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Piecewise-Koszul Complexes*

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Abstract. In this paper, the so called piecewise-Koszul complex is defined, which is a natural generalization of Koszul and higher Koszul complexes. We give a necessary and sufficient condition for a special class of graded algebras to be piecewise-Koszul in terms of piecewise-Koszul complex. Moreover, for a special piecewise-Koszul algebra, we give a description of the Yoneda-Ext algebra by using its dual algebra.

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1. Introduction

The Koszul algebra, introduced by Priddy in [8], is one of quadratic algebras with a linear resolution. Many nice homological properties and applications of Koszul algebras have been shown in different branches of mathematics, such as algebraic topology, algebraic geometry, quantum group, and Lie algebra ([2], etc.). Motivated by the cubic Artin-Schelter regular algebras [1], Berger extended the concept to higher homogeneous algebras [3], one can find more discussions under the name d-Koszul algebras in [5], or higher Koszul algebras in [6], the

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latter explained Koszulity by A_∞-language. Recently, a new class of Koszul-type algebras, named *piecewise-Koszul* algebras, was introduced by Lü, He, and Lu in [7]. Generally, it is a new class of quadratic algebras and is different from Koszul algebras. It was turned out that this class of algebras possess many perfect homological properties similar to Koszul and higher Koszul algebras, and it provided a negative answer to the following question by constructing a piecewise-Koszul algebra of any period, which was raised by Green and Marcos in [4]:

- *Is there a bound $N_0 \in \mathbb{N}$, such that if $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ is a δ-Koszul algebra, then the Ext algebra $E(A) = \bigoplus_{n\geq 0} \mathrm{Ext}_A^n(A_0, A_0)$, is generated by $\mathrm{Ext}_A^0(A_0, A_0)$, $\mathrm{Ext}_A^1(A_0, A_0)$, $\cdots$, $\mathrm{Ext}_A^{N_0}(A_0, A_0)$?*

It is well-known that Koszul and higher Koszul complexes play an important role in discussing the properties of Koszul and higher Koszul algebras, which were introduced in [2] and [3] (also [9]), and we have the following statements:

- *A is a (d-)Koszul algebra if and only if the (d-)Koszul complex related to A is a projective resolution of A_0.*

In this paper, we generalize the notion of (high-)Koszul complex to the case of piecewise-Koszul, i.e., *piecewise-Koszul complex* is defined. It is shown that for our new objects, we have a similar result for a class of special piecewise-Koszul algebras and we discuss them in Sec. 3.

Using the Yoneda-Ext algebra $E(A) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^i(A_0, A_0)$ to investigate the Koszulity of a graded algebra A is an effective method. Generally speaking, the Yoneda-Ext algebra $E(A)$ is more complicated and it is quite significant to give a description for $E(A)$. In Sec. 4., as an application of the piecewise-Koszul complex, we describe the Yoneda-Ext algebra in terms of the dual algebra $A^!$.

Throughout, $\mathbb{Z}$ denotes the set of integers and we work over a fixed field $\mathbb{F}$ and p, $d \in \mathbb{Z}$ are fixed integers.

2. Definitions and Properties

Throughout, all the graded $\mathbb{F}$-algebras $A = \bigoplus_{i\geq 0} A_i$ are assumed with the following properties:

(a) A_0 is a semisimple Artin algebra;

(b) A is generated in degrees 0 and 1; that is, $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$;

(c) A_1 is of finite dimension as an $\mathbb{F}$-space.

Given a pair of integers d and p ($d > p \geq 3$), we introduce a function $\delta_p^d : \mathbb{N} \to \mathbb{N}$ by

$$\delta_p^d(n) = \begin{cases} \dfrac{nd}{p}, & \text{if } n \equiv 0 (\operatorname{mod} p), \\ \dfrac{(n-1)d}{p} + 1, & \text{if } n \equiv 1 (\operatorname{mod} p), \\ \cdots & \cdots \\ \dfrac{(n-p+1)d}{p} + p-1, & \text{if } n \equiv p-1 (\operatorname{mod} p). \end{cases}$$

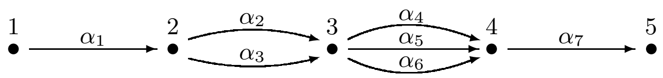
Definition 2.1. A graded algebra $A = \bigoplus_{i\geq 0} A_i$ is called a *piecewise-Koszul algebra* if the trivial A-module A_0 admits a minimal graded projective resolution

$$\mathbf{P}: \quad \cdots \to P_n \to \cdots \to P_1 \to P_0 \to A_0 \to 0,$$

such that each P_n is generated in degree $\delta_p^d(n)$ for all $n \geq 0$.

The following is an example of piecewise-Koszul algebra [7].

Example 2.2. [7] Let Γ be the quiver:



Now let $A = \mathbb{F}\Gamma/R$, where R is the ideal generated by the following relations:

$$\alpha_1\alpha_2 - \alpha_1\alpha_3, \quad \alpha_4\alpha_7 - \alpha_5\alpha_7, \quad \alpha_5\alpha_7 - \alpha_6\alpha_7, \quad \alpha_2\alpha_4, \quad \alpha_3\alpha_6.$$

Under some routine computations, we get a minimal projective resolution of the trivial A-module A_0:

$$0 \to P_3 \to P_2 \to P_1 \to P_0 \to A_0 \to 0,$$

with $P_3 = A \cdot (P_3)_4$, $P_2 = A \cdot (P_2)_2$, $P_1 = A \cdot (P_1)_1$ and $P_0 = A$. Therefore, A is a piecewise-Koszul algebra with $p = 3$, $d = 4$.

Definition 2.3. Let $A = T_{A_0}(A_1)/\langle R\rangle$ be a quadratic algebra, define the so called *piecewise-Koszul complex*

$$\mathbf{PK}^*: \cdots \to \mathcal{PK}^{pm+p-1} \stackrel{\partial^{pm+p-1}}{\to} \cdots \to \mathcal{PK}^{pm} \stackrel{\partial^{pm}}{\to} \mathcal{PK}^{pm-1} \stackrel{\partial^{pm-1}}{\to} \cdots \to \mathcal{PK}^1 \stackrel{\partial^1}{\to} \mathcal{PK}^0$$

as follows. For any $i \geq 0$, $\mathcal{PK}^i$ is a graded projective A-module given by

$$\mathcal{PK}^i = A \otimes_{A_0} \mathcal{PK}^i_{\delta_p^d(i)},$$

where $\mathcal{PK}^i_{\delta_p^d(i)}$ is an A_0-A_0-bimodule concentrated in degree $\delta_p^d(i)$, i.e., for $i \geq 3$,

$$\mathcal{PK}^i_{\delta^d_p(i)} = \bigcap_{u+v+2=\delta^d_p(i)} A_1^{\otimes u} \otimes R \otimes A_1^{\otimes v} \subseteq A_1^{\otimes \delta^d_p(i)}.$$

In particular, we have

$$\mathcal{PK}^0 = A, \;\; \mathcal{PK}^1 = A \otimes A_1, \;\; \mathcal{PK}^2 = A \otimes R.$$

The differential of $\mathbf{PK}^*$, $\partial^i : \mathcal{PK}^i \to \mathcal{PK}^{i-1}$ is the restriction of the map

$$\widetilde{\partial^i} : \; A \otimes A_1^{\otimes \delta^d_p(i)} \to A \otimes A_1^{\otimes \delta^d_p(i-1)}$$

via

$$a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{\delta^d_p(i)} \mapsto aa_1 \cdots a_{\delta^d_p(i)-\delta^d_p(i-1)} \otimes a_{\delta^d_p(i)-\delta^d_p(i-1)+1} \otimes \cdots \otimes a_{\delta^d_p(i)}.$$

More precisely,

$$\partial^{pm}(a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{\delta^d_p(pm)}) = aa_1a_2 \cdots a_{d-p+1} \otimes a_{d-p+2} \otimes \cdots \otimes a_{\delta^d_p(pm)};$$

$$\partial^{pm+k}(a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{\delta^d_p(pm+k)}) = aa_1 \otimes a_2 \otimes \cdots \otimes a_{\delta^d_p(pm+k)},$$

$(k = 1, \; 2, \ldots, \; p-1)$, where $a \in A$ and $a_i \in A_1, \; i = 1, \; 2, \ldots$.

Remark 2.4. (1) If $\mathbf{PK}^* \to A_0 \to 0$ is exact, then A is a piecewise-Koszul algebra;
(2) Let $\mathbf{Z}^i = \ker \partial^i$. It is clear that the support of $\mathbf{Z}^i$ is $\{n | n \geq \delta^d_p(i) + 1\}$. More precisely, the support of $\mathbf{Z}^{pm+j}$ is $\{n | n \geq md + j + 1\}$ for $0 \leq j \leq p-2$ and that of $\mathbf{Z}^{pm+p-1}$ is $\{n | n \geq (m+1)d\}$.

Proposition 2.5. *Let A be a graded $\mathbb{F}$-algebra, then the following are equivalent,*

(1) *A is a piecewise-Koszul algebra;*
(2) *For all $n \geq 0$, $\mathrm{Ext}^n_A(A_0, A_0) = \mathrm{Ext}^n_A(A_0, A_0)_{\delta^d_p(n)}$;*
(3) *$\mathrm{Ext}^n_A(A_0, A_0[m]) = 0$ unless $m = \delta^d_p(n)$, where $[\;\;]$ is the shift functor.*

Proof. By the definition of piecewise-Koszul algebra and the bigrading on $\mathrm{Ext}^*_A(A_0, A_0)$, the proof is immediate. ∎

Proposition 2.6. *The opposed graded algebra of a piecewise-Koszul algebra is also piecewise-Koszul.*

Proof. It is similar to [9] and we only give a sketch of the proof. Consider a resolution

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to A_0 \to 0$$

of the trivial A-module A_0 such that $P_i = A \cdot (P_i)_{\delta^d_p(i)}$.

Let $V^* := \mathrm{Hom}_{A_0}(V, A_0)$ and $(V^\sharp)_i = (V_{-i})^*$. Thus, we get an injective resolution of A_0:

$$0 \to A_0 \to (P_0)^\sharp \to (P_1)^\sharp \to \cdots ,$$

which implies that $\mathrm{Ext}^i_{A^{opp}}(A_0, A_0) = \mathrm{Ext}^i_{A^{opp}}(A_0, A_0)_{\delta_p^d(i)}$ for all $i \geq 0$. ■

Definition 2.7. Let $A = T_{A_0}(A_1)/\langle R\rangle$ be a quadratic algebra, where $T_{A_0}(A_1)$ is the tensor algebra of A_0-A_0-bimodule ${}_{A_0}A_{1A_0}$. Define

$$R^{\perp} := \{f \in (A_1^*)^{\otimes 2} | f(R) = 0\}$$

and

$$A^! := T_{A_0}(A_1^*)/\langle R^{\perp}\rangle,$$

where $(A_1)^* := \mathrm{Hom}_{A_0}(A_1, A_0)$ and $\langle R^{\perp}\rangle$ is the ideal of $T_{A_0}(A_1^*)$ generated by $R^{\perp}$. We call $A^!$ the *dual algebra* of A.

Proposition 2.8. *Let $A^!$ be the same as in Definition 2.7. Then $(A^!_{\delta_p^d(i)})^* := ((A^!)_{\delta_p^d(i)})^* = \mathcal{PK}^i_{\delta_p^d(i)}$ for all $i \geq 0$.*

Proof. It is clear that

$$A^!_{\delta_p^d(i)} = \frac{(A_1)^{*\otimes \delta_p^d(i)}}{\sum_{i+j+2=\delta_p^d(i)} (A_1)^{*\otimes i} \otimes R^{\perp} \otimes (A_1)^{*\otimes j}}.$$

Note that

$$(V/W)^* = \{f \in V^* | f(W) = 0\},$$

we have

$$\begin{aligned}
(A^!_{\delta_p^d(i)})^* &= \left(\frac{(A_1)^{*\otimes \delta_p^d(i)}}{\sum\limits_{i+j+2=\delta_p^d(i)} (A_1)^{*\otimes i} \otimes R^{\perp} \otimes (A_1)^{*\otimes j}} \right)^* \\
&= \left\{ f \in ((A_1)^{*\otimes \delta_p^d(i)})^* = A_1^{\delta_p^d(i)} \,\middle|\, f\left(\sum_{i+j+2=\delta_p^d(i)} (A_1)^{*\otimes i} \otimes R^{\perp} \otimes (A_1)^{*\otimes j} \right) = 0 \right\} \\
&= \bigcap_{i+j+2=\delta_p^d(i)} A_1^i \otimes R \otimes A_1^j = \mathcal{PK}^i_{\delta_p^d(i)}.
\end{aligned}$$

■

3. Piecewise-Koszul Complexes

In this section, we will study the piecewise-Koszul complex in detail and give a criteria for a graded algebra to be piecewise-Koszul by using the piecewise-Koszul complex under certain conditions.

Lemma 3.1. *Let A be a graded $\mathbb{F}$-algebra and $\mathbf{P}$ be the minimal graded projective resolution of the trivial A-module A_0,*

$$\mathbf{P}: \quad \cdots \to P_n \to \cdots \to P_1 \to P_0 \to A_0 \to 0.$$

Then

$$\operatorname{Ext}^n_A(A_0, A_0) \cong \operatorname{Hom}_A(\ker(P_{n-1} \to P_{n-2}), A_0).$$

Proof. Note that A_0 is semisimple and the resolution $\mathbf{P}$ is minimal, now the lemma is clear. ∎

Theorem 3.2. *Let A be a piecewise-Koszul algebra and $\mathbf{PK}^*$ be the piecewise-Koszul complex. Then*

$$\mathbf{PK}^* \longrightarrow A_0 \longrightarrow 0$$

is exact if and only if

$$\mathbf{Z}^{pm+p-1}_{(m+1)d} \subseteq \partial^{pm+p}(\mathcal{PK}^{pm+p}_{(m+1)d}), \ \ \forall m \geq 0.$$

Proof. Suppose A is a piecewise-Koszul algebra and

$$\mathbf{PK}^* \longrightarrow A_0 \longrightarrow 0$$

is exact, then it is clear that

$$\mathbf{Z}^{pm+p-1}_{(m+1)d} \subseteq \partial^{pm+p}(\mathcal{PK}^{pm+p}_{(m+1)d}), \ \ \forall m \geq 0.$$

Conversely, we only need to prove $H^i(\mathcal{PK}^*) = 0$ for all $i \geq 0$. Since we have the exact sequence,

$$\mathcal{PK}^2 \to \mathcal{PK}^1 \to \mathcal{PK}^0 \to A_0 \to 0,$$

to finish the proof, we need to prove $H^i(\mathcal{PK}^*) = 0$ for all $i \geq 2$.

Firstly, we claim $\mathbf{Z}^i = A \cdot \mathbf{Z}^i_{\delta^d_p(i)}$. In fact, it suffices to prove that

$\operatorname{Hom}_A(\mathbf{Z}^i, A_0[n]) = 0$ unless $n = \delta^d_p(i+1)$. By Lemma 3.1,

$$\operatorname{Ext}^{i+1}_A(A_0, A_0[n]) = \operatorname{Hom}_A(\mathbf{Z}^i, A_0[n]).$$

Since A is a piecewise-Koszul algebra, by Proposition 2.5, $\operatorname{Ext}^{i+1}_A(A_), A_0[n]) = 0$ unless $n = \delta^d_p(i{+}1)$, which implies that $\operatorname{Hom}_A(\mathbf{Z}^i, A_0[n]) = 0$ unless $n = \delta^d_p(i{+}1)$. This implies that $\mathbf{Z}^i$ is generated in degree $\delta^d_p(i+1)$.

Secondly, we claim $H^{pm}(\mathcal{PK}^*) = 0$, it suffices to show that $\mathbf{Z}^{pm} = \operatorname{Im}\partial^{pm+1}$. We only need to show

$$\mathbf{Z}^{pm} \subseteq \operatorname{Im}\partial^{pm+1}$$

since

$$\mathbf{Z}^{pm} \supseteq \operatorname{Im}\partial^{pm+1}$$

is clear. In fact, if

$$\sum a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{md} \in \mathbf{Z}^{pm}_{md+1} \subseteq A_1 \otimes \mathcal{PK}^{pm}_{md},$$

then

$$\sum aa_1 \cdots a_{d-2} \otimes a_{d-1} \otimes \cdots \otimes a_{md} = 0,$$

hence

$$\sum (a \otimes a_1) \otimes a_2 \otimes \cdots \otimes a_{md} \in R \otimes A_1^{md-1}.$$

It follows that

$$\sum 1 \otimes a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{md} \in A_0 \otimes \mathcal{PK}_{md+1}^{pm+1} = \mathcal{PK}_{md+1}^{pm+1}$$

and

$$\partial^{pm+1}(\sum 1 \otimes a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{md}) = \sum a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{md}.$$

Thirdly, we claim $H^{pm+k}(\mathcal{PK}^*) = 0$ for all $k = 1, 2, \ldots, p-2$, whose proof is similar to that of $H^{pm}(\mathcal{PK}^*) = 0$, so we omit it.

Finally, by assumption, $\mathbf{Z}_{(m+1)d}^{pm+p-1} \subseteq \partial^{pm+p}(\mathcal{PK}_{(m+1)d}^{pm+p})$, $\forall m \geq 0$, we have $\mathbf{Z}^{pm+p-1} = \operatorname{Im} \partial^{pm+p}$, thus $H^{pm+p-1}(\mathbf{PK}^*) = 0$ for all $m \geq 0$. ■

Lemma 3.3. ([9]) *Let S be a semisimple ring. Then*
(1) *Let M be a right S-module and N a left S-module. For any submodules $H, L \in N$, we have*

$$M \otimes (H \cap L) = (M \otimes H) \cap (M \otimes L);$$

(2) *Let M, N be right S-modules, $f \in \operatorname{Hom}_S(M, N)$, and H a left S-module. Considering the map $f \otimes 1 : M \otimes H \longrightarrow N \otimes H$, we have* $\ker(f \otimes 1) = \ker f \otimes H$.

Similarly, Let M, N be left S-modules, $f \in \operatorname{Hom}_S(M, N)$, and H a right S-module. Then $\ker(1 \otimes f) = H \otimes \ker f$.

Definition 3.4. Let A be a piecewise-Koszul algebra, writing $A = T_{A_0}(A_1)/\langle R \rangle$. If we have

$$(A_1^{\otimes n} \otimes R) \bigcap (R \otimes A_1^{\otimes n}) = \bigcap_{i+j=n} \left(A_1^{\otimes i} \otimes R \otimes A_1^{\otimes j} \right)$$

for all $1 \leq n \leq d-2$, then A will be called a *special piecewise-Koszul* algebra.

Theorem 3.5. *Let A be a graded $\mathbb{F}$-algebra.*
(1) *If A is a special piecewise-Koszul algebra, then $\mathbf{PK}^* \to A_0 \to 0$ is exact;*
(2) *Conversely, if $\mathbf{PK}^* \to A_0 \to 0$ is exact, then A is a piecewise-Koszul algebra.*

Proof. We only prove assertion (1), since (2) is obvious. In order to prove (1), by Theorem 3.2, we only need to check

$$\mathbf{Z}_{(m+1)d}^{pm+p-1} \subseteq \partial^{pm+p}(\mathcal{PK}_{(m+1)d}^{pm+p}), \ \ \forall m \geq 0.$$

Similarly, we only consider the case of $p = 3$, i.e., we need to show

$$\mathbf{Z}^{3m+2}_{(m+1)d} \subseteq \partial^{3m+3}(\mathcal{PK}^{3m+3}_{(m+1)d}), \ \ \forall m \geq 0.$$

Let $x \in \mathbf{Z}^{3m+2}_{(m+1)d}$, then $x \in A_{d-2} \otimes \mathcal{PK}^{3m+2}_{md+2}$. Consider the following map

$$A_1^{\otimes (m+1)d} \longrightarrow A_{d-2} \otimes A_1^{\otimes md+2} \longrightarrow A_{d-1} \otimes A_1^{\otimes md+1},$$

where the first map is the multiplication μ of the algebra A, it is surjective and the second map is $\widetilde{\partial}^{3m+2}$. It is clear that there exists $\overline{x} \in A_1^{\otimes (m+1)d}$, such that $x = \mu(\overline{x})$. Note that $x \in A_{d-2} \otimes \mathcal{PK}^{3m+2}_{md+2}$, which implies that $\overline{x} \in A_1^{\otimes d-2} \otimes \mathcal{PK}^{3m+2}_{md+2}$. Since

$$\widetilde{\partial}^{3m+2}|_{A_{d-2}\otimes \mathcal{PK}^{3m+2}_{md+2}} = \partial^{3m+2}|_{A_{d-2}\otimes \mathcal{PK}^{3m+2}_{md+2}},$$

it follows that

$$\widetilde{\partial}^{3m+2}(\mu(\overline{x})) = \partial^{3m+2}(\mu(\overline{x})) = \partial^{3m+2}(x) = 0$$

since $x \in \mathbf{Z}^{3m+2}_{(m+1)d}$. Thus $\overline{x} \in R \otimes A_1^{\otimes (m+1)d-2}$.

$$\begin{aligned}
&\overline{x} \in \left(R \otimes A_1^{\otimes (m+1)d-2}\right) \bigcap \left(A_1^{\otimes d-2} \otimes \mathcal{PK}^{3m+2}_{md+2}\right) \\
&= \left(R \otimes A_1^{\otimes (m+1)d-2}\right) \bigcap \left(A_1^{\otimes d-2} \otimes \left(\bigcap_{i+j=md} A_1^{\otimes i} \otimes R \otimes A_1^{\otimes j}\right)\right) \\
&= \left(R \otimes A_1^{\otimes (m+1)d-2}\right) \bigcap \left(\bigcap_{i+j=(m+1)d-2,\ i\geq d-2} A_1^{\otimes i} \otimes R \otimes A_1^{\otimes j}\right) \\
&\subseteq \left(R \otimes A_1^{\otimes (m+1)d-2}\right) \bigcap \left(A_1^{\otimes d-2} \otimes R \otimes A_1^{\otimes md}\right) \\
&= \left(R \otimes A_1^{\otimes d-2} \otimes A_1^{\otimes md}\right) \bigcap \left(A_1^{\otimes d-2} \otimes R \otimes A_1^{\otimes md}\right) \\
&= \left(R \otimes A_1^{\otimes d-2}\right) \bigcap \left(A_1^{\otimes d-2} \otimes R\right) \otimes A_1^{\otimes md} \\
&= \bigcap_{i+j=d-2} \left(A_1^{\otimes i} \otimes R \otimes A_1^{\otimes j}\right) \otimes A_1^{\otimes md}.
\end{aligned}$$

Hence

$$\overline{x} \in \bigcap_{i+j=(m+1)d-2} (A_1^{\otimes i} \otimes R \otimes A_1^{\otimes j}) = \mathcal{PK}^{3m+3}_{(m+1)d}.$$

Now consider the element $1 \otimes \overline{x} \in A_0 \otimes \mathcal{PK}^{3m+3}_{(m+1)d}$, and we have

$$x = \partial^{3m+3}\left(1 \otimes \overline{x}\right) \in \partial^{3m+3}(\mathcal{PK}^{3m+3}_{(m+1)d}\Big).$$

■

4. The Yoneda-Ext Algebra $E(A)$

In this section, we will study some easy properties of the Yoneda-Ext algebra $E(A)$, where A is a piecewise-Koszul algebra. In particular, for special piecewise-Koszul algebras, we give a description of $E(A)$ by using the dual algebra $A^!$, which is an application of piecewise-Koszul complexes.

Proposition 4.1. *Let A be a piecewise Koszul algebra and $E(A)$ be the Yoneda-Ext algebra. Then*

$$\mathrm{Ext}_A^{pm+i}(A_0, A_0)\cdot \mathrm{Ext}_A^{pm'+j}(A_0, A_0) = 0$$

for all $p \le i+j \le 2p-2$, where $i,\ j = 1,\ 2, \dots, p-1$.

Proof. It is obvious that $\mathrm{Ext}_A^{pm+i}(A_0, A_0)$ and $\mathrm{Ext}_A^{pm'+j}(A_0, A_0)$ are concentrated in the bidegrees $(pm+i, md+i)$ and $(pm'+j, m'd+j)$, respectively. Therefore,

$$\begin{aligned}
&\mathrm{Ext}_A^{pm+i}(A_0, A_0)\cdot \mathrm{Ext}_A^{pm'+j}(A_0, A_0)\\
&= \mathrm{Ext}_A^{pm+i}(A_0, A_0)_{md+i}\cdot \mathrm{Ext}_A^{pm'+j}(A_0, A_0)_{m'd+j}\\
&\subseteq \mathrm{Ext}_A^{p(m+m')+i+j}(A_0, A_0)_{(m+m')d+i+j}\\
&= \mathrm{Ext}_A^{p(m+m'+1)+i+j-p}(A_0, A_0)_{(m+m')d+i+j} = 0
\end{aligned}$$

since

$$\mathrm{Ext}_A^{p(m+m'+1)+i+j-p}(A_0, A_0) = \mathrm{Ext}^{p(m+m'+1)+i+j-p}_{(m+m'+1)d+i+j-p}(A_0, A_0)$$

and $d > p$ in general. ∎

Theorem 4.2. *Let A be a special piecewise-Koszul algebra and $E(A)$ its Yoneda-Ext algebra. Let $B = \bigoplus_{i\ge 0, j\in\mathbb{Z}} B_j^i$, where $B^i = B^i_{\delta_p^d(i)} := A^!_{\delta_p^d(i)}$ and $B^i_j = 0$ if $j \ne \delta_p^d(i)$. Then $E(A) \cong B$ as bigraded $\mathbb{F}$-algebras.*

Proof. Firstly we claim

$$\mathrm{Ext}_A^i(A_0, A_0) = B^i_{\delta_p^d(i)},\ \ \forall\, i \ge 0.$$

In fact, since A is a special piecewise-Koszul algebra, by Theorem 3.5, the piecewise-Koszul complex $\mathbf{PK}^*$ of A is a projective resolution of the trivial A-module A_0. Hence

$$\begin{aligned}
\mathrm{Ext}_A^i(A_0, A_0) &= \frac{\ker(\mathrm{Hom}_A(\mathcal{PK}^i, A_0) \to \mathrm{Hom}_A(\mathcal{PK}^{i+1}, A_0))}{\mathrm{Im}(\mathrm{Hom}_A(\mathcal{PK}^{i-1}, A_0) \to \mathrm{Hom}_A(\mathcal{PK}^i, A_0))}\\
&= \mathrm{Hom}_A(\mathcal{PK}^i, A_0) = \mathrm{Hom}_A(A\otimes_{A_0}\mathcal{PK}^i_{\delta_p^d(i)},\ A_0)\\
&= \mathrm{Hom}_{A_0}(\mathcal{PK}^i_{\delta_p^d(i)},\ \mathrm{Hom}_A(A,\ A_0))\\
&= \mathrm{Hom}_{A_0}(\mathcal{PK}^i_{\delta_p^d(i)},\ A_0)
\end{aligned}$$

$$= \mathrm{Hom}_{A_0}((A^!_{\delta^d_p(i)})^*,\ A_0)$$
$$= A^!_{\delta^d_p(i)} = B^i_{\delta^d_p(i)}.$$

In order to finish the proof, define a multiplication $\star$ on B as follows: for $\psi \in B^i$ and $\varphi \in B^j$,

$$\psi \star \varphi = \begin{cases} \psi \cdot \varphi, & \text{at least one of } i \text{ and } j \text{ is of the form } kp,\ k \in \mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\cdot$ is the product of the dual algebra $A^!$. Under the multiplication $\star$, it is easy to see that B is a bigraded algebra. Note that we have the following isomorphisms

$$A^!_{\delta^d_p(n)} = (\mathcal{PK}^n_{\delta^d_p(n)})^*,\ \ \mathrm{Ext}^n_A(A_0, A_0) = A^!_{\delta^d_p(n)}$$

and

$$\mathrm{Ext}^n_A(A_0, A_0) = \mathrm{Hom}_A\left(A \otimes_{A_0} \mathcal{PK}^n_{\delta^d_p(n)}, A_0\right).$$

Let $\varphi \in \mathrm{Ext}^n_A(A_0, A_0) = A^!_{\delta^d_p(n)}$, $\psi \in \mathrm{Ext}^m_A(A_0, A_0) = A^!_{\delta^d_p(m)}$. Let $\overline{\varphi} : A \otimes_{A_0} \mathcal{PK}^n_{\delta^d_p(n)} \to A_0$ and $\overline{\psi} : A \otimes_{A_0} \mathcal{PK}^m_{\delta^d_p(m)} \to A_0$ be the A-module morphisms induced by φ and ψ, respectively. Now we show that

$$\psi \star \varphi = \psi \cdot \varphi,$$

where $\star$ denotes the product of $\mathrm{Ext}^*_A(A_0, A_0)$, $\cdot$ the product of $A^!$. Consider the following diagram

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial_{n+m+1}} & A \otimes \mathcal{PK}^{n+m}_{\delta^d_P(n+m)} & \xrightarrow{\partial_{n+m}} & \cdots & \xrightarrow{\partial_{n+1}} & A \otimes \mathcal{PK}^n_{\delta^d_P(n)} & \xrightarrow{\partial_n} & \cdots & \\
& \downarrow \overline{\varphi}_m & & & & \overline{\varphi}_0 \downarrow & \searrow \overline{\varphi} & & \\
\cdots \xrightarrow{\partial_{m+1}} & A \otimes \mathcal{PK}^m_{\delta^d_P(m)} & \xrightarrow{\partial_m} & \cdots & \xrightarrow{\partial_1} & A \otimes \mathcal{PK}^0_{\delta^d_P(0} & \xrightarrow{\epsilon} & A_0 \to 0 & \\
& \downarrow \overline{\psi} & & & & & & & \\
& A_0 & & & & & & &
\end{array}$$

where $\overline{\varphi}_i$ for $i = 0, 1, \ldots, m$ is defined as follows.

(i) If $n = pk$, define

$$\overline{\varphi}_l(a \otimes v_1 \otimes \cdots \otimes v_{\delta^d_P(n+l)}) = a \otimes v_1 \otimes \cdots \otimes v_{\delta^d_P(l)} \varphi(v_{\delta^d_P(l)+1} \otimes \cdots \otimes v_{\delta^d_P(n+l)}),$$

where $l = 0,\ 1,\ \ldots, m$. Clearly, $\partial^l \overline{\varphi_l} = \overline{\varphi}_{l-1} \partial^{n+l}$. Thus,

$$\begin{aligned}
&\overline{\psi}\overline{\varphi}_m\left(a \otimes v_1 \otimes \cdots \otimes v_{\delta^d_P(n+m)}\right) \\
&= \overline{\psi}\left(a \otimes v_1 \otimes \cdots \otimes v_{\delta^d_P(m)}\right) \varphi\left(v_{\delta^d_P(m)+1} \otimes \cdots \otimes v_{\delta^d_P(m+n)}\right) \\
&= a \otimes \psi\left(a \otimes v_1 \otimes \cdots \otimes v_{\delta^d_P(m)}\right) \varphi\left(v_{\delta^d_P(m)+1} \otimes \cdots \otimes v_{\delta^d_P(m+n)}\right)
\end{aligned}$$

$$= a\psi \cdot \varphi \left(v_1 \otimes \cdots \otimes v_{\delta_P^d(n+m)} \right).$$

(ii) If $n = pk + i$, $i = 1,\ 2,\ \ldots, p-1$ and $l = pk' + j$, $j = 1,\ 2,\ \ldots, p-1$, then define

$$\overline{\varphi}_l(a \otimes v_1 \otimes \cdots \otimes v_{\delta_P^d(n+l)})$$
$$= av_1 \cdots v_{d-p+1} \otimes \cdots \otimes v_{\delta_P^d(n+l)-\delta_P^d(n)} \varphi(v_{\delta_P^d(n+l)-\delta_P^d(n)+1} \otimes \cdots \otimes v_{\delta_P^d(n+l)}).$$

Similarly, $\partial^l \overline{\varphi_l} = \overline{\varphi}_{l-1} \partial^{n+l}$. If $\psi \star \varphi \neq 0$, then $\delta_p^d(m+n) = \delta_p^d(m) + \delta_p^d(n)$. Since $n = pk+i$, $i = 1,\ 2, \ldots, p-1$, clearly $m = pk'$. By the computations similar to (i), we have $\psi \star \varphi = \psi \cdot \varphi$. Therefore, $E(A) \cong B$ as bigraded $\mathbb{F}$-algebras. ■

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