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Some Uniqueness Results Related to Certain Non-linear Differential Polynomials Sharing the Same 1-points

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Abstract. The purpose of the paper is to study the uniqueness of meromorphic functions when certain non-linear differential polynomials share the same 1-points. Though the main concern of the paper is to improve a result of Fang [5] but as a consequence of the main result we improve and supplement some former results of Lahiri-Sahoo [15], Lin-Yi [17], Fang-Fang [6] and a recent result of the first author [3].

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1. Introduction, Definitions, and Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities). Let m be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{m)}(a; f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_{\infty)}(a; f)$=$E_{\infty)}(a; g)$ we say that f, g share the value a CM.

We will use the standard notations of value distribution theory:

$$T(r,f),\ \ m(r,f),\ \ N(r,\infty;f),\ \ \overline{N}(r,\infty;f),\dots$$

(see [8]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\to\infty$, outside of a possible exceptional set of finite linear measure. For any constant a, we define

$$\Theta(a;f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}.$$

In 1999 Lahiri [9] studied the problem of uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points. In the same paper [9] regarding the nonlinear differential polynomials Lahiri asked the following question.

What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 *CM?*

Since then the progress to investigate the uniqueness of meromorphic functions which are the generating functions of different types of non-linear differential polynomials is remarkable and continuous efforts are being put in to relax the hypothesis of the results. (cf. [1]-[7], [13]-[18]).

In 2001 Fang and Hong [7] proved the following result.

Theorem A. *Let f and g be two transcendental entire functions and $n(\geq 11)$ be an integer. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share* 1 *CM, then $f\equiv g$.*

In 2002 Fang and Fang [6] improved and supplemented the above theorem by proving the following theorems.

Theorem B. *Let f and g be two nonconstant entire functions and $m(\geq 3)$, $n(\geq 8)$ be two positive integers. If $E_{m)}(1;f^n(f-1)f^{'})=E_{m)}(1;g^n(g-1)g^{'})$, then $f\equiv g$.*

Theorem C. *Let f and g be two nonconstant entire functions and $n(\geq 9)$ be an integer. If $E_{2)}(1;f^n(f-1)f^{'})=E_{2)}(1;g^n(g-1)g^{'})$, then $f\equiv g$.*

Theorem D. *Let f and g be two nonconstant entire functions and $n(\geq 14)$ be an integer. If $E_{1)}(1;f^n(f-1)f^{'})=E_{1)}(1;g^n(g-1)g^{'})$, then $f\equiv g$.*

In 2004 Lin and Yi [17] further improved Theorem A as follows.

Theorem E. *Let f and g be two transcendental entire functions and $n(\geq 7)$ be an integer. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share* 1 *CM, then $f\equiv g$.*

The following example shows that the above theorems are not valid when f and g are two meromorphic functions.

Example 1.1.

$$f(z) = \frac{(n+2)}{(n+1)} \frac{e^z + \ldots + e^{(n+1)z}}{1 + e^z + \ldots + e^{(n+1)z}}$$

and

$$g(z) = \frac{(n+2)}{(n+1)} \frac{1 + e^z + \ldots + e^{nz}}{1 + e^z + \ldots + e^{(n+1)z}}.$$

Clearly $f(z) = e^z g(z)$. Also $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$ share 1 CM but $f \not\equiv g$.

We note that in the above example $\Theta(\infty; f) = \Theta(\infty; g) = 0$. So to replace entire functions by meromorphic functions in the above mentioned theorems definitely some extra conditions are required.

For meromorphic functions Lin and Yi [17] proved the following result.

Theorem F. *Let f and g be two nonconstant meromorphic functions such that $\Theta(\infty; f) > \frac{2}{n+1}$ and $n(\geq 11)$ be an integer. If $f^n(f-1)f^{'}$ and $g^n(g-1)g^{'}$, share the value* 1*-CM, then $f \equiv g$.*

In 2005 Lahiri and Sahoo [15] proved the following theorem for the uniqueness of non-linear differential polynomials.

Theorem G. *Let f and g be two transcendental meromorphic functions such that $\Theta(\infty; f) > 0$, $\Theta(\infty; g) > 0$, $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1}$ and $n(\geq 11)$ be an integer. If $E_{3)}(1; f^n(f-1)f^{'}) = E_{3)}(1; g^n(g-1)g^{'})$, then $f \equiv g$.*

Lahiri and Sahoo [15] also gave the following example to show that the condition $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1}$ is sharp in Theorem G.

Example 1.2. Let

$$f = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \; g = h\frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})} \text{ and } h = \frac{\alpha^2(e^z-1)}{e^z-\alpha},$$

where $\alpha = \exp(\frac{2\pi i}{n+2})$ and n is a positive integer.

Clearly $T(r,f) = (n+1)T(r,h) + O(1)$ and $T(r,g) = (n+1)T(r,h) + O(1)$. Further we see that $h \neq \alpha, \alpha^2$ and a root of $h = 1$ is not a pole of f and g. Hence $\Theta(\infty; f) = \Theta(\infty; g) = \frac{2}{n+1}$. Also $f^{n+1}(\frac{f}{n+1} - \frac{1}{n+1}) \equiv g^{n+1}(\frac{g}{n+1} - \frac{1}{n+1})$ and $f^n(f-1)f^{'} \equiv g^n(g-1)g^{'}$ but $f \not\equiv g$.

In the direction of Theorem C Lahiri and Mandal [13] proved the following theorem for meromorphic functions.

Theorem H. *Let f and g be two transcendental meromorphic functions such that $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1}$ and $n(\geq 17)$ be an integer. If $E_{2)}(1; f^n(f-1)f^{'}) = E_{2)}(1; g^n(g-1)g^{'})$, then $f \equiv g$.*

Recently the present first author [3] has improved Theorem H by significantly reducing the lower bound of n. In [3] Banerjee proved the following theorem.

Theorem I. *Let f and g be two transcendental meromorphic functions such that $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n+1}$ and $n(\geq 14)$ be an integer. If $E_{2)}(1;f^{n}(f-1)f^{'})=E_{2)}(1;g^{n}(g-1)g^{'})$, then $f\equiv g$.*

In 2002 Fang [5] first considered the uniqueness of entire functions corresponding to more generalized non-linear differential polynomials and proved the following result.

Theorem J. *Let f and g be two nonconstant entire functions and let n, k be two positive integers with $n\geq 2k+8$. If $[f^{n}(f-1)]^{(k)}$ and $[g^{n}(g-1)]^{(k)}$ share 1 CM, then $f\equiv g$.*

Recently Bhoosnurmath and Dyavanal [4] also considered the uniqueness of meromorphic functions corresponding to the k-th derivative of a linear polynomial expression.

In the paper we will prove two theorems, the second of which will not only improve Theorem J by reducing the lower bound of n and at the same time relaxing the nature of sharing the value 1 but also improve Theorem D and Theorem C. Our first theorem will improve and supplement Theorem G and Theorem I. The following theorems are the main results of the paper.

Theorem 1.3. *Let f and g be two transcendental meromorphic functions and $n(\geq 1)$, $k(\geq 1)$, $m(\geq 1)$ be three integers such that $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n}$. Suppose for two finite nonzero constants a and b $E_{m)}\left(1;[f^{n}(af+b)]^{(k)}\right)=E_{m)}\left(1;[g^{n}(ag+b)]^{(k)}\right)$. If $m\geq 3$, $\Theta(\infty;f)>0$, $\Theta(\infty;g)>0$ and $n\geq 3k+9$ or if $m=2$ and $n\geq 4k+11$ or if $m=1$ and $n\geq 7k+15$, then either $f\equiv g$ or $[f^{n}(af+b)]^{(k)}[g^{n}(ag+b)]^{(k)}\equiv 1$. When $k=1$ the possibility $[f^{n}(af+b)]^{(k)}[g^{n}(ag+b)]^{(k)}\equiv 1$ does not occur.*

Putting $n=l+1$, $a=\frac{1}{l+2}$, $b=-\frac{1}{l+1}$ and $k=1$ in the above theorem we can immediately deduce the following corollary.

Corollary 1.4. *Let f and g be two transcendental meromorphic functions and $l(\geq 1)$, $m(\geq 1)$ be two integers such that $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{l+1}$. Suppose for two finite non-zero constants a and b $E_{m)}\left(1;f^{l}(f-1)f^{'}\right)=E_{m)}\left(1;g^{l}(g-1)g^{'}\right)$. If $m\geq 3$, $\Theta(\infty;f)>0$, $\Theta(\infty;g)>0$ and $l\geq 11$ or if $m=2$ and $l\geq 14$ or if $m=1$ and $l\geq 21$, then $f\equiv g$.*

Remark 1.5. Since Theorems G and I can be obtained as special cases of Theorem 1.3, clearly Theorem 1.3 improves them.

Remark 1.6. In Theorem 1.3 for $m=3$ if we take $n\geq 3k+10$ then the conditions $\Theta(\infty;f)>0$ and $\Theta(\infty;g)>0$ can be removed.

Theorem 1.7. *Let f and g be two non-constant entire functions and $n(\geq 1)$, $k(\geq 1)$, $m(\geq 1)$ be three integers. Suppose for two non-zero constants a and b $E_{m)}\left(1;[f^{n}(af+b)]^{(k)}\right)=E_{m)}\left(1;[g^{n}(ag+b)]^{(k)}\right)$. If $m\geq 3$ and $n\geq 2k+6$ or if $m=2$ and $n\geq\frac{5k}{2}+7$ or if $m=1$ and $n\geq 4k+10$, then $f\equiv g$.*

Putting $n = l+1$, $a = \frac{1}{l+2}$, $b = -\frac{1}{l+1}$ and $k = 1$ in the above theorem we can immediately deduce the following corollary.

Corollary 1.8. *Let f and g be two nonconstant entire functions and $l(\geq 1)$, $m(\geq 1)$ be two integers. Suppose for two non-zero constants a and b*

$$E_{m)}\left(1; f^l(f-1)f^{'}\right) = E_{m)}\left(1; g^l(g-1)g^{'}\right).$$

If $m \geq 3$ and $l \geq 7$ or if $m = 2$ and $l \geq 9$ or if $m = 1$ and $l \geq 13$, then $f \equiv g$.

Remark 1.9. Clearly Corollary 1.8 improve and supplement Theorem C and Theorem D.

Though we use the standard notations and definitions of the value distribution theory available in [8], we explain some definitions and notations which are used in the paper.

Definition 1.10. ([15]) For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)$ $(N(r, a; f \mid\geq m))$ the counting function of those a points of f whose multiplicities are not greater (less) than m where each a point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid\leq m)$ $(\overline{N}(r, a; f \mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Definition 1.11. Let m be a positive integer and for $a \in \mathbb{C}$, $E_{m)}(a; f) = E_{m)}(a; g)$. Let z_0 be a zero of $f(z) - a$ of multiplicity p and a zero of $g(z) - a$ of multiplicity q. We denote by $\overline{N}_L(r, a; f)$ $(\overline{N}_L(r, a; g))$ the reduced counting function of those a-points of f and g for which $p > q \geq m+1 (q > p \geq m+1)$, by $\overline{N}_E^{(m+1}(r, a; f)$ the reduced counting function of those a-points of f and g for which $p = q \geq m+1$, by $\overline{N}_{f>m+1}(r, 1; g)$ the reduced counting function of f and g for which $p \geq m+2$ and $q = m+1$. Also by $\overline{N}_{f\geq m+1}(r, a; f \mid g \neq a)$ $(\overline{N}_{g\geq m+1}(r, a; g \mid f \neq a))$ we denote the reduced counting functions of those a-points of f and g for which $p \geq m+1$ and $q = 0$ $(q \geq m+1$ and $p = 0)$.

Definition 1.12. We denote by $\overline{N}(r, a; f \mid= k)$ the reduced counting function of those a-points of f whose multiplicities are exactly k where $k \geq 2$ is an integer. For $k = 1$ we refer Definition 1.10.

Definition 1.13. ([11]) Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.14. ([11]) Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid \ g \neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

Definition 1.15. ([11], cf.[20]) For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots + \overline{N}(r, a; f \mid\geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.16. Let $a, b \in \mathbb{C} \cup \{\infty\}$. Let p be a positive integer. We denote by $\overline{N}(r, a; f \mid \geq p \mid \ g = b)$ ($\overline{N}(r, a; f \mid \geq p \mid \ g \neq b)$) the reduced counting function of those a-points of f with multiplicities $\geq p$, which are the b-points (not the b-points) of g.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two nonconstant meromorphic functions. Henceforth we shall denote by H the following function

$$H = \left(\frac{F^{(k+2)}}{F^{(k+1)}} - \frac{2F^{(k+1)}}{F^{(k)} - 1} \right) - \left(\frac{G^{(k+2)}}{G^{(k+1)}} - \frac{2G^{(k+1)}}{G^{(k)} - 1} \right). \tag{1}$$

Lemma 2.1. ([8]) *Let f be a non-constant meromorphic function, k a positive integer and let c be a non-zero finite complex number. Then*

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, \infty; f) + N(r, 0; f) + N\left(r, c; f^{(k)}\right) - N\left(r, 0; f^{(k+1)}\right) + S(r, f) \\ &\leq \overline{N}(r, \infty; f) + N_{k+1}(r, 0; f) + \overline{N}\left(r, c; f^{(k)}\right) - N_0\left(r, 0; f^{(k+1)}\right) \\ &\quad + S(r, f), \end{aligned}$$

where $N_0\left(r, 0; f^{(k+1)}\right)$ is the counting function of the zeros of $f^{(k+1)}$ which are not the zeros of $f(f^{(k)} - c)$.

Lemma 2.2. ([12]) *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid< k) + k\overline{N}(r, 0; f \mid\geq k) + S(r, f).$$

Lemma 2.3. ([21]) *Let f be a nonconstant meromorphic function and p, k be positive integers, then*

$$N_p(r, 0; f^{(k)}) \leq N_{p+k}(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.4. *Let $E_{m)}(1; f) = E_{m)}(1; g)$ and $2 \leq m < \infty$. Then*

$$\begin{aligned}&\overline{N}(r,1;f\mid=2)+2\overline{N}(r,1;f\mid=3)+\ldots+(m-1)\overline{N}(r,1;f\mid=m)\\&\quad+m\overline{N}_{E}^{(m+1}(r,1;f)+m\overline{N}_{L}(r,1;f)+(m+1)\overline{N}_{L}(r,1;g)\\&\quad+m\overline{N}_{g\geq m+1}(r,1;g\mid\ f\neq 1)\\&\leq\ N(r,1;g)-\overline{N}(r,1;g).\end{aligned}$$

Proof. Since $E_{m)}(1;f)=E_{m)}(1;g)$, we note that common zeros of $f-1$ and $g-1$ up to multiplicity m are same. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. If $q=m+1$ the possible values of p are as follows (i) $p=m+1$, (ii) $p\geq m+2$, (iii) $p=0$. Similarly when $q=m+2$ the possible values of p are (i) $p=m+1$, (ii) $p=m+2$, (iii) $p\geq m+3$, (iv) $p=0$. If $q\geq m+3$ we can similarly find the possible values of p. Now the lemma follows from the above explanation. ■

Lemma 2.5. *Let* $E_{1)}(1;f)=E_{1)}(1;g)$*. Then*

$$\begin{aligned}&2\,\overline{N}_{L}(r,1;f)+2\,\overline{N}_{L}(r,1;g)+\overline{N}_{E}^{(2}(r,1;f)+\overline{N}_{g\geq 2}(r,1;g\mid f\neq 1)-\overline{N}_{f>2}(r,1;g)\\&\leq\ N(r,1;g)-\overline{N}(r,1;g).\end{aligned}$$

Proof. Since $E_{1)}(1;f)=E_{1)}(1;g)$ the simple 1-points of f and g are same. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. If $q=2$ the possible values of p are as follows (i) $p=2$, (ii) $p\geq 3$, (iii) $p=0$. Similarly when $q=3$ the possible values of p are (i) $p=2$, (ii) $p=3$, (iii) $p\geq 4$, (iv) $p=0$. If $q\geq 4$ we can similarly find the possible values of p. Now the lemma follows from the above discussion. ■

Lemma 2.6. *Let* $E_{2)}(1;f)=E_{2)}(1;g)$*. Then*

$$\overline{N}_{f\geq 3}(r,1;f\mid\ g\neq 1)\leq\frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}N_{\oplus}(r,0;f^{'})+S(r,f),$$

where $N_{\oplus}(r,0;f^{'})$ *is the counting function of those zeros of* $f^{'}$ *which are not the zeros of* $f(f-1)$*, each point is counted according to its multiplicity.*

Proof. Using Lemma 2.2 we get

$$\begin{aligned}&\overline{N}_{f\geq 3}(r,1;f\mid\ g\neq 1)\\&\leq\ \overline{N}(r,1;f\mid\geq 3)\\&\leq\ \frac{1}{2}N(r,0;f^{'}\mid f=1)\\&\leq\ \frac{1}{2}N(r,0;f^{'}\mid\ f\neq 0)-\frac{1}{2}N_{\oplus}(r,0;f^{'})\\&\leq\ \frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}N_{\oplus}(r,0;f^{'})+S(r,f).\end{aligned}$$

■

Lemma 2.7. *Let* $E_{1)}(1;f) = E_{1)}(1;g)$*. Then*

$$\begin{aligned}&\overline{N}_{f>2}(r,1;g) + \overline{N}_{f\geq 2}(r,1;f \mid g \neq 1)\\ &\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_{\oplus}(r,0;f^{'}) + S(r,f).\end{aligned}$$

Proof. We note that a 1-point of f with multiplicity 2 is counted atmost once in the counting function $\overline{N}_{f\geq 2}(r,1;f \mid g \neq 1)$. Also since a 1- point of f with multiplicity ≥ 3 may or may not be a 1 point of g, those 1-points of f are counted only once, either in $\overline{N}_{f>2}(r,1;g)$ or $\overline{N}_{f\geq 2}(r,1;f \mid g \neq 1)$. So using Lemma 2.2 we get

$$\begin{aligned}\overline{N}_{f>2}(r,1;g) + \overline{N}_{f\geq 2}(r,1;f \mid g \neq 1) &\leq \overline{N}(r,1;f \mid\geq 2)\\ &\leq N(r,0;f^{'} \mid f = 1)\\ &\leq \overline{N}(r,0;f^{'} \mid f \neq 0) - N_{\oplus}(r,0;f^{'})\\ &\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f)\\ &\quad - N_{\oplus}(r,0;f^{'}) + S(r,f).\end{aligned}$$

■

Lemma 2.8. *Let* $E_{1)}(1;f) = E_{1)}(1;g)$*. Then*

$$\overline{N}_{f\geq 2}(r,1;f \mid g \neq 1) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_{\oplus}(r,0;f^{'}) + S(r,f).$$

Proof. Using Lemma 2.2 we get

$$\begin{aligned}\overline{N}_{f\geq 2}(r,1;f \mid g \neq 1) &\leq \overline{N}(r,1;f \mid\geq 2)\\ &\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_{\oplus}(r,0;f^{'}) + S(r,f).\end{aligned}$$

■

Lemma 2.9. ([19]) *Let f be a nonconstant meromorphic function and* $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$*, where* $a_0, a_1, a_2 \ldots, a_n$ *are constants and* $a_n \neq 0$*. Then* $T(r,P(f)) = nT(r,f) + O(1)$*.*

Lemma 2.10. *Let f and g be two nonconstant meromorphic functions. Then*

$$[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} \not\equiv 1,$$

where n, $k = 1$ are two positive integers and $n(\geq 3k+9)$*.*

Proof. We note that when $k = 1$, according to the statement of the lemma we have to prove

$$f^{n-1}\left[a(n+1)f + nb\right] f^{'} \;\; g^{n-1}\left[a(n+1)g + nb\right] g^{'} \not\equiv 1,$$

which can be proved in the line of the proof Lemma 2.7 in [15]. ■

Lemma 2.11. *Let f and g be two nonconstant entire functions. Then*

$$[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} \not\equiv 1,$$

where a and b are nonzero complex numbers; n, k are two positive integers and $n(>k)$.

Proof. We omit the proof since the proof can be found in the proof of Theorem 2 in [5]. ■

Lemma 2.12. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n},$$

where $n(\geq 3)$ is an integer. Then

$$f^n(af+b) \equiv g^n(ag+b)$$

implies $f \equiv g$, where a, b are non-zero constants.

Proof. We omit the proof since it can be carried out in the line of Lemma 6 in [10]. ■

3. Proofs of the Theorems

Proof of Theorem 1.3. Let $F = f^n(af+b)$ and $G = g^n(ag+b)$. It follows that $E_{m)}\left(1; F^{(k)}\right) = E_{m)}\left(1; G^{(k)}\right)$.
Case 1. Let $H \not\equiv 0$.
From (1) we get

$$\begin{aligned} & N(r,\infty;H) \qquad (2)\\ & \leq \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_L\left(r,1;F^{(k)}\right) \\ & \quad + \overline{N}_L\left(r,1;G^{(k)}\right) + \overline{N}_{F^{(k)}\geq m+1}\left(r,1;F^{(k)} \mid G^{(k)} \neq 1\right) \\ & \quad + \overline{N}_{G^{(k)}\geq m+1}\left(r,1;G^{(k)} \mid F^{(k)} \neq 1\right) + \overline{N}\left(r,0;F^{(k)} \mid \geq 2\right) \\ & \quad + \overline{N}\left(r,0;G^{(k)} \mid \geq 2\right) + \overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right) + \overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right), \end{aligned}$$

where $\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)$ is the reduced counting function of those zeros of $F^{(k+1)}$ which are not the zeros of $F^{(k)}\left(F^{(k)}-1\right)$ and $\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)$ is similarly defined.

Let z_0 be a simple zero of $F^{(k)}-1$. Then z_0 is a simple zero of $G^{(k)}-1$ and a zero of H. So

$$N\left(r,1;F^{(k)} \mid = 1\right) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,F) + S(r,G) \qquad (3)$$

Subcase 1.1. $m \geq 2$
Using Lemma 2.4, (2) and (3) we get

$$\begin{aligned}
&\overline{N}\left(r,1;F^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right) && (4)\\
&\leq N\left(r,1;F^{(k)}\mid=1\right)+\overline{N}\left(r,1;F^{(k)}\mid=2\right)\ldots+\overline{N}\left(r,1;F^{(k)}\mid=m\right)\\
&\quad+\overline{N}_L\left(r,1;F^{(k)}\right)+\overline{N}_L\left(r,1;G^{(k)}\right)+\overline{N}_{F^{(k)}\geq m+1}\left(r,1;F^{(k)}\mid\ G^{(k)}\neq 1\right)\\
&\quad+\overline{N}_E^{(m+1}\left(r,1;G^{(k)}\right)+\overline{N}\left(r,1;G^{(k)}\right)\\
&\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}\left(r.0;F^{(k)}\mid\geq 2\right)+\overline{N}\left(r.0;G^{(k)}\mid\geq 2\right)\\
&\quad+T\left(r,G^{(k)}\right)+2\overline{N}_{F^{(k)}\geq m+1}\left(r,1;F^{(k)}\mid\ G^{(k)}\neq 1\right)\\
&\quad-(m-1)\overline{N}_{G^{(k)}\geq m+1}\left(r,1;G^{(k)}\mid\ F^{(k)}\neq 1\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)\\
&\quad+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)+S(r,F)+S(r,G).
\end{aligned}$$

So in view of (4), from Lemma 2.1 we have

$$\begin{aligned}
&T(r,F)+T(r,G) && (5)\\
&\leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_{k+1}(r,0;F)+N_{k+1}(r,0;G)\\
&\quad+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}\left(r,0;G^{(k)}\mid\geq 2\right)+T(r,G)+k\overline{N}(r,\infty;G)\\
&\quad+2\overline{N}_{F^{(k)}\geq m+1}\left(r,1;F^{(k)}\mid\ G^{(k)}\neq 1\right)\\
&\quad-(m-1)\overline{N}_{G^{(k)}\geq m+1}\left(r,1;G^{(k)}\mid\ F^{(k)}\neq 1\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)\\
&\quad+\overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right)-N_0\left(r,0;F^{(k+1)}\right)-N_0\left(r,0;G^{(k+1)}\right)\\
&\quad+S(r,F)+S(r,G).
\end{aligned}$$

We note that

$$\begin{aligned}
&N_{k+1}(r,0;F)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right) && (6)\\
&\leq N_{k+1}(r,0;F)+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\mid F=0\right)\\
&\quad+\overline{N}\left(r,0;F^{(k)}\mid\geq 2\mid F\neq 0\right)+\overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right)\\
&\leq N_{k+1}(r,0;F)+\overline{N}\left(r,0;F\mid\geq k+2\right)+\overline{N}_0\left(r,0;F^{(k+1)}\right)\\
&\leq N_{k+2}(r,0;F)+\overline{N}_0\left(r,0;F^{(k+1)}\right).
\end{aligned}$$

Clearly a similar expression holds for G also.
Using (6) in (5) we get

$$\begin{aligned} T(r,F) \leq\ & 2\overline{N}(r,\infty;F) + (k+2)\overline{N}(r,\infty;G) + N_{k+2}(r,0;F) \\ & + N_{k+2}(r,0;G) + 2\overline{N}_{F^{(k)}\geq m+1}\left(r,1;F^{(k)} \mid\ G^{(k)} \neq 1\right) \\ & - (m-1)\overline{N}_{G^{(k)}\geq m+1}\left(r,1;G^{(k)} \mid\ F^{(k)} \neq 1\right) + S(r,F) + S(r,G). \end{aligned} \tag{7}$$

In a similar way we can obtain

$$\begin{aligned} T(r,G) \leq\ & (k+2)\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + N_{k+2}(r,0;F) \\ & + N_{k+2}(r,0;G) + 2\overline{N}_{G^{(k)}\geq m+1}\left(r,1;G^{(k)} \mid\ F^{(k)} \neq 1\right) \\ & - (m-1)\overline{N}_{F^{(k)}\geq m+1}\left(r,1;F^{(k)} \mid\ G^{(k)} \neq 1\right) \\ & + S(r,F) + S(r,G). \end{aligned} \tag{8}$$

While $m \geq 3$, in view of Lemma 2.9, adding (7) and (8) we get for $\varepsilon > 0$

$$\begin{aligned} & (n+1)\{T(r,f) + T(r,g)\} \\ & \leq\ (k+4)\overline{N}(r,\infty;f) + 2\{(k+2)\overline{N}(r,0;f) + N_{k+2}(r,0;af+b)\} \\ & \quad + (k+4)\overline{N}(r,\infty;g) + 2\{(k+2)\overline{N}(r,0;g) + N_{k+2}(r,0;ag+b)\} \\ & \quad + S(r,f) + S(r,g) \\ & \leq\ (3k+10-(k+4)\Theta(\infty;f)+\varepsilon)\,T(r,f) + (3k+10-(k+4) \\ & \quad \Theta(\infty;g)+\varepsilon)\,T(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{9}$$

That is

$$\begin{aligned} & (n-3k-9+(k+4)\Theta(\infty;f)-\varepsilon)\,T(r,f) \\ & \quad + (n-3k-9+(k+4)\Theta(\infty;g)-\varepsilon)\,T(r,g) \\ & \leq\ S(r,f) + S(r,g). \end{aligned}$$

Since $n \geq 3k+9$, choosing $0 < \varepsilon < \min\{\Theta(\infty;f);\Theta(\infty;g)\}$, we get a contradiction from above. While $m = 2$, in view of Lemmas 2.3, 2.6 and 2.9, adding (7) and (8) we get

$$\begin{aligned} & (n+1)\,[T(r,f) + T(r,g)] \\ & \leq\ \left(\frac{3k}{2} + \frac{9}{2}\right)\overline{N}(r,\infty;f) + 2\{(k+2)\overline{N}(r,0;f) + N_{k+2}(r,0;af+b)\} \\ & \quad + \frac{1}{2}\left((k+1)\overline{N}(r,0;f) + N_{k+2}(r,0;af+b)\right) + \left(\frac{3k}{2} + \frac{9}{2}\right)\overline{N}(r,\infty;g) \\ & \quad + 2\{(k+2)\overline{N}(r,0;g) + N_{k+2}(r,0;ag+b)\} \\ & \quad + \frac{1}{2}\left((k+1)\overline{N}(r,0;g) + N_{k+2}(r,0;ag+b)\right) + S(r,f) + S(r,g) \\ & \leq\ \left(4k + \frac{23}{2}\right)T(r,f) + \left(4k + \frac{23}{2}\right)T(r,g) + S(r,f) + S(r,g), \end{aligned} \tag{10}$$

which is a contradiction since $n \geq 4k + 11$.
Subcase 1.2. $m = 1$. Using Lemma 2.5, (2) and (3) we get

$$\begin{aligned}
&\overline{N}\left(r,1;F^{(k)}\right) + \overline{N}\left(r,1;G^{(k)}\right) \qquad (11)\\
&\leq N\left(r,1;F^{(k)} \mid= 1\right) + \overline{N}_L\left(r,1;F^{(k)}\right) + \overline{N}_L\left(r,1;G^{(k)}\right)\\
&\quad + \overline{N}_{F^{(k)}\geq 2}\left(r,1;F^{(k)} \mid\; G^{(k)} \neq 1\right) + \overline{N}_E^{(2}\left(r,1;G^{(k)}\right) + \overline{N}\left(r,1;G^{(k)}\right)\\
&\leq \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}\left(r.0;F^{(k)} \mid\geq 2\right) + \overline{N}\left(r.0;G^{(k)} \mid\geq 2\right)\\
&\quad + T\left(r,G^{(k)}\right) + 2\overline{N}_{F^{(k)}\geq 2}\left(r,1;F^{(k)} \mid\; G^{(k)} \neq 1\right) + \overline{N}_{F^{(k)}>2}\left(r,1;G^{(k)}\right)\\
&\quad + \overline{N}_{\otimes}\left(r,0;F^{(k+1)}\right) + \overline{N}_{\otimes}\left(r,0;G^{(k+1)}\right) + S(r,F) + S(r,G).
\end{aligned}$$

So in view of (6) and (11) from Lemmas 2.1, 2.3, 2.7 and 2.8 we have

$$\begin{aligned}
&T(r,F) + T(r,G)\\
&\leq 4\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + N_{k+2}(r,0;F) + N_{k+2}(r,0;G)\\
&\quad + T(r,G) + k\overline{N}(r,\infty;G) + 2\overline{N}\left(r,0;F^{(k)}\right) + S(r,F) + S(r,G)\\
&\leq (2k+4)\overline{N}(r,\infty;F) + (k+2)\overline{N}(r,\infty;G) + N_{k+2}(r,0;F)\\
&\quad + 2N_{k+1}(r,0;F) + N_{k+2}(r,0;G) + T(r,G) + S(r,F) + S(r,G).
\end{aligned}$$

Using Lemma 2.9 we get from above for $\varepsilon > 0$

$$\begin{aligned}
&(n+1)T(r,f) \qquad (12)\\
&\leq (5k + 11 - (2k+4)\Theta(\infty;f) + \varepsilon)\, T(r,f) + (2k + 5 - (k+2)\Theta(\infty;g) + \varepsilon)\\
&\quad T(r,g) + S(r,f) + S(r,g)\\
&\leq (7k + 16 - (k+2)\Theta(\infty;f) - (k+2)\Theta(\infty;g)\\
&\quad - (k+2)\min\{\Theta(\infty;f),\Theta(\infty;g)\} + 2\varepsilon)\, T(r) + S(r,f) + S(r,g).
\end{aligned}$$

In a similar manner we can obtain

$$\begin{aligned}
&(n+1)T(r,g) \qquad (13)\\
&\leq (7k + 16 - (k+2)\Theta(\infty;f) - (k+2)\Theta(\infty;g)\\
&\quad - (k+2)\min\{\Theta(\infty;f),\Theta(\infty;g)\} + 2\varepsilon)\, T(r) + S(r,f) + S(r,g).
\end{aligned}$$

Combining (12) and (13) we get

$$\begin{aligned}
&(n - 7k - 15 + (k+2)\Theta(\infty;f) \qquad (14)\\
&\quad + (k+2)\Theta(\infty;g) + (k+2)\min\{\Theta(\infty;f),\Theta(\infty;g)\} - 2\varepsilon)\, T(r)\\
&\leq S(r).
\end{aligned}$$

Since $n \geq 7k+15$, $\Theta(\infty; f)+\Theta(\infty; g) > \frac{4}{n}$ and $\varepsilon > 0$ are arbitrary, (14) implies a contradiction.

Case 2. Next we suppose that $H \equiv 0$. Then by integration we get from (1)

$$\frac{1}{F^{(k)}-1} \equiv \frac{bG^{(k)}+a-b}{G^{(k)}-1}, \tag{15}$$

where a, b are constants and $a \neq 0$. From (15) it is clear that $F^{(k)}$ and $G^{(k)}$ share 1 CM and hence $E_{3)}\left(1; F^{(k)}\right) = E_{3)}\left(1; G^{(k)}\right)$. So in this case always $n \geq 3k+9$. We now consider the following subcases.

Subcase 2.1. Let $b \neq 0$ and $a \neq b$.

If $b = -1$, then from (15) we have

$$F^{(k)} = \frac{-a}{G^{(k)}-a-1}.$$

Therefore

$$\overline{N}\left(r, a+1; G^{(k)}\right) = \overline{N}\left(r, \infty; F^{(k)}\right) = \overline{N}(r, \infty; f).$$

Since $a \neq b = -1$, from Lemma 2.1 we have

$$\begin{aligned}
&(n+1)T(r,g)+O(1)\\
&= T(r,G)\\
&\leq \overline{N}(r,\infty;G)+N_{k+1}(r,0;G)+\overline{N}(r,a+1;G^{(k)})+S(r,G)\\
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+N_{k+1}(r,0;G)+S(r,G)\\
&\leq T(r,f)+(k+3)T(r,g)+S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$.

So for $r \in I$ we have

$$(n-k-3)T(r,g) \leq S(r,g),$$

which is a contradiction for $n \geq 3k+9$.

If $b \neq -1$, from (15) we obtain that

$$F^{(k)} - \left(1+\frac{1}{b}\right) = \frac{-a}{b^2[G^{(k)}+(a-b)/b]}.$$

Therefore

$$\overline{N}\left(r, (b-a)/b; G^{(k)}\right) = \overline{N}\left(r, \infty; F^{(k)}-(1+1/b)\right) = \overline{N}(r, \infty; f).$$

Using Lemma 2.1 and the same argument as used in the case when $b = -1$ we can get a contradiction.

Subcase 2.2. Let $b \neq 0$ and $a = b$.

If $b = -1$, then from (15) we have

$$F^{(k)}G^{(k)} \equiv 1,$$

that is

$$[f^n(af+b)]^{(k)}[g^n(ag+b)]^{(k)} \equiv 1,$$

which is impossible by Lemma 2.10 when $k=1$.
If $b \neq -1$, from (15) we have

$$\frac{1}{F^{(k)}} = \frac{bG^{(k)}}{(1+b)G^{(k)}-1}.$$

Hence from Lemma 2.3 we have

$$\begin{aligned}
&\overline{N}\left(r, 1/(1+b); G^{(k)}\right) \\
&= \overline{N}\left(r, 0; F^{(k)}\right) \\
&\leq N_{k+1}(r,0;F) + k\overline{N}(r,\infty;f).
\end{aligned}$$

From Lemma 2.1 we have

$$\begin{aligned}
&(n+1)T(r,g) + O(1) \\
&= T(r,G) \\
&\leq \overline{N}(r,\infty;G) + N_{k+1}(r,0;G) + \overline{N}\left(r, \frac{1}{b+1}; G^{(k)}\right) + S(r,G) \\
&\leq k\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + N_{k+1}(r,0;F) + N_{k+1}(r,0;G) + S(r,G) \\
&\leq (2k+2)T(r,f) + (k+3)T(r,g) + S(r,g),
\end{aligned}$$

which is a contradiction for $n \geq 3k+9$ for $r \in I$.
Subcase 2.3. Let $b=0$. From (15) we obtain

$$F = \frac{1}{a}G + p(z), \tag{16}$$

where $p(z)$ is a polynomial of degree at most k. We claim that $p(z) \equiv 0$. Otherwise noting that f is transcendental in view of Lemma 2.3 we have

$$\begin{aligned}
&(n+1)T(r,f) + O(1) \qquad (17)\\
&= T(r,F) \\
&\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;f) + \overline{N}(r,p;F) + S(r,F) \\
&\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;f) + \overline{N}(r,0;G) + S(r,F) \\
&\leq 3T(r,f) + 2T(r,g) + S(r,f).
\end{aligned}$$

Also from (16) we get

$$T(r,f) = T(r,g) + S(r,f),$$

which together with (17) implies a contradiction. So

$$F = \frac{1}{a} G. \tag{18}$$

Differentiating (18) k times we get

$$F^{(k)} = \frac{1}{a} G^{(k)}.$$

The above equation together with the fact that $F^{(k)}$ and $G^{(k)}$ share 1 CM yields $a = 1$. Hence (18) becomes

$$F \equiv G.$$

So from Lemma 2.12 we get $f \equiv g$. ■

Proof of Theorem 1.7. We omit the proof since instead of Lemma 2.10 using Lemma 2.11 and proceeding in the same way the proof of the theorem can be carried out in the line of the proof of Theorem 1.3 and Theorem 2 of [5]. ■

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