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A Condition for the Properness of Polynomial Maps

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Abstract. In this paper we present a condition for a polynomial map $F := (f_1, f_2, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ to be a global polynomial diffeomorphism. To do this we express a sufficient condition in terms of the Newton polyhedron for a polynomial function $f : \mathbb{R}^n \to \mathbb{R}$ to be a proper map. We also prove that the fiber $f^{-1}(A)$, with large value $|A|$, of a proper polynomial f is diffeomorphic to the unit sphere $\mathbb{S}^{n-1}$, if it is non-empty.

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1. Introduction

Let $F := (f_1, f_2, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map. If F is a local diffeomorphism, then it is not sure that F is a global diffeomorphism [6]. By Hadamard's classical theorem, F is a global diffeormorphism if and only if F is a local diffeomorphism and F is proper. However, it is not trivial to find out whether the polynomial map F is proper or not.

Let us mention some previous results on the properness of real polynomial maps. In 1983, Randall [7] has shown that the polynomial map F is proper if *the highest order homogeneous terms of* f_i *have no non-trivial common real zeros.* In 1996, Cima et al. [2] replaced the notion of homogeneity in the above condition with quasi-homogeneity and he received the same result. Recently, by an approach based on the estimation from below of Lojasiewicz exponent at infinity of F, Ausina [1] gave a sufficient condition for the properness of F in terms of the Newton polyhedra of the component functions of F.

In this paper we consider the polynomial function $f := \sum_{i=1}^{n} f_i^{2k_i} : \mathbb{R}^n \to \mathbb{R}$, where k_i, $i = 1, 2, \ldots, n$, are positive integers. We will show that F is proper if the polynomial f satisfies a non-degenerate condition with respect to its Newton polyhedron at infinity. It seems to us that our condition is close but not identified with the condition of Ausina [1]. In addition, our proof is based on a completely different idea. By the way, we also show that the fiber $f^{-1}(A)$, with large value $|A|$, of a proper polynomial f is diffeomorphic to the unit sphere $\mathbb{S}^{n-1}$, if it is non-empty.

2. Proper Polynomial Functions on $\mathbb{R}^n$

For every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by x^α the monomial $x_1^{\alpha_1}.x_2^{\alpha_2}\ldots x_n^{\alpha_n}$. We first recall some notations about Newton polyhedra.

Definition 2.1. Let

$$f(x) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha$$

be a polynomial. Put $\operatorname{supp}(f) := \{\alpha \in \mathbb{N}^n |\ a_\alpha \neq 0\}$. The *Newton polyhedron at infinity* $\Gamma_\infty(f)$ of f is the convex hull in $\mathbb{R}^n$ of $\{0\} \cup \operatorname{supp}(f)$. Clearly, $\Gamma_\infty(f)$ is a compact convex polyhedron of dimension at most n.

A *supporting hyperplane* of $\Gamma_\infty(f)$ is a hyperplane minimizing the value of some linear functions on $\Gamma_\infty(f)$. The *faces* of the boundary of the Newton polyhedron $\Gamma_\infty(f)$ are the intersection of $\Gamma_\infty(f)$ with supporting hyperplanes. They are compact convex polyhedra of dimension at most $n-1$. The *vertices* are faces of dimension 0 (i.e., points). The *Newton diagram at infinity* of f, denoted by $\Gamma(f)$, is defined to be the union of the closed faces which do not contain 0. For a face $\sigma \in \Gamma(f)$, the polynomial

$$f_\sigma(x) := \sum_{\alpha \in \sigma} a_\alpha x^\alpha$$

is called the *quasi-homogeneous component* of f with respect to σ.

Definition 2.2. A polynomial $f \in \mathbb{R}[x]$ is called *convenient* if for any $i = 1, 2, \ldots, n$, there exists some $\alpha_i \geq 1$ such that the monomial $x_i^{\alpha_i}$ appears in f with non-zero coefficient. Note that f is convenient if and only if the Newton diagram at infinity $\Gamma(f)$ of f intersects each coordinate axis in a point different from the origin.

A continuous map $F : \mathbb{R}^n \to \mathbb{R}^m$ is called *proper* if the inverse image of a compact set is also compact. Equivalently, $F : \mathbb{R}^n \to \mathbb{R}^m$ is proper if and only if $\|x_k\| \to \infty$ implies $\|f(x_k)\| \to \infty$.

We denote by $\mathbb{R}^*$ the set of non-zero real numbers.

Theorem 2.3. *Let $f \in \mathbb{R}[x]$ be a convenient polynomial. If all quasi-homogeneous components of f do not vanish outside the coordinate planes, then f is proper.*

Proof. There is no loss of generality in assuming that α^0 is a vertex of $\Gamma(f)$ such that the coefficient a_{α^0} of x^{α^0} in f is positive. It is shown that all quasi-homogeneous components of f are strictly positive outside the coordinate planes. The proof is by induction on the dimension of faces of $\Gamma(f)$.

- Assume that α is a vertex of $\Gamma(f)$. Let $\sigma \in \Gamma(f)$ be a face of dimension 1 whose boundary is the union of the vertex α and another one β of $\Gamma(f)$. Then, there exists some $i \in \{1, 2, \ldots, n\}$ such that $\alpha_i \neq \beta_i$. Without loss of generality, we can assume that $\alpha_i > \beta_i$. Then

$$f_\sigma(1, \ldots, x_i, \ldots, 1) = (a_\alpha x_i^{\alpha_i} + \cdots + a_\beta x_i^{\beta_i}) = x_i^{\beta_i}(a_\alpha x_i^{\alpha_i - \beta_i} + \cdots + a_\beta).$$

  By assumption, the polynomial

$$h(x_i) := a_\alpha x_i^{\alpha_i - \beta_i} + \cdots + a_\beta$$

  has no non-trivial real zeros. Hence $a_\alpha . a_\beta > 0$ and $\alpha_i - \beta_i$ is even. Thus $a_\alpha . a_\beta > 0$ and $\alpha_i - \beta_i$ is even for all $i \in \{1, 2, \ldots, n\}$. This implies that the coefficient of x^α in f is positive, because the union of faces of dimension 1 of $\Gamma(f)$ is connected and $a_{\alpha^0} > 0$. Moreover, since f is convenient, there are vertices $(0, \ldots, e_i, \ldots, 0) \in \Gamma(f)$ with $e_i > 0$ for all $i \in \{1, 2, \ldots, n\}$. This implies that α_i is even for all $i \in \{1, 2, \ldots, n\}$. Consequently, $f_{\{\alpha\}} = a_\alpha x^\alpha$ is strictly positive outside the coordinate planes.
- Assume that $\sigma \in \Gamma(f)$ is a face of dimension 1 whose boundary is the union of two vertices α and β of $\Gamma(f)$, and that $\alpha_i > \beta_i$ for some $i \in \{1, 2, \ldots, n\}$. Then, for every point $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in (\mathbb{R}^*)^n$, we have

$$\begin{aligned} &f_\sigma\left(x_1^0, \ldots, x_i, \ldots, x_n^0\right) \\ &= x_i^{\beta_i}\left[a_\alpha (x_1^0)^{\alpha_1} \ldots x_i^{\alpha_i - \beta_i} \ldots (x_n^0)^{\alpha_n} + \cdots + a_\beta (x_1^0)^{\beta_1} \ldots 1 \ldots (x_n^0)^{\beta_n}\right]. \end{aligned}$$

  By assumption, the polynomial $f_\sigma(x_1^0, \ldots, x_i, \ldots, x_n^0)$ has no trivial real zeros. It follows that $f_\sigma(x_1^0, \ldots, x_i, \ldots, x_n^0)$ is strictly positive outside the coordinate plane $x_i = 0$. Hence $f_\sigma(x^0) > 0$. So f_σ is strictly positive outside the coordinate planes.
- Assume that the claim holds for the dimension k, $1 \le k < n-1$, we will prove it for $k+1$.

  Let $\sigma \in \Gamma(f)$ be a face of dimension $k+1$. Then, there exist $\sigma_1 = H_1 \cap \Gamma(f)$ and $\sigma_2 = H_2 \cap \Gamma(f)$ which are contained in the boundary of σ such that

$$H_1 := \{\alpha \in \mathbb{R}^n | \alpha_i = \nu_1\} \text{ and } H_2 := \{\alpha \in \mathbb{R}^n | \alpha_i = \nu_2\}$$

  are supporting hyperplanes of $\Gamma_\infty(f)$ with $\nu_1 \neq \nu_2$. Without loss of generality, we can assume that $\nu_1 > \nu_2$. Then, for every point $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in$

$(\mathbb{R}^*)^n$, we have

$$\begin{aligned} &f_\sigma\left(x_1^0,\dots,x_i,\dots,x_n^0\right)\\ &= f_{\sigma_1}\left(x_1^0\dots 1\dots x_n^0\right)x_i^{\nu_1}+\cdots+f_{\sigma_2}\left(x_1^0\dots 1\dots x_n^0\right)x_i^{\nu_2}\\ &= x_i^{\nu_2}\left[f_{\sigma_1}\left(x_1^0\dots 1\dots x_n^0\right)x_i^{\nu_1-\nu_2}+\cdots+f_{\sigma_2}\left(x_1^0\dots 1\dots x_n^0\right)\right].\end{aligned}$$

Since σ_1 and σ_2 are faces of dimension at most k, we have $f_{\sigma_1}(x_1^0\dots 1\dots x_n^0) > 0$ and $f_{\sigma_2}(x_1^0\dots 1\dots x_n^0) > 0$. Clearly, ν_1 and ν_2 are even. Thus $f_\sigma(x_1^0,\dots,x_i,\dots,x_n^0)$ is strictly positive outside the coordinate plane $x_i = 0$, because, by assumption, the polynomial function $f_\sigma(x_1^0,\dots,x_i,\dots,x_n^0)$ has no trivial real zeros. Hence $f_\sigma(x^0) > 0$. This shows that f_σ is strictly positive outside the coordinate planes.

Now, we can show the properness of f. Assume on the contrary that f is not proper. Then there is a sequence $\{x_k\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty}\|x_k\| = +\infty \quad \text{and} \quad \lim_{k\to\infty} f(x_k) = t_0.$$

Since f is convenient and all quasi-homogeneous components of f are strictly positive outside the coordinate planes, by [4, Theorem 3.1], we have f is bounded below and there are constants c_1, c_2 $(c_2 > 0)$ such that

$$f(x) \geq c_1 + c_2 \sum_{\alpha\in V(f)} x^\alpha,$$

where $V(f)$ is the set of vertices of $\Gamma_\infty(f)$. Hence

$$\lim_{k\to\infty} f(x_k) \geq c_1 + c_2 \lim_{k\to\infty} \sum_{\alpha\in V(f)} x_k^\alpha.$$

By the above, for every $\alpha \in V(f)$ we have α_i, $i = 1,2,\dots,n$, are even. Moreover, since f is convenient, for any $i = 1,2,\dots,n$ there exists an $\alpha_i \geq 1$ such that the monomial $x_i^{\alpha_i}$ appears in $\sum_{\alpha\in V(f)} x^\alpha$. Therefore $t_0 \geq c_1 + c_2(+\infty)$, a contradiction. So f is a proper map. ■

Remark 2.4. Suppose a polynomial f satisfies the following condition: $f(0) = 0$, and the highest order homogeneous term $f_d(x)$ is strictly positive if and only if $x \neq 0$. Then f satisfies the condition of Theorem 2.3, and hence f is a proper map. Thus Theorem 2.3 is a generalization of Sakkalis's result in [9].

The following result is also a generalization of Sakkalis's one in [9].

Theorem 2.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper polynomial function, $n \geq 2$. Then, f is bounded from one side. Moreover, the fiber $f^{-1}(A)$ is diffeomorphic to the unit sphere $\mathbb{S}^{n-1}$ for $|A|$ sufficiently large, if it is non-empty.*

Here we state the result of Shiota, whose corollary is the last conclusion of Theorem 2.5.

Remark 2.6. ([8]) Let f_1, f_2 be positive proper polynomials on $\mathbb{R}^n$. Then there exists τ a C^∞ diffeomorphism of $\mathbb{R}^n$ such that τ is the identity on a given bounded subset and that $f_1 \circ \tau$ and f_2 are equal outside a bounded subset.

Remark 2.7. Let f be a positive proper C^∞ function on $\mathbb{R}^n$. Assume $n \neq 4, 5$ and that the set of critical points is bounded. It follows from [8, Lemma 7] that there exists τ a C^∞ diffeomorphism of $\mathbb{R}^n$ such that τ is the identity on a given bounded subset and $f \circ \tau = \sum_{i=1}^n x_i^2$ outside a bounded subset. Thus the fiber $f^{-1}(A)$, with large value $|A|$, is diffeomorphic to the unit sphere $\mathbb{S}^{n-1}$, if it is non-empty. However, it is not sure that the statement of Theorem 2.5 is true if we replace polynomials with C^∞ functions.

Theorem 2.5 follows from the following two claims.

Claim 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper polynomial function, $n \geq 2$. Then f is bounded below or f is bounded above.

Proof. We will prove that if f is not bounded above, then f must be bounded below.

Let $t_0 \in \mathbb{R}$. Since f is proper, the set $f^{-1}(t_0)$ is compact. Hence it is contained in a large enough ball $\mathbb{B}_r$. Then the set

$$\{x \in \mathbb{R}^n | f(x) \leq t_0\}$$

is bounded. Indeed, if it is not so, then there exists a point $x_1 \in \mathbb{R}^n \backslash \mathbb{B}_r$ such that $f(x_1) < t_0$. By assumption, there exists a point $x_2 \in \mathbb{R}^n \backslash \mathbb{B}_r$ such that $f(x_2) > t_0$. Since $n \geq 2$, we can pick a continuous curve l in $\mathbb{R}^n \backslash \mathbb{B}_r$ connecting x_1 and x_2. Since $f|_l$ is continuous, $f(x_1) < t_0$ and $f(x_2) > t_0$, there exists a point $x_0 \in l$ such that $f(x_0) = t_0$, a contradiction. This implies that f is bounded below. ■

Without loss of generality, we now can assume that f is bounded below.

Claim 2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper polynomial function. If f is bounded below and $\mathrm{grad} f(x) = \lambda x$, $\|x\| \geq r \gg 1$, then $\lambda > 0$.

Proof. Put

$$W = \{x \in \mathbb{R}^n | \mathrm{grad} f(x) = \lambda x, \lambda \leq 0\}.$$

Note that W is a semi-algebraic set. It is sufficient to show that W is bounded. On the contrary, suppose that W is not bounded. By Curve Selection Lemma at infinity [5], there is a meromorphic mapping $\rho : [M, +\infty) \to \mathbb{R}^n$ such that $\rho(\tau) \in W$ for $\tau > M$ and $\lim_{\tau\to\infty} \|\rho(\tau)\| = +\infty$. We can write

$$\rho(\tau) = a\tau^\alpha + \text{lower order terms in } \tau,$$
$$f(\rho(\tau)) = b\tau^\beta + \text{lower order terms in } \tau.$$

Since $\mathrm{grad} f(\rho(\tau)) = \lambda(\tau)\rho(\tau)$, we have

$$\begin{aligned}\frac{d}{d\tau}f(\rho(\tau)) &= \langle \text{grad}f(\rho(\tau)), \frac{d\rho}{d\tau}(\tau)\rangle \\ &= \langle \lambda(\tau)\rho(\tau), \frac{d\rho}{d\tau}(\tau)\rangle \\ &= \lambda(\tau)\langle \rho(\tau), \frac{d\rho}{d\tau}(\tau)\rangle.\end{aligned}$$

Thus

$$\beta b\tau^{\beta-1} + \cdots = \lambda(\tau)\langle a\tau^{\alpha} + \cdots, \alpha a\tau^{\alpha-1} + \cdots\rangle.$$

Hence

$$\lambda(\tau) = \frac{\beta b}{\|a\|^2\alpha}\tau^{\beta-2\alpha} + \cdots.$$

It is clear that $\alpha > 0$; and since f is a proper map, $\beta > 0$. Since f is bounded below and f is proper, we have $f(\rho(\tau)) \to +\infty$ as $\tau \to +\infty$, hence $b > 0$. Thus $\lambda(\tau) > 0$ for τ sufficiently large, a contradiction. ■

Proof of Theorem 2.5. For $r > 0$, let

$$\mathbb{B}_r := \{x \in \mathbb{R}^n \mid \|x\| \le r\}, \ \text{ and } \ \mathbb{S}_r^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = r\}.$$

We now put

$$v(x) = \|\text{grad}f(x)\|x + \|x\|\text{grad}f(x).$$

Then

$$\langle \text{grad}f(x), v(x)\rangle = \|\text{grad}f(x)\|\Big[\langle \text{grad}f(x), x\rangle + \|x\|\|\text{grad}f(x)\|\Big] \ge 0,$$

$$\langle x, v(x)\rangle = \|x\|\Big[\langle \text{grad}f(x), x\rangle + \|x\|\|\text{grad}f(x)\|\Big] \ge 0.$$

By Claim 2, if $\text{grad}f(x) = \lambda x$, $\|x\| \ge r \gg 1$, then $\lambda > 0$. Hence the sign "=" in the above inequalities cannot happen for all x outside some large ball $\mathbb{B}_r$. Thus for all $x \in \mathbb{R}^n \backslash \mathbb{B}_r$, $r \gg 1$, we have

$$\begin{aligned}\langle \text{grad}f(x), v(x)\rangle &> 0, \\ \langle x, v(x)\rangle &> 0.\end{aligned}$$

Let

$$\omega(x) = \frac{v(x)}{2\langle x, v(x)\rangle}.$$

We have

$$\langle \text{grad}f(x), \omega(x)\rangle > 0, \qquad (*)$$

$$\langle x, \omega(x)\rangle = \frac{1}{2}. \qquad (**)$$

Let $A > \max_{x\in\mathbb{B}_r} f(x)$. For a fixed point $x_0 \in \mathbb{S}_r^{n-1}$, consider integral curve $x(x_0;\tau)$ of the vector field $w(x)$ starting from the point x_0. It follows from (**)

that $\|x(x_0;\tau)\|^2 = \tau + r^2$. Consequently, $x(x_0;\tau)$ is defined on the whole of $[0,+\infty)$.

Since $f \circ x(x_0;0) < A$ and $\lim_{t\to\infty} f \circ x(x_0;\tau) = +\infty$, there exists a point $\tau(x_0) \in (0,+\infty)$ such that $f \circ x(x_0;\tau(x_0)) = A$. From (*) it follows that the function $f \circ x(x_0;\tau)$ is increasing when $\tau > 0$, hence the point $\tau(x_0)$ is unique.

Consider the map

$$G : \mathbb{S}^{n-1} \to f^{-1}(A),\ v \mapsto x(rv;\tau(rv)).$$

It is easy to see that G is a diffeomorphism. ∎

3. Global Polynomial Diffeomorphisms on $\mathbb{R}^n$

Lemma 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function and k be a positive integer. Then the Newton polyhedron at infinity $\Gamma_\infty(f^k)$ of f^k is the convex hull of $\{0\} \cup \{k\alpha | \alpha \in supp(f)\}$. Moreover, for every face $\overline{\sigma} \in \Gamma(f^k)$ there is a unique face $\sigma \in \Gamma(f), k\alpha \in \overline{\sigma}$ for all $\alpha \in \sigma$, such that $\left(f^k\right)_{\overline{\sigma}} = (f_\sigma)^k$.*

Proof. (i) Let B be the convex hull of $\{0\} \cup \{k\alpha | \alpha \in \text{supp}(f)\}$. We will show that $\Gamma_\infty(f^k) = B$.

We can write

$$f^k(x) = \sum_{\substack{\alpha^1,\alpha^2,\dots,\alpha^r \in supp(f)\\ i_1+i_2+\cdots+i_r=k}} c_{\overline{\alpha}} \left(x^{\alpha^1}\right)^{i_1} \left(x^{\alpha^2}\right)^{i_2} \cdots \left(x^{\alpha^r}\right)^{i_r}.$$

Then for every $\overline{\alpha} \in \text{supp}(f^k)$

$$\overline{\alpha} = i_1\alpha^1 + i_2\alpha^2 + \cdots + i_r\alpha^r,$$

where $\alpha^1, \alpha^2, \dots, \alpha^r \in \text{supp}(f)$, $i_1 + i_2 + \cdots + i_r = k$, $i_1, i_2, \cdots, i_r > 0$. Hence

$$\overline{\alpha} = \frac{i_1}{k}(k\alpha^1) + \frac{i_2}{k}(k\alpha^2) + \cdots + \frac{i_r}{k}(k\alpha^r) \in B.$$

Thus $\Gamma_\infty(f^k) \subset B$.

We now prove that $\{k\alpha | \alpha \in \text{supp}(f)\} \subset \Gamma_\infty(f^k)$. There are two cases to be considered:
Case1. $\alpha \in \text{supp}(f)$ and α is a vertex of $\Gamma_\infty(f)$. Let

$$H := \{\alpha \in \mathbb{R}^n | \langle m, \alpha\rangle := m_1\alpha_1 + m_2\alpha_2 + \cdots + m_n\alpha_n = \nu\}$$

be a supporting hyperplane of $\Gamma_\infty(f)$ such that $H \cap \Gamma_\infty(f) = \{\alpha\}$. By definition of supporting hyperplanes, we have

$$\langle m, \alpha\rangle = \nu \ \text{ and } \ \langle m, \alpha'\rangle > \nu \text{ for all } \alpha' \in \Gamma_\infty(f) \backslash \{\alpha\}.$$

Hence

$$\langle m, k\alpha\rangle = k\nu \text{ and } \langle m, \overline{\alpha}\rangle > k\nu \text{ for all } \overline{\alpha} \in \operatorname{supp}(f^k)\backslash\{k\alpha\}.$$

Thus

$$\langle m, k\alpha\rangle = k\nu \text{ and } \langle m, \overline{\alpha}\rangle > k\nu \text{ for all } \overline{\alpha} \in \Gamma_\infty(f^k)\backslash\{k\alpha\}.$$

This implies that

$$\overline{H} := \{\overline{\alpha} \in \mathbb{R}^n | \langle m, \overline{\alpha}\rangle := m_1\overline{\alpha}_1 + m_2\overline{\alpha}_2 + \cdots + m_n\overline{\alpha}_n = k\nu\}$$

is a supporting hyperplane of $\Gamma_\infty(f^k)$ and $\overline{H} \cap \Gamma_\infty(f^k) = \{k\alpha\}$. Hence $k\alpha \in \Gamma_\infty(f^k)$.

Case 2. $\alpha \in \operatorname{supp}(f)$ and α is not any vertex of $\Gamma_\infty(f)$. Then

$$\alpha = i_1\alpha^1 + i_2\alpha^2 + \cdots + i_r\alpha^r,$$

where $\alpha^1, \alpha^2, \ldots, \alpha^r$ are vertices of $\Gamma_\infty(f)$, $i_1+i_2+\cdots+i_r = 1$, $i_1, i_2, \cdots, i_r > 0$. Hence

$$k\alpha = i_1(k\alpha^1) + i_2(k\alpha^2) + \cdots + i_r(k\alpha^r) \in \Gamma_\infty(f^k).$$

Thus $B \subset \Gamma_\infty(f^k)$, and hence $B = \Gamma_\infty(f^k)$.

(ii) Let $\overline{\sigma}$ be a face of $\Gamma(f^k)$. Assume that $\overline{\sigma} = \overline{H} \cap \Gamma_\infty(f^k)$, where

$$\overline{H} := \{\overline{\alpha} \in \mathbb{R}^n | \langle m, \overline{\alpha}\rangle := m_1\overline{\alpha}_1 + m_2\overline{\alpha}_2 + \cdots + m_n\overline{\alpha}_n = k\nu\}$$

is a supporting hyperplane of $\Gamma_\infty(f^k)$. By the above result, we have

$$H := \{\alpha \in \mathbb{R}^n | \langle m, \alpha\rangle := m_1\alpha_1 + m_2\alpha_2 + \cdots + m_n\alpha_n = \nu\}$$

is a supporting hyperplane of $\Gamma_\infty(f)$. Hence $\sigma = H \cap \Gamma_\infty(f)$ is a face of $\Gamma(f)$.

For every $\overline{\alpha} \in \operatorname{supp}(f^k)$, we can write

$$\overline{\alpha} = i_1\alpha^1 + i_2\alpha^2 + \cdots + i_r\alpha^r,$$

where $\alpha^1, \alpha^2, \ldots, \alpha^r \in \operatorname{supp}(f)$, $i_1 + i_2 + \cdots + i_r = k$, $i_1, i_2, \cdots, i_r > 0$. By definition of supporting hyperplanes, we have

$$\langle m, \alpha\rangle = \nu \text{ for all } \alpha \in \sigma \text{ and } \langle m, \alpha\rangle > \nu \text{ for all } \alpha \in \operatorname{supp}(f)\backslash\sigma.$$

It follows that $\langle m, \overline{\alpha}\rangle = k\nu$ if and only if $\alpha^1, \alpha^2, \ldots, \alpha^r \in \sigma$. Hence $\overline{\alpha} \in \overline{\sigma}$ if and only if $\alpha^1, \alpha^2, \ldots, \alpha^r \in \sigma$. Thus $\left(f^k\right)_{\overline{\sigma}} = (f_\sigma)^k$. ■

Lemma 3.2. *Let* $F := (f_1, f_2, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ *be a polynomial map and* k_i, $i = 1, 2, \ldots, n$, *be positive integers. Then the Newton polyhedron at infinity of* $f := \sum_{i=1}^{n} f_i^{2k_i} : \mathbb{R}^n \to \mathbb{R}$ *is the convex hull of* $\{0\} \cup_{i=1}^n \{2k_i\alpha | \alpha \in supp(f_i)\}$. *Moreover, for every face* $\overline{\sigma} \in \Gamma(f)$

$$f_{\overline{\sigma}} = \sum f_{i,\sigma_i}^{2k_i},$$

where σ_i is the face of $\Gamma(f_i)$ such that $2k_i\alpha \in \overline{\sigma}$ for all $\alpha \in \sigma_i$.

Proof. (i) Put

$$X := \cup_{i=1}^{n}\{2k_i\alpha | \alpha \in \mathrm{supp}(f_i)\}, \text{ and } C := \text{the convex hull of } \{0\} \cup X.$$

First, we prove that $X \subset \Gamma_\infty(f)$. In fact, if it is not so, then there exists an $\alpha \in \mathrm{supp}(f_i)$ such that $\overline{\alpha} = 2k_i\alpha \notin \Gamma_\infty(f)$. For any $j = 1, 2, \ldots, n$, we can write

$$\big(f_j(x)\big)^{2k_j} = \sum_{\substack{\alpha^1,\alpha^2,\ldots,\alpha^m \in supp(f_j)\\ j_1+j_2+\cdots+j_m=2k_j}} c_{\alpha^1,\alpha^2,\ldots,\alpha^m}^{j_1,j_2,\ldots,j_m} \left(x^{\alpha^1}\right)^{j_1} \left(x^{\alpha^2}\right)^{j_2} \cdots \left(x^{\alpha^m}\right)^{j_m}.$$

Then the coefficient of $x^{\overline{\alpha}}$ in f can be written as a sum of the real numbers $c_{\alpha^1,\alpha^2,\ldots,\alpha^m}^{j_1,j_2,\ldots,j_m}$ with $j_1\alpha^1 + j_2\alpha^2 + \cdots + j_m\alpha^m = \overline{\alpha}$. Since the real number $c_{\alpha}^{2k_i}$ is positive, it follows that there exists some $j \in \{1, 2, \ldots, n\}$ such that

$$\overline{\alpha} = 2k_i\alpha = j_1\alpha^1 + j_2\alpha^2 + \cdots + j_m\alpha^m,$$

where $\alpha^1, \alpha^2, \ldots, \alpha^m \in \mathrm{supp}(f_j)$, $j_1 + j_2 + \cdots + j_m = 2k_j$, $j_1, j_2, \ldots, j_m > 0$, and $c_{\alpha^1,\alpha^2,\ldots,\alpha^m}^{j_1,j_2,\ldots,j_m} < 0$. Hence

$$\overline{\alpha} = \frac{j_1}{2k_j}(2k_j\alpha^1) + \frac{j_2}{2k_j}(2k_j\alpha^2) + \cdots + \frac{j_m}{2k_j}(2k_j\alpha^m) = i_1'\overline{\alpha}^1 + i_2'\overline{\alpha}^2 + \cdots + i_m'\overline{\alpha}^m,$$

where $\overline{\alpha}^1, \overline{\alpha}^2, \ldots, \overline{\alpha}^m \in X$, $j_1' + j_2' + \cdots + j_m' = 1$, $j_1', j_2', \ldots, j_m' > 0$, and $m \geq 2$.

We can rewrite

$$\overline{\alpha} = \lambda_1\overline{\alpha}^1 + \lambda_2\overline{\alpha}^2 + \cdots + \lambda_r\overline{\alpha}^r,$$

where $\overline{\alpha}^1, \overline{\alpha}^2, \ldots, \overline{\alpha}^r \in X$, $\lambda_1 + \lambda_2 + \cdots + \lambda_r = 1$, $\lambda_1, \lambda_2, \ldots, \lambda_r > 0$, and $\overline{\alpha}^l \neq \overline{\alpha}$ for all $l = 1, 2, \ldots, r$.

Without loss of generality, we can suppose that the coefficient of $x^{\overline{\alpha}^1}$ in f is zero. Then we can write

$$\overline{\alpha} = \mu_1\overline{\beta}^1 + \mu_2\overline{\beta}^2 + \cdots + \mu_s\overline{\beta}^s,$$

where $\overline{\beta}^1, \overline{\beta}^2, \ldots, \overline{\beta}^s \in X$, $\mu_1 + \mu_2 + \cdots + \mu_s = 1$, $\mu_1, \mu_2, \ldots, \mu_s > 0$, $\overline{\alpha}^l = y_1^l\overline{\beta}^1 + y_2^l\overline{\beta}^2 + \cdots + y_s^l\overline{\beta}^s$, $y_1^l + y_2^l + \cdots + y_s^l = 1$, $y_1^l, y_2^l, \ldots, y_s^l \geq 0$, and $\overline{\beta}^l \notin \{\overline{\alpha}, \overline{\alpha}^1\}$ for all $l = 1, 2, \ldots, s$.

Assume that the coefficient of $x^{\overline{\beta}^1}$ in f is zero. Then we can write

$$\overline{\alpha} = \epsilon_1\overline{\gamma}^1 + \epsilon_2\overline{\gamma}^2 + \cdots + \epsilon_p\overline{\gamma}^p$$

where $\overline{\gamma}^1, \overline{\gamma}^2, \ldots, \overline{\gamma}^p \in X$, $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_p = 1$, $\epsilon_1, \epsilon_2, \ldots, \epsilon_p > 0$, $\overline{\beta}^l = z_1^l \overline{\gamma}^1 + z_2^l \overline{\gamma}^2 + \cdots + z_p^l \overline{\gamma}^p$, $z_1^l + z_2^l + \cdots + z_p^l = 1, z_1^l, z_2^l, \ldots, z_p^l \geq 0$, and $\overline{\gamma}^l \notin \{\overline{\alpha}, \overline{\alpha}^1, \overline{\beta}^1\}$ for all $l = 1, 2, \ldots, p$.

We continue in this fashion to obtain

$$\overline{\alpha} = \zeta_1 \overline{\delta}^1 + \zeta_2 \overline{\delta}^2 + \cdots + \zeta_q \overline{\delta}^q,$$

where $\overline{\delta}^1, \overline{\delta}^2, \ldots, \overline{\delta}^q \in X$, $\zeta_1 + \zeta_2 + \cdots + \zeta_q = 1$, $\zeta_1, \zeta_2, \ldots, \zeta_q > 0$, and all coefficients of $x^{\overline{\delta}^l}$ in f are non-zero. Hence $\overline{\alpha} \in \Gamma_\infty(f)$, a contradiction. Thus $X \subset \Gamma_\infty(f)$. This implies that $C \subset \Gamma_\infty(f)$. Now, by definition of f and Lemma 3.1, $\Gamma_\infty(f) \subset C$. Therefore $\Gamma_\infty(f) = C$.

(ii) Let $\overline{\sigma} \in \Gamma(f)$. Assume that $\overline{\sigma} = \overline{H} \cap \Gamma_\infty(f)$, where $\overline{H}$ is a supporting hyperplane of $\Gamma_\infty(f)$. By the above result and Lemma 3.1, $\overline{H}$ is a supporting hyperplane of some $\Gamma_\infty(f_i^{2k_i})$. We have $f_{\overline{\sigma}} = \sum (f_i^{2k_i})_{\overline{\sigma}_i}$, where $\overline{\sigma}_i = \overline{H} \cap \Gamma_\infty(f_i^{2k_i})$. By Lemma 3.1, $(f_i^{2k_i})_{\overline{\sigma}_i} = f_{i,\sigma_i}^{2k_i}$, where σ_i is the face of $\Gamma(f_i)$ such that $2k_i\alpha \in \overline{\sigma}_i$ for all $\alpha \in \sigma_i$. Thus $f_{\overline{\sigma}} = \sum f_{i,\sigma_i}^{2k_i}$. ■

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map and k_i, $i = 1, 2, \ldots, n$, be positive integers. We denote by $\Gamma_\infty(f_1^{2k_1}, f_2^{2k_2}, \ldots, f_n^{2k_n})$ the Newton polyhedron at infinity of the polynomial f, and $\Gamma(f_1^{2k_1}, f_2^{2k_2}, \ldots, f_n^{2k_n})$ the Newton diagram at infinity of f, where $f = \sum\limits_{i=1}^{n} f_i^{2k_i}$.

Definition 3.3. A polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *convenient* if for any $i = 1, 2, \ldots, n$, there exist $\alpha_i \geq 1$ and $j \in \{1, 2, \ldots, n\}$ such that the monomial $x_i^{\alpha_i}$ appears in the polynomial f_j with non-zero coefficient. Note that F is convenient if and only if $\Gamma(f_1^{2k_1}, f_2^{2k_2}, \ldots, f_n^{2k_n})$ intersects each coordinate axis in a point different from the origin, where k_i, $i = 1, 2, \ldots, n$, are positive integers.

Theorem 3.4. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map. Assume that*
(i) *The Jacobian $J(F)(x) \neq 0$ for all $x \in \mathbb{R}^n$;*
(ii) *F is convenient; and*
(iii) *There are positive integers k_i, $i = 1, 2, \ldots, n$, such that for any face $\overline{\sigma} \in \Gamma(f_1^{2k_1}, \ldots, f_n^{2k_n})$ the terms f_{i,σ_i} have no common real zeros outside the coordinate planes, where σ_i is the face of $\Gamma(f_i)$ such that $k_i\alpha \in \overline{\sigma}$ for all $\alpha \in \sigma_i$. Then F is a proper map and hence a diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.*

Proof. Let us consider the polynomial function

$$f = \sum_{i=1}^{n} f_i^{2k_i} : \mathbb{R}^n \to \mathbb{R}.$$

By the assumption (ii) and Lemma 3.2, f is convenient. Moreover, it follows from Lemma 3.2 that if $\overline{\sigma}$ is a face of $\Gamma(f)$, then $f_{\overline{\sigma}} = \sum f_{i,\sigma_i}^{2k_i}$, where σ_i is the face of

$\Gamma(f_i)$ such that $2k_i\alpha \in \overline{\sigma}$ for all $\alpha \in \sigma_i$. By the assumption (iii), $f_{\overline{\sigma}}(x) > 0$ for all $x \in (\mathbb{R}^*)^n$. So, by Theorem 2.3, f is proper. This implies that F is also proper. Consequently, F is a diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$ because of the assumption (i). ■

Remark 3.5. (1) We consider the polynomial map $F = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ which is given by

$$f_1(x, y) = x^4 - y^4 + x^{12}y^8 + 2x^8y^{12}, \quad f_2(x, y) = xy^3 - x^3y.$$

A simple computation shows that F satisfies Theorem 3.4 (iii) for $k_1 = 1$ and $k_2 = 2$, but not for $k_1 = 1$ and $k_2 = 1$.
(2) Let $F = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map. We will find conditions for positive integers k_1, k_2 to be satisfying Theorem 3.4 (iii).

Let us consider quasi-homogeneous components f_{1,σ_1} and f_{2,σ_2} with $\sigma_1 = H_1 \cap \Gamma(f_1)$ and $\sigma_2 = H_2 \cap \Gamma(f_2)$, where

$$H_1 := \{\alpha \in \mathbb{R}^2 | \langle m, \alpha\rangle := m_1\alpha_1 + m_2\alpha_2 = \nu_1\}$$

and

$$H_2 := \{\beta \in \mathbb{R}^2 | \langle m, \beta\rangle := m_1\beta_1 + m_2\beta_2 = \nu_2\}$$

are supporting hyperplanes of $\Gamma_\infty(f_1)$ and $\Gamma_\infty(f_2)$, respectively. Note that H_1 and H_2 are hyperplanes minimizing the values of the linear function $L(\alpha) := m_1\alpha_1 + m_2\alpha_2$ on $\Gamma_\infty(f_1)$ and $\Gamma_\infty(f_2)$, respectively. Then

- If f_{1,σ_1} and f_{2,σ_2} have some common zeros in $(\mathbb{R}^*)^2$, then there do not exist positive integers k_1 and k_2 which satisfy the condition (iii) of Theorem 3.4.
- If f_{1,σ_1} and f_{2,σ_2} have no common zeros in $(\mathbb{R}^*)^2$ and every f_{i,σ_i}, $i = 1, 2$, has some zeros in $(\mathbb{R}^*)^2$, then it is necessary to take k_1 and k_2 such that $\nu_1k_1 = \nu_2k_2$.
- If f_{1,σ_1} has some zeros in $(\mathbb{R}^*)^2$ and f_{2,σ_2} has no zeros in $(\mathbb{R}^*)^2$, then it is necessary to take k_1 and k_2 such that there exists a vertex $\beta \in \Gamma(f_2)$ with $\langle m, k_2\beta\rangle \le k_1\nu_1$. If, on the other hand, f_{2,σ_2} has some zeros in $(\mathbb{R}^*)^2$ and f_{1,σ_1} has no zeros in $(\mathbb{R}^*)^2$, a similar reasoning applies. We observe that if σ_i is a vertex of $\Gamma(f_i)$, then f_{i,σ_i} has no zeros in $(\mathbb{R}^*)^2$.

Remark 3.6. In [1], Ausina gave a condition on the properness of a polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$, which seems to be similar to our condition. This condition is the following:

(B1) f_i is convenient for all $i = 1, 2, \ldots, n$; and

(B2) F is non-degenerate at infinity.

Note that

- The condition (iii) of Theorem 3.4 is close but not identified with (B2). In fact, let $F = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ with $f_1(x, y) = x^2 + y^2 - x^6y^4 + 2x^4y^6$,

$f_2(x,y) = xy^3 + x^2y^4 - x^4y^2$. Then F is non-degenerate at infinity, but F does not satisfy Theorem 3.4 (iii). On the other hand, let us consider $G = (g_1, g_2) : \mathbb{R}^2 \to \mathbb{R}^2$ with $g_1 = xy + 2x^2y^2 + y^2$, $g_2 = x^3y^3 + x^4y^4 + y^6$. Then G satisfies Theorem 3.4 (iii) for $k_1 = 1$ and $k_2 = 1$, but G does not satisfy (B2).

- The condition (ii) of Theorem 3.4 is strictly weaker than (B1). In fact, let $H = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2$ with $h_1 = x^2 + y^2 + x^6y^4 + 2x^4y^6$, $h_2 = xy^3 + x^2y^4 - x^4y^2$. It is clear that H is convenient, but h_2 is not convenient.

Thus, our result does not coincide with Bivià-Ausina's one.

Corollary 3.7. (Randall, 1983) *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map. Assume that*
(i) *The Jacobian $J(F)(x) \neq 0$ for all $x \in \mathbb{R}^n$; and*
(ii) *The highest order homogeneous terms of f_i have no non-trivial common real zeros.*
Then F is a proper map and hence a diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Proof. Since the highest order homogeneous terms of f_i have no non-trivial common real zeros, it follows that F is convenient. Moreover, F satisfies the condition (iii) of Theorem 3.4 for $k_i = \frac{\deg f_1 . \deg f_2 \ldots \deg f_n}{\deg f_i}$, $i = 1, 2, \ldots, n$. Therefore F is a proper map and hence a diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$. ■

Let $f \in \mathbb{R}[x]$ be a real polynomial and let r, ω_1, ω_2, $\ldots$, ω_n be positive integers. We will say that f is quasi-homogeneous of quasi-degree r with weight $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ if $f(\lambda^{\omega_1}x_1, \lambda^{\omega_2}x_2, \ldots, \lambda^{\omega_n}x_n) = \lambda^r f(x_1, x_2, \ldots, x_n)$. Then if one fixes a weight ω, every polynomial can be decomposed as a sum of quasi-homogeneous polynomials of different quasi-degrees, with weight ω. We denote by f_ω the quasi-homogeneous component, with weight ω, of maximum quasi-degree of f. If $F = (f_1, f_2, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map, we also denote by F_ω the map with components $(f_{1,\omega}, f_{2,\omega}, \ldots, f_{n,\omega})$.

Corollary 3.8. (Cima et al., 1996) *Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map such that the Jacobian $J(F)(x) \neq 0$ for all $x \in \mathbb{R}^n$, and F_ω has no non-trivial real zeros, for some weight w. Then F is a proper map and hence a diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.*

Proof. Since F_ω has no non-trivial real zeros, for some weight w, it follows that F is convenient. Moreover, F satisfies the condition (iii) of Theorem 3.4 for $k_i = \frac{\deg_\omega f_1 . \deg_\omega f_2 \ldots \deg_\omega f_n}{\deg_\omega f_i}$, where $\deg_\omega f_i$ is the quasi-degree of the quasi-homogeneous component $f_{i,\omega}$. Therefore F is a proper map and hence a diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$. ■

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