodic solution.

Case (ii). If (A_4) holds, multiplying $Z^{(n-1)}(t)$ and (10) and then integrating it from 0 to T, together with (13), we obtain

$$\begin{aligned}C_2|Z^{(n-1)}|_2^2 = & \int_0^T C_2|x_1^{(n-1)}(t) - x_2^{(n-1)}(t)|^2 dt\\ \leq & \int_0^T (f(x_1^{(n-1)}(t) - f(x_2^{(n-1)}(t)))(x_1^{(n-1)}(t) - x_2^{(n-1)}(t))dt\\ = & -\int_0^T (g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t))))Z^{(n-1)}(t)dt\\ \leq & b\int_0^T |x_1(t-\tau(t)) - x_2(t-\tau(t))||Z^{(n-1)}(t)|dt\\ \leq & b|Z|_\infty\sqrt{T}|Z^{(n-1)}|_2\\ \leq & b\frac{T^{n-1}}{(2\pi)^{n-2}}|Z^{(n-1)}|_2^2. \end{aligned} \tag{16}$$

From (12) and (A_4), (16) implies that

$$Z(t) \equiv Z'(t) \equiv \ldots \equiv Z^{(n-1)}(t) \equiv 0, \quad \text{for all} \quad t \in \mathbb{R}.$$

Hence, $x_1(t) \equiv x_2(t)$, for all $t \in \mathbb{R}$. Therefore, Equation (1) has at most one T-periodic solution. The proof of Lemma 2.4 is now complete. ∎

3. Main Results

Theorem 3.1. *Let (A_1) (or (A_2)) hold. Assume that either the condition (A_3) or the condition (A_4) is satisfied. Then Equation (1) has a unique T-periodic solution.*

Proof. By Lemma 2.4, together with (A_3) and (A_4), it is easy to see that Equation (1) has at most one T-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that Equation (1) has at least one T-periodic solution. To do this, we shall apply Lemma 2.1. Firstly, we will claim that the set of all possible T-periodic solutions of Equation (5) are bounded. In view of (A_3) and (A_4), we consider two cases as follows.

Case (1). Let (A_3) hold and $x(t)$ be a T-periodic solution of Equation (5). Multiplying $x^{(n)}(t)$ and Equation (5) and then integrating it from 0 to T, in view of (6), (7), (A_3) and the inequality of Schwarz, we have

$$\begin{aligned}
|x^{(n)}|_2^2 =& \int_0^T |x^{(n)}(t)|^2 dt \\
=& -\lambda \int_0^T f(x^{(n-1)}(t))x^{(n)}(t)dt - \lambda \int_0^T g(t, x(t-\tau(t)))x^{(n)}(t)dt \\
&+ \lambda \int_0^T p(t)x^{(n)}(t)dt \\
=& -\lambda \int_0^T g(t, x(t-\tau(t)))x^{(n)}(t)dt + \lambda \int_0^T p(t)x^{(n)}(t)dt \\
\leq& \int_0^T |g(t, x(t-\tau(t)))| \cdot |x^{(n)}(t)|dt + \int_0^T |p(t)| \cdot |x^{(n)}(t)|dt \\
\leq& \int_0^T [|g(t, x(t-\tau(t))) - g(t,0)| + |g(t,0)|] \cdot |x^{(n)}(t)|dt \\
&+ \int_0^T |p(t)| \cdot |x^{(n)}(t)|dt \\
\leq& b \int_0^T |x(t-\tau(t))| \cdot |x^{(n)}(t)|dt + \int_0^T |g(t,0)| \cdot |x^{(n)}(t)|dt \\
&+ \int_0^T |p(t)| \cdot |x^{(n)}(t)|dt \\
\leq& b|x|_\infty \sqrt{T}|x^{(n)}|_2 + [\max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty] \sqrt{T}|x^{(n)}|_2 \\
\leq& b\sqrt{T}|x'|_2\sqrt{T}|x^{(n)}|_2 + [bd + \max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty] \\
&\times \sqrt{T}|x^{(n)}|_2 \\
\leq& b\frac{T^n}{(2\pi)^{n-1}}|x^{(n)}|_2^2 + [bd + \max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty] \\
&\times \sqrt{T}|x^{(n)}|_2,
\end{aligned} \tag{17}$$

which, together with (A_3), implies that there exists a positive constant D_1 such that

$$|x^{(j)}|_2 \leq \left(\frac{T}{2\pi}\right)^{n-j} |x^{(n)}|_2 < D_1, \ j = 1, 2, \ldots, n. \tag{18}$$

Since $x^{(j)}(0) = x^{(j)}(T)(j = 0, 1, 2, \ldots, n-1)$, it follows that there exists a constant $\zeta_j \in [0, T]$ such that

$$x^{(j+1)}(\zeta_j) = 0$$

and

$$|x^{(j+1)}(t)| = |x^{(j+1)}(\zeta_j) + \int_{\zeta_j}^{t} x^{(j+2)}(s)ds| \leq \int_0^T |x^{(j+2)}(t)|dt \leq \sqrt{T}|x^{(j+2)}|_2, \tag{19}$$

where $j = 0, 1, 2, \ldots, n-2, \quad t \in [0,\ T]$.

Together with (7) and (18), (19) implies that there exist a positive constant D_2 such that

$$|x^{(j)}|_\infty \leq \sqrt{T}|x^{(j+1)}|_2 + d \leq D_2,\ j = 0, 1, 2, \ldots, n-1. \tag{20}$$

which implies that, for all possible T-periodic solutions $x(t)$ of (5), there exists a constant M_1 such that

$$||x|| = \sum_{j=0}^{n-1} |x^{(j)}|_\infty < M_1, \tag{21}$$

with $M_1 > 0$ independent of λ.

Case (2). Let (A_4) hold and $x(t)$ be a T-periodic solution of Equation (5). Multiplying Equation (5) by $x^{(n-1)}(t)$ and then integrating it from 0 to T, by (A_4), (7) and the inequality of Schwarz, we have

$$\begin{aligned}
C_2|x^{(n-1)}|_2^2 &= \int_0^T C_2 x^{(n-1)}(t)x^{(n-1)}(t)dt \\
&\leq \int_0^T f(x^{(n-1)}(t))x^{(n-1)}(t)dt \\
&= -\int_0^T g(t, x(t-\tau(t)))x^{(n-1)}(t)dt + \int_0^T p(t)x^{(n-1)}(t)dt \\
&\leq \int_0^T [|g(t, x(t-\tau(t))) - g(t,0)| + |g(t,0)|] \cdot |x^{(n-1)}(t)|dt \\
&\quad + \int_0^T |p(t)| \cdot |x^{(n-1)}(t)|dt \\
&\leq b\int_0^T |x(t-\tau(t))| \cdot |x^{(n-1)}(t)|dt + \int_0^T |g(t,0)| \cdot |x^{(n-1)}(t)|dt \\
&\quad + \int_0^T |p(t)| \cdot |x^{(n-1)}(t)|dt \\
&\leq b|x|_\infty \sqrt{T}|x^{(n-1)}|_2 + [\max\{|g(t,0)| : 0 \leq t \leq T\} + |p|_\infty] \\
&\quad \times \sqrt{T}|x^{(n-1)}|_2
\end{aligned}$$

$$\le bT|x'|_2|x^{(n-1)}|_2 + [bd + \max\{|g(t,0)| : 0 \le t \le T\} + |p|_\infty]$$
$$\times\sqrt{T}|x^{(n-1)}|_2$$
$$\le b\frac{T^{n-1}}{(2\pi)^{n-2}})|x^{(n-1)}|_2^2 + [bd + \max\{|g(t,0)| : 0 \le t \le T\} + |p|_\infty]$$
$$\times\sqrt{T}|x^{(n-1)}|_2. \tag{22}$$

This implies that there exists a constant $\overline{D_2} > 0$ such that

$$|x^{(j)}|_\infty \le \sqrt{T}|x^{(j+1)}|_2 + d \le \overline{D_2},\ j = 0, 1, 2, \ldots, n-2. \tag{23}$$

Multiplying Equation (5) by $x^{(n)}(t)$ and then integrating it from 0 to T, by (A_4), (7), (17) and the inequality of Schwarz, therefore, from (23), we obtain

$$\begin{aligned}
|x^{(n)}|_2^2 &= \int_0^T |x^{(n)}(t)|^2 dt \\
&\le \int_0^T [|g(t, x(t-\tau(t))) - g(t,0)| + |g(t,0)|] \cdot |x^{(n)}(t)| dt \\
&\quad + \int_0^T |p(t)| \cdot |x^{(n)}(t)| dt \\
&\le b\int_0^T |x(t-\tau(t))| \cdot |x^{(n)}(t)| dt + \int_0^T |g(t,0)| \cdot |x^{(n)}(t)| dt \\
&\quad + \int_0^T |p(t)| \cdot |x^{(n)}(t)| dt \\
&\le b\sqrt{T}|x'|_2\sqrt{T}|x^{(n)}|_2 + [bd + \max\{|g(t,0)| : 0 \le t \le T\} + |p|_\infty] \\
&\quad \times\sqrt{T}|x^{(n)}|_2 \\
&\le bT\overline{D_2}|x^{(n)}|_2 + [bd + \max\{|g(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}|x^{(n)}|_2.
\end{aligned}$$

It follows from (19) that there exists a positive constant $\overline{D_1}$

$$\left|x^{(n-1)}(t)\right| \le \sqrt{T}\left|x^{(n)}\right|_2 \le \overline{D_1}. \tag{24}$$

Therefore, in view of (23) and (24), for all possible T-periodic solutions $x(t)$ of (5), there exists a constant $\widetilde{M_1}$ such that

$$||x|| = \sum_{j=0}^{n-1}\left|x^{(j)}\right|_\infty < \widetilde{M_1}. \tag{25}$$

If $x \in \Omega_1 = \{x | x \in \mathrm{Ker}L \cap X,\ \text{and}\ \ Nx \in \mathrm{Im}L\}$, then there exists a constant M_2 such that

$$x(t) \equiv M_2, \quad \text{and} \quad \int_0^T [g(t,\ M_2) - p(t)] dt = 0. \tag{26}$$

Thus,

$$|x(t)| \equiv |M_2| < d, \quad \text{for all} \quad x(t) \in \Omega_1. \tag{27}$$

Let $M = M_1 + \widetilde{M_1} + d + 1$. Set

$$\Omega = \left\{ x | x \in X, \; ||x|| = \sum_{j=0}^{n-1} |x^{(j)}|_\infty < M \right\}.$$

It is easy to see from (3) and (4) that N is L-compact on $\overline{\Omega}$. We have from (26), (27) and the fact $M > \max\{M_1 + \widetilde{M_1}, \; d\}$ that the conditions (a) and (b) in Lemma 2.1 hold.

Furthermore, define continuous functions $H_1(x,\mu)$ and $H_2(x,\mu)$ by setting

$$H_1(x,\mu) = (1-\mu)x - \mu \cdot \frac{1}{T}\int_0^T [g(t,x) - p(t)]dt; \; \mu \in [0,1],$$

$$H_2(x,\mu) = -(1-\mu)x - \mu \cdot \frac{1}{T}\int_0^T [g(t,x) - p(t)]dt; \; \mu \in [0,1].$$

If (A_1) holds, then

$$xH_1(x,\mu) \neq 0 \text{ for all } x \in \partial\Omega \cap \mathrm{Ker}L.$$

Hence, using the homotopy invariance theorem, we have

$$\begin{aligned} &\deg\{QN, \; \Omega \cap \mathrm{Ker}L, 0\} \\ &= \deg\left\{ -\frac{1}{T}\int_0^T [g(t,x) - p(t)]dt, \; \Omega \cap \mathrm{Ker}L, 0 \right\} \\ &= \deg\{x, \; \Omega \cap \mathrm{Ker}L, 0\} \neq 0. \end{aligned}$$

If (A_2) holds, then

$$xH_2(x,\mu) \neq 0 \text{ for all } x \in \partial\Omega \cap \mathrm{Ker}L.$$

Hence, using the homotopy invariance theorem, we obtain

$$\begin{aligned} &\deg\{QN, \; \Omega \cap \mathrm{Ker}L, 0\} \\ &= \deg\left\{ -\frac{1}{T}\int_0^T [g(t,x) - p(t)]dt, \; \Omega \cap \mathrm{Ker}L, 0 \right\} \\ &= \deg\{-x, \; \Omega \cap \mathrm{Ker}L, 0\} \neq 0. \end{aligned}$$

From the discussions above and Lemma 2.1, Theorem 3.1 is proved. ■

4. Example and Remark

Example 4.1. Let $g(t,x) = \frac{1}{6\pi}x$, for all $t \in \mathbb{R}$, $x > 0$, and $g(t,x) = \frac{1}{12\pi}\arctan x$, for all $t \in \mathbb{R}$, $x \le 0$. Then the Rayleigh equation

$$x''(t) + \frac{1}{8}x'(t) + \frac{1}{8}\sin x'(t) + g(t, x(t - \sin^2 t) = \frac{1}{40}e^{\sin t} \qquad (28)$$

has a unique 2π-periodic solution.

Indeed by (28), we have $b = \frac{1}{6\pi}, C_1 = \frac{1}{4}$, $\tau(t) = \sin^2 t$ and $p(t) = \frac{1}{40}e^{\sin t}$. It is obvious that the assumptions (A_1) and (A_3) hold. Hence, by Theorem 3.1, Equation (28) has a unique 2π-periodic solution.

Remark 4.2. Equation (28) is a very simple version of Rayleigh equation. Since $f(x) = \frac{1}{8}x + \frac{1}{8}\sin x$ and $\tau(t) = \sin^2 t$, all the results in [1]-[4], [6]-[18] and the references therein cannot be applicable to Equation (28) to obtain the existence and uniqueness of 2π-periodic solutions. This implies that the results of this paper are essentially new.

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