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Some Ostrowski Type Inequalities and Applications

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Abstract. New generalizations of Ostrowski type inequality for functions of Lipschitzian type are established. Applications for cumulative distribution functions are given.

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1. Introduction

In [4], Dragomir, Cerone and Roumeliotis have proved the following generalization of Ostrowski inequality.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b) and whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b). Denote $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)| < \infty$. Then*

$$\left|\int_a^b f(t)\,dt - \left[f(x)(1-\lambda) + \frac{f(a)+f(b)}{2}\lambda\right](b-a)\right|$$
$$\le \left[\frac{1}{4}(b-a)^2(\lambda^2 + (1-\lambda)^2) + (x - \frac{a+b}{2})^2\right] \|f'\|_\infty, \tag{1}$$

for all $\lambda \in [0,1]$ *and* $a + \lambda\frac{b-a}{2} \leq x \leq b - \lambda\frac{b-a}{2}$.

In [7], Ujević derived a perturbation of (1) as follows:

Theorem 1.2. *Let* $I \subset \mathbb{R}$ *be an open interval and* $a, b \in I$, $a < b$. *If* $f : I \to \mathbb{R}$ *is a differentiable function such that* $\gamma \leq f'(t) \leq \Gamma$, $\forall t \in [a,b]$, *for some constants* $\gamma, \Gamma \in \mathbb{R}$, *then we have*

$$\left|(b-a)\left[\frac{\lambda}{2}(f(a)+f(b)) + (1-\lambda)f(x) - \frac{\Gamma+\gamma}{2}(1-\lambda)\left(x - \frac{a+b}{2}\right)\right] - \int_a^b f(t)\,dt\right| \leq \frac{\Gamma-\gamma}{2}\left[\frac{(b-a)^2}{4}(\lambda^2 + (1-\lambda)^2) + \left(x - \frac{a+b}{2}\right)^2\right], \qquad (2)$$

where $a + \lambda\frac{b-a}{2} \leq x \leq b - \lambda\frac{b-a}{2}$ *and* $\lambda \in [a,b]$.

Both (1) and (2) have been used to estimate the remainder term of the midpoint, trapezoid, and Simpson quadrature formulae. Applications in numerical integration are given respectively.

In this paper, we will generalize Theorem 1.1 and Theorem 1.2 to functions of some larger classes. For convenience, we define functions of Lipschitzian type as follows:

Definition 1.3. The function $f : [a,b] \to \mathbb{R}$ is said to be *L-Lipschitzian* on $[a,b]$ if

$$|f(x) - f(y)| \leq L|x-y| \text{ for all } x, y \in [a,b],$$

where $L > 0$ is given.

Definition 1.4. The function $f : [a,b] \to \mathbb{R}$ is said to be (l,L)-*Lipschitzian* on $[a,b]$ if

$$l(x_2 - x_1) \leq f(x_2) - f(x_1) \leq L(x_2 - x_1) \; for \; a \leq x_1 \leq x_2 \leq b,$$

where $l, L \in \mathbb{R}$ with $l < L$.

We will also need the following well-known result.

Lemma 1.5. *Let* $g, v : [a,b] \to \mathbb{R}$ *be such that* g *is Riemann-integrable on* $[a,b]$ *and* v *is* L*-Lipschitzian on* $[a,b]$. *Then*

$$\left|\int_a^b g(t)\,dv(t)\right| \leq L\int_a^b |g(t)|\,dt. \qquad (3)$$

The purpose of this paper is to generalize Theorem 1.1 and Theorem 1.2 to functions which are L-Lipschitzian and (l, L)-Lipschitzian, respectively. Applications for cumulative distribution functions are given.

2. Main Results

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be (l, L)-Lipschitzian on $[a, b]$. Then we have*

$$\left|(b-a)\left[\frac{\lambda}{2}(f(a)+f(b))+(1-\lambda)f(x)-\frac{L+l}{2}(1-\lambda)\left(x-\frac{a+b}{2}\right)\right]\right.$$
$$\left.-\int_a^b f(t)\,dt\right|$$
$$\leq \frac{L-l}{2}\left[\frac{(b-a)^2}{4}(\lambda^2+(1-\lambda)^2)+\left(x-\frac{a+b}{2}\right)^2\right], \tag{4}$$

where $a + \lambda((b-a)/2) \leq x \leq b - \lambda((b-a)/2)$ and $\lambda \in [0, 1]$.

Proof. Let us define the function

$$K(x,t) := \begin{cases} t-\left(a+\lambda\dfrac{b-a}{2}\right), & t \in [a,x], \\ \\ t-\left(b-\lambda\dfrac{b-a}{2}\right), & t \in (x,b]. \end{cases}$$

Put

$$g(t) := f(t) - \frac{L+l}{2}t. \tag{5}$$

It is easy to find that the function $g : [a, b] \to \mathbb{R}$ is K-Lipschitzian on $[a, b]$ with $K = \frac{L-l}{2}$. So, the Riemann-Stieltjes integral $\int_a^b K(x,t)\,dg(t)$ exists. Using the integration by parts formula for Riemann-Stieltjes integral, we have

$$\int_a^b K(x,t)\,dg(t) = (b-a)\left[\frac{\lambda}{2}(g(a)+g(b))+(1-\lambda)g(x)\right]-\int_a^b g(t)\,dt. \tag{6}$$

From (3) of Lemma 1.5 we have

$$\left|\int_a^b K(x,t)\,dg(t)\right| \leq \frac{L-l}{2}\int_a^b |K(x,t)|\,dt. \tag{7}$$

We also have

$$\int_a^b |K(x,t)|\,dt = \frac{(b-a)^2}{4}\left[\lambda^2+(1-\lambda)^2\right]+\left(x-\frac{a+b}{2}\right)^2. \tag{8}$$

(6), (7) and (8) imply that

$$\left|(b-a)\left[\frac{\lambda}{2}(g(a)+g(b))+(1-\lambda)g(x)\right]-\int_a^b g(t)\,dt\right|$$
$$\leq \frac{L-l}{2}\left[\frac{(b-a)^2}{4}(\lambda^2+(1-\lambda)^2)+\left(x-\frac{a+b}{2}\right)^2\right]. \tag{9}$$

Consequently, the inequality (4) follows from substituting (5) to the left hand side of the inequality (9). ■

Corollary 2.2. *Under the assumptions of Theorem 2.1, we have*

$$\left|f(x)(b-a)-\frac{L+l}{2}(b-a)\left(x-\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right|$$
$$\leq \frac{L-l}{2}\left[\frac{(b-a)^2}{4}+\left(x-\frac{a+b}{2}\right)^2\right] \tag{10}$$

for all $x \in [a,b]$, *and especially*

$$\left|f\left(\frac{a+b}{2}\right)(b-a)-\int_a^b f(t)\,dt\right| \leq \frac{L-l}{8}(b-a)^2. \tag{11}$$

Proof. We set $\lambda = 0$ in (4) to get (10) and $x = \frac{a+b}{2}$ in (10) to get (11). ■

Corollary 2.3. *Under the assumptions of Theorem 2.1, we have*

$$\left|\frac{b-a}{2}\left[f(a)+f(b)\right]-\int_a^b f(t)\,dt\right| \leq \frac{L-l}{8}(b-a)^2. \tag{12}$$

Proof. We set $\lambda = 1$ in (4) to get (12). (Note that $\lambda = 1$ implies $x = \frac{a+b}{2}$.) ■

Corollary 2.4. *Under the assumptions of Theorem 2.1, we have*

$$\left|(b-a)\left[\frac{f(a)+f(b)}{4}+\frac{1}{2}f(x)-\frac{L+l}{4}\left(x-\frac{a+b}{2}\right)\right]-\int_a^b f(t)\,dt\right|$$
$$\leq \frac{L-l}{2}\left[\frac{(b-a)^2}{8}+\left(x-\frac{a+b}{2}\right)^2\right] \tag{13}$$

for all $x \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]$, *and especially*

$$\left|\frac{b-a}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\int_a^b f(t)\,dt\right| \leq \frac{L-l}{16}(b-a)^2. \tag{14}$$

Proof. We set $\lambda = \frac{1}{2}$ in (4) to get (13) and $x = \frac{a+b}{2}$ in (13) to get (14). ∎

Corollary 2.5. *Under the assumptions of Theorem 2.1, we have*

$$\left| \frac{b-a}{6} [f(a) + 4f(x) + f(b)] - \frac{L+l}{3} \left(x - \frac{a+b}{2} \right) - \int_a^b f(t)\, dt \right| \leq \frac{L-l}{2} \left[\frac{5}{36}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right], \tag{15}$$

for all $x \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right]$, *and especially*

$$\left| \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2} \right) + f(b) \right] - \int_a^b f(t)\, dt \right| \leq \frac{5}{72}(L-l)(b-a)^2. \tag{16}$$

Proof. We set $\lambda = \frac{1}{3}$ in (4) to get (15) and $x = \frac{a+b}{2}$ in (15) to get (16). ∎

Remark 2.6. Inequalities (11), (12) and (16) were first appeared in [6].

Remark 2.7. It is interesting to note that the smallest bound for (4) is obtained at $x = \frac{a+b}{2}$ and $\lambda = \frac{1}{2}$. Thus the quadrature rule (14) comprised of the linear combination of the mid-point and trapezoidal rule is optimal and has a lower bound than Simpson's rule (16).

Theorem 2.8. *Let* $f : [a,b] \to \mathbb{R}$ *be* L*-Lipschitzian on* $[a,b]$. *Then we have*

$$\left| \int_a^b f(t)\, dt - \left[f(x)(1-\lambda) + \frac{f(b)+f(a)}{2}\lambda \right](b-a) \right| \leq \left[\frac{1}{4}(b-a)^2(\lambda^2 + (1-\lambda)^2) + \left(x - \frac{a+b}{2} \right)^2 \right] L, \tag{17}$$

where $a + \lambda\frac{b-a}{2} \leq x \leq b - \lambda\frac{b-a}{2}$ *and* $\lambda \in [0,1]$.

Proof. It is immediate to get the inequality (17) by taking $l = -L$ in Theorem 2.1. ∎

Remark 2.9. Estimates for the remainder term of the mid-point, trapezoid and Simpson formulae for functions of L-Lipschitzian which have been given in [2, 3, 5] can now be recaptured from (17) as special cases.

Remark 2.10. It is clear that Theorem 2.1 and Theorem 2.8 can be regarded as generalizations of Theorem 1.2 and Theorem 1.1, respectively.

3. Applications for Cumulative Distribution Functions

Now we consider some applications for cumulative distribution functions. Let X be a random variable having the probability density function $f : [a,b] \to \mathbb{R}_+$ and the cumulative distribution function $F(x) = P_r(X \le x)$, i.e.,

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a,b].$$

$E(X)$ is the expectation of X. Then we have the following inequality.

Theorem 3.1. *With the above assumptions and if there exist the constants M, m such that $0 \le m \le f(t) \le M \le 1$ for all $t \in [a,b]$, then we have the inequality*

$$\left|(1-\lambda)\left[P_r(X \le x) - \frac{M+m}{2}\left(x - \frac{a+b}{2}\right)\right] + \frac{\lambda}{2} - \frac{b-E(X)}{b-a}\right| \le \frac{M-m}{2(b-a)}\left[\frac{(b-a)^2}{4}(\lambda^2 + (1-\lambda)^2) + \left(x - \frac{a+b}{2}\right)^2\right], \tag{18}$$

for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \le x \le b - \lambda\frac{b-a}{2}$.

Proof. It is easy to find that the function $F(x) = \int_a^x f(t)\,dt$ is (m,M)-Lipschitzian on $[a,b]$. So, by Theorem 2.1 we get

$$\left|(b-a)\left[\frac{\lambda}{2}(F(a)+F(b)) + (1-\lambda)F(x) - \frac{M+m}{2}(1-\lambda)\left(x - \frac{a+b}{2}\right)\right] - \int_a^b F(t)\,dt\right| \le \frac{M-m}{2}\left[\frac{(b-a)^2}{4}(\lambda^2 + (1-\lambda)^2) + \left(x - \frac{a+b}{2}\right)^2\right],$$

for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \le x \le b - \lambda\frac{b-a}{2}$.

As $F(a) = 0, F(b) = 1$, and

$$\int_a^b F(t)\,dt = b - E(X),$$

then we can easily deduce the inequality (18). ∎

Corollary 3.2. *Under the assumptions of Theorem 3.1, we have*

$$\left|E(X) - \frac{a+b}{2}\right| \le \frac{M-m}{8}(b-a)^2. \tag{19}$$

Proof. We set $\lambda = 1$ (which also implies $x = \frac{a+b}{2}$) in (18) to get (19). ∎

Remark 3.3. It should be noted that the inequality (19) improves the inequality (5.4) in [1].

Corollary 3.4. *Under the assumptions of Theorem 3.1, we have*

$$\left| P_r(X \le x) - \frac{M+m}{2}\left(x - \frac{a+b}{2}\right) - \frac{b-E(X)}{b-a} \right| \le \frac{M-m}{2(b-a)}\left[\frac{(b-a)^2}{4} + \left(x - \frac{a+b}{2}\right)^2\right], \tag{20}$$

and especially

$$\left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{b-E(X)}{b-a} \right| \le \frac{M-m}{8}(b-a). \tag{21}$$

Proof. We set $\lambda = 0$ in (18) to get (20) and $x = \frac{a+b}{2}$ in (20) to get (21). ∎

Corollary 3.5. *Under the assumptions of Theorem 3.1, we have*

$$\left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{1}{2} \right| \le \frac{M-m}{4}(b-a). \tag{22}$$

Proof. Using the triangle inequality, we get

$$\begin{aligned} &\left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{1}{2} \right| \\ &= \left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{1}{2} + \frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right) - \frac{1}{b-a}\left(E(X) - \frac{a+b}{2}\right) \right| \\ &\le \left| P_r\left(X \le \frac{a+b}{2}\right) - \frac{b-E(X)}{b-a} \right| + \frac{1}{b-a}\left| E(X) - \frac{a+b}{2} \right|, \end{aligned}$$

and then the inequality (22) follows from (19) and (21). ∎

Remark 3.6. Finally, we would like to point out that the inequalities (4) and (17) can also be applied in numerical integrations as in [4, 7].

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