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Characterization of Linearly Convex Domains in $\mathbb{C}^n$ by their Noncompact Automorphism Groups

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Abstract. In this paper, a theorem on the characterization of linearly convex domains in $\mathbb{C}^n$ by their noncompact automorphism groups is given.

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1. Introduction

Let Ω be a domain (connected, open set) in a $\mathbb{C}^n$. Let the *automorphism group* of Ω (denoted by $\mathrm{Aut}(\Omega)$) be the collection of biholomorphic self-maps of Ω with composition of mappings as its binary operation. The topology on $\mathrm{Aut}(\Omega)$ is that of uniform convergence on compact sets (i.e., the compact-open topology). The problem of characterizing domains in $\mathbb{C}^n$ by their noncompact automorphism groups have received much attention in the last few decades, and they are related to many problems of Analysis in several complex variables (see the reference in [2, 3, 4, 5, 12, 18, 19] for the development in related subjects). More precisely, a "characterization of a domain in $\mathbb{C}^n$ by its noncompact automorphism group" means finding a biholomorphic equivalence between our original domain and some rigid polynomial domain.

Let Ω be a domain in $\mathbb{C}^n$. We say that $p_\infty \in \partial\Omega$ is an *accumulating point* for an orbit of $\mathrm{Aut}(\Omega)$ if there exists a family $(h_\nu)_\nu$ of automorphisms of Ω and a point q in Ω such that

$$\lim_{\nu\to\infty} h_\nu(q) = p_\infty.$$

Recall that if $\partial\Omega$ is smooth, pseudoconvex and of finite type in the sense of D'Angelo [8] near p_∞, then for each compact subset $K \subset \Omega$ and each neighborhood U of p_∞, there exists an integer ν_0 such that $h_\nu(K) \subset \Omega \cap U$ for every $\nu \geq \nu_0$ (see [4, Proposition 2.1]).

Recently, in [18, 19] we showed theorems of characterizing Ω in the case where $\partial\Omega$ is pseudoconvex, finite type in the sense of D'Angelo [8] and smooth of class C^∞ in some neighborhood of $p_\infty \in \partial\Omega$ and the Levi form has rank at least $n-2$ at p_∞.

For convex domains in $\mathbb{C}^n$, by using the scaling technique and the construction of polydiscs given by McNeal [14, 15], in [10] H. Gaussier proved the following theorem.

Gaussier theorem. *Let Ω be a domain in $\mathbb{C}^n$, and let p_∞ be a point of $\partial\Omega$. Assume that p_∞ is an accumulating point for a sequence of automorphisms of Ω. If $\partial\Omega$ is smooth, convex, and of finite type $2m$ near p_∞, then Ω is biholomorphically equivalent to a rigid polynomial domain*

$$D = \{z \in \mathbb{C}^n : \operatorname{Re} z_1 + P(z') < 0\},$$

where P is a real nondegenerate convex polynomial of degree less than or equal to $2m$.

The nondegeneracy of P is given by condition "$\{P = 0\}$ without nontrivial analytic set".

We would like to emphasize here that the assumption on convexity of domains in the above-mentioned theorem is essential in his proofs. Thus, there is a natural question that whether the Gaussier theorem is true for more general domains in $\mathbb{C}^n$. The aim in this paper is to show that the above theorem holds for linearly convex domains (not necessary bounded) in $\mathbb{C}^n$.

Recall that $\partial\Omega$ is *linearly convex near* $p_\infty \in \partial\Omega$ if there exists a neighborhood U of p_∞ such that, for all $z \in \partial\Omega \cap U$, the intersection

$$(z + T_z^{10}\partial\Omega) \cap (\Omega \cap U) = \emptyset.$$

We prove the following

Theorem 1.1. *Let Ω be a domain in $\mathbb{C}^n$, and let p_∞ be a point of $\partial\Omega$. Assume that p_∞ is an accumulating point for a sequence of automorphisms of Ω. If $\partial\Omega$ is smooth, linearly convex, and of finite type $2m$ near p_∞, then Ω is biholomorphically equivalent to a rigid polynomial domain*

$$D = \{z \in \mathbb{C}^n : \operatorname{Re} z_1 + P(z') < 0\},$$

where P is a real nondegenerate plurisubharmonic polynomial of degree less than or equal to $2m$.

We recall the concept of Carathéodory kernel convergence of domains which is relevant to the discussion of scaling methods (see [9]). Note that, the local Hausdorff convergence can replace the normal convergence in case the domains in consideration are convex domains.

Definition 1.2 (Carathéodory Kernel Convergence). Let $\{\Omega_\nu\}$ be a sequence of domains in $\mathbb{C}^n$ such that $p \in \bigcap_{\nu=1}^{\infty} \Omega_\nu$. If p is an interior point of $\bigcap_{\nu=1}^{\infty} \Omega_\nu$, the Carathéodory kernel $\hat{\Omega}$ at p of the sequence $\{\Omega_\nu\}$ is defined to be the largest domain containing p having the property that each compact subset of $\hat{\Omega}$ lies in all but a finite number of the domains Ω_ν. If p is not an interior point of $\bigcap_{\nu=1}^{\infty} \Omega_\nu$, the Carathéodory kernel $\hat{\Omega}$ is $\{p\}$. The sequence $\{\Omega_\nu\}$ is said to converge to its kernel at p if every subsequence of $\{\Omega_\nu\}$ has the same kernel at p.

We shall also say that a sequence $\{\Omega_\nu\}$ of domains in $\mathbb{C}^n$ converges normally to $\hat{\Omega}$ (denoted by $\lim \Omega_\nu = \hat{\Omega}$) if there exists a point $p \in \bigcap_{\nu=1}^{\infty} \Omega_\nu$ such that $\{\Omega_\nu\}$ converges to its Carathéodory kernel $\hat{\Omega}$ at p.

The paper is organized as follows. In Sec. 2, we discribe the construction of polydiscs around points near the boundary of a linearly convex domain, and give some of their properties. In Sec. 3, we localize the polydiscs centered at q^ν, where $q \in \Omega$ and $(h_\nu)_\nu$ is a noncompact sequence of automorphisms of Ω accumulating to p_∞ and $q^\nu = h_\nu(q)$. This allows us to rescale the domain Ω by a dilation we define there. Then we show that the scaled domains converges to a rigid polynomial domain. In Sec. 4, we prove the biholomorphic equivalence between Ω and the limit rigid polynomial domain.

2. Coordinates and Polydiscs of Conrad

In his thesis (see [7]), M. Conrad gave the construction of polydiscs on a linearly convex domain in $\mathbb{C}^n$ and also gave their properties. But these results have not been published elsewhere. So we give the detailed proofs here for convenience.

The coordinates in $\mathbb{C}^n$ are denoted by $z = (z_1, z')$, where $z_1 \in \mathbb{C}$ and $z' \in \mathbb{C}^{n-1}$.

Let Ω be a domain in $\mathbb{C}^n$. Assume that $\partial\Omega$ is linearly convex, of finite type $2m$ near a point p_∞ of $\partial\Omega$. We may also assume that $p_\infty = 0$. There exists a neighborhood U of $p_\infty = 0$ in $\mathbb{C}^n$ such that $\Omega \cap U$ is linearly convex and is defined by a smooth function

$$r(z_1, z') = \operatorname{Re} z_1 + h(\operatorname{Im} z_1, z'),$$

where h is a function of class C^∞. We may also assume that there exists a real positive number ϵ_0 such that for every $-\epsilon_0 < \epsilon < \epsilon_0$, the level sets $\{r(z) = \epsilon\}$ are linearly convex.

For each $\epsilon \in (0, \epsilon_0/2)$, $q \in \Omega \cap U$ with $|r(q)| < \epsilon_0/2$ and each unit vector $v \in \mathbb{S}^{n-1} := \{v \in \mathbb{C}^n : |v| = 1\}$, we set

$$\tau(q, v, \epsilon) := \sup\{\rho > 0 : r(q + \lambda v) - r(q) < \epsilon \text{ for all } \lambda \in \mathbb{C} \text{ with } |\lambda| < \rho\}.$$

It is easy to see that $\tau(q, v, \epsilon)$ is the distance from q to $S_{q,\epsilon} := \{r(z) = r(q)+\epsilon\}$ along the complex line $\{q + \lambda v : \lambda \in \mathbb{C}\}$. To every point $q \in \Omega \cap U$ and every sufficiently small positive constant ϵ we associate

(a) A holomorphic coordinate system $(z_1, z_2, \ldots, z_n)$ centered at q and preserving orthogonality,
(b) Points $p_1, p_2, \ldots, p_n$ on the hypersurface $S_{q,\epsilon}$ and,
(c) Positive real numbers $\tau_1(q, \epsilon), \tau_2(q, \epsilon), \ldots, \tau_n(q, \epsilon)$.

The construction proceeds as follows. We first set

$$e_1 := \frac{\nabla r(q)}{|\nabla r(q)|} \text{ and } \tau_1(q, \epsilon) := \tau(q, e_1, \epsilon).$$

Working with sufficiently small ϵ, there exists a unique point p_1 in $S_{q,\epsilon}$ where this distance is achieved. Choose a parameterization of the complex line from q to p_1 such that $z_1(0) = q$ and p_1 lies on the positive $\operatorname{Re} z_1$ axis. By the choice of real axis for z_1, we have $\frac{\partial r}{\partial x_1}(q) = 1$ and thus, if U is small enough,

$$\frac{\partial r}{\partial x_1}(z) \approx 1 \quad \text{for all} \quad z \in U.$$

We also have

$$\tau_1(q, \epsilon) \approx \epsilon, \tag{1}$$

where the constant is independent of q and ϵ. Now consider the orthogonal complement H_1 of the span of the coordinate z_1 in $\mathbb{C}^n$. For any $\gamma \in H_1 \cap \mathbb{S}^{n-1}$, compute $\tau(q, \gamma, \epsilon)$. Because of the assumption of finite type, the largest such distance is finite and is achieved at a vector $e_2 \in H_1 \cap \mathbb{S}^{n-1}$. Set $\tau_2(q, \epsilon) := \tau(q, e_2, \epsilon)$. Let $p_2 \in S_{q,\epsilon}$ be a point such that $p_2 = q + \tau_2(q, \epsilon)e_2$. The coordinate z_2 is defined by parameterizing the complex line from q to p_2 in such a way that $z_2(0) = q$ and p_2 lies on the positive $\operatorname{Re} z_2$ axis. For the next step, define H_2 as the orthogonal complement of the span of z_1 and z_2 and repeat the above construction. Continuing this process, we obtain n coordinate functions z_j, vectors e_j, the numbers $\tau_j(q, \epsilon)$ and the distinguished points $p_j (1 \le j \le n)$. Let $z_j = x_j + iy_j$ $(1 \le j \le n)$ denote the underlying real coordinates. By our parameterization, we immediatly have

$$\frac{\partial r}{\partial z_j}(p_k) = 0 \text{ and } \frac{\partial r}{\partial y_k}(p_k) = 0 \text{ for } j > k \ge 2. \tag{2}$$

The ϵ-distinguished polydiscs and their versions scaled by $c > 0$ are defined as

$$cP_\epsilon(q) = \{z \in \mathbb{C}^n : |z_k - q_k| < c\tau_k(q, \epsilon) \text{ for } 1 \le k \le n\}.$$

Lemma 2.1. *There exists a positive constant c such that*
(i) $\tau_1(q,\epsilon) \leq c\epsilon$,
(ii) *For every* $j \geq 2$, $\tau_j(q,\epsilon) \leq c\epsilon^{1/2m}$.

Proof. (i) It is obvious.

(ii) Suppose that there exist a constant C, a number j, a point $q \in \Omega \cap U$ and $\epsilon > 0$ such that $\tau_j(q,\epsilon)^{2m+1} \geq C.\epsilon$. Since $\epsilon \geq r(q+\lambda e_j) - r(q)$ for all $\lambda \in \mathbb{C}$ with $|\lambda| < \tau_j(q,\epsilon)$, it follows that $\tau_j(q,\epsilon)^{2m+1} \geq C.(r(q+\lambda e_j) - r(q))$ for all $\lambda \in \mathbb{C}$ with $|\lambda| < \tau_j(q,\epsilon)$.

Consequently, there exists the complex line $\mathbb{C} \ni \lambda \mapsto q + \lambda e_j$ which has order of contact greater than $2m$ with $S_{q,0}$. This is absurd hence (ii) is proved. ■

Using the linear convexity of the level sets $\{r(z) = \epsilon\}$, we obtain a complete localization of polydiscs given by the following two lemmas.

Lemma 2.2. *There exists a positive constant* c_1 *such that, for all* $q \in \Omega \cap U$ *and* $0 < \epsilon < \epsilon_0/2$,

$$c_1 P_\epsilon(q) \subset \{r(z) < r(q) + \epsilon\}. \tag{3}$$

Proof. We show that (3) holds for $c_1 = \frac{1}{4^n}$. Let $(z_1, z_2, \ldots, z_n)$ be the coordinates associated to q and ϵ by the above construction. For each $z \in P_\epsilon(q)$, we define

$$h(z) = \sum_{k=1}^{n} \frac{|x_k| + |y_k|}{\tau_k(q,\epsilon)},$$

where $x_k = \operatorname{Re} z_k,\ y_k = \operatorname{Im} z_k$. It is easy to see that $h(z) < 1,\ \forall z \in c_1 P_\epsilon(q)$. We now prove that, for any boundary point $Q \in S_{q,\epsilon} \cap U \cap P_\epsilon(q)$, $h(Q) \geq 1$ and this completed the proof. Suppose that $h(Q) < 1$. Let H be the complex tangent space to $S_{q,\epsilon}$ at Q. Then, there exists a vector $X \in \mathbb{C}^n,\ X \neq 0$ such that $H = \{z \in \mathbb{C}^n : < z, X > = < Q, X >\}$. Without loss of generality, we assume that $X_k \neq 0$ for $k = 1, 2, \ldots, n$. Otherwise, we consider the point $\widetilde{Q} \in H \cap P_\epsilon(q)$ in $U \cap \{z : z_k = 0 \text{ if } X_k = 0\}$ with

$$\widetilde{Q} := \begin{cases} 0 \text{ if } X_k = 0 \\ Q_k \text{ otherwise.} \end{cases}$$

Since the complex dimension of H is $n-1$ and $X_k \neq 0$ for $1 \leq k \leq n$, there exist points $z^k = \lambda_k e_k \in H$ with $\lambda_k \in \mathbb{C}$ for $1 \leq k \leq n$. Thus

$$|z^k| \geq \tau_k(q,\epsilon)\ (1 \leq k \leq n). \tag{4}$$

The hyperplane is now represented by

$$H = \left\{ z = z^1 + \sum_{k=2}^{n} \alpha_k (z^1 - z^k),\ \alpha_k \in \mathbb{C} \right\}.$$

Since $Q \in H$, $Q = z^1 + \sum_{k=2}^n \alpha_k(z^1 - z^k)$ with some complex numbers $\alpha_2, \alpha_3, \dots, \alpha_n$. Thus

$$Q_1 = \left(1 + \sum_{k=2}^n \alpha_k\right) z_1^1 \text{ and } Q_k = -\alpha_k z_k^k,\ k = 2, \dots, n. \tag{5}$$

If $h(Q) < 1$, then $1 > h(Q) = \sum_{k=1}^n \frac{|x_k|+|y_k|}{\tau_k(q,\epsilon)} = \sum_{k=1}^n \frac{|Q_k|}{\tau_k(q,\epsilon)}$. By (5) and (4), we have

$$\frac{|Q_1|}{\tau_1(q,\epsilon)} < 1 - \sum_{k=2}^n \frac{|Q_k|}{\tau_k(q,\epsilon} \leq 1 - \sum_{k=2}^n |\alpha_k|. \tag{6}$$

However, by (5) we have

$$|Q_1| = |1 + \sum_{k=2}^n \alpha_k||z^1| \geq |1 + \sum_{k=2}^n \alpha_k|\tau_1(q,\epsilon). \tag{7}$$

By (6) and (7) we also have

$$1 - \sum_{k=2}^n |\alpha_k| > |1 + \sum_{k=2}^n \alpha_k| \geq 1 - \sum_{k=2}^n |\alpha_k|.$$

This is a contradiction. ■

Lemma 2.3. *There exists a constant* $c_2 > 0$ *such that*

$$c_2 P_\epsilon(q) \supset \{r(z) > r(q) - \epsilon\}.$$

Proof. By using the implicit function theorem and a linear change of coordinates which does not destroy the linear convexity, it follows that

$$r(z) - r(q) = \operatorname{Re} z_1 + h_1(z') + h_2(\operatorname{Im} z_1, z'). \tag{8}$$

Since $S_{q,0}$ is linearly convex, $h_1(z') \geq 0$. Moreover,

$$|h_2(\operatorname{Im} z_1, z')| \lesssim |\operatorname{Im} z_1|. \tag{9}$$

By (1), for $z \in P_\epsilon(q)$, we have

$$r(z) - r(q) \geq \operatorname{Re} z_1 + 0 - C|\operatorname{Im} z_1| \gtrsim -|z_1| \gtrsim \tau_1(q,\epsilon) \approx -\epsilon.$$

The proof is complete. ■

Lemma 2.4. *There exists a constant* $C > 0$ *such that, for every* $q \in \Omega \cap U$,

$$\Gamma_q \cap \Omega_{q,0} \cap U = \emptyset,$$

where $\Gamma_q := \{z \in \mathbb{C}^n : \gamma_q(z) = \operatorname{Re}[\partial r(q)(z-q)] - C|\operatorname{Im}[\partial r(q)(z-q)]| \geq 0\}$*;* $\Omega_{q,\epsilon} := \{r(z) < r(q) + \epsilon\}$.

Proof. By (9) and (8), there exists a positive constant A such that, for every $z \in \Omega_{q,0} \cap U$, we have

$$0 \geq r(z) - r(q) \geq \operatorname{Re} z_1 - A|\operatorname{Im} z_1|.$$

By the choice of coordinates, $\partial r(q) = (1, 0, \ldots, 0)$ and thus, $\gamma_q(z) = \operatorname{Re} z_1 - C|\operatorname{Im} z_1|$. Taking $C \geq A$, we get $\gamma_q(z) \leq \operatorname{Re} z_1 - A|\operatorname{Im} z_1| < 0$, $\forall z \in \Omega_{q,0} \cap U$ and hence, $\Gamma_q \cap \Omega_{q,0} \cap U = \emptyset$. ∎

Remark 2.5. The constant c is invariant under any linear change of coordinates.

Lemma 2.6. (i) *For every* $j \leq n$, $\frac{\partial r}{\partial z_j}(p_j)$ *is real.*
(ii) *There exists a positive constant* c *such that, for every* $j \leq n$

$$\left|\frac{\partial r}{\partial z_j}(p_j)\right| \geq c\frac{\tau_1(q,\epsilon)}{\tau_j(q,\epsilon)}.$$

(iii) *If* $j \leq n-1$, *then* $\frac{\partial r}{\partial z_k}(p_j) = 0$ *for all* $k > j$.

Proof. (i) and (iii) is deduced from the above construction.

(ii) In the coordinates $(z_1, z_2, \ldots, z_n)$, x_1, which is the real normal axis to $S_{q,\epsilon}$ at p_1, is a small perturbation of the real normal axis to $\partial\Omega$ at p_∞. Restricting U if necessary and using the form of the function r, we may assume that for all q in U and all sufficiently small ϵ,

$$\frac{1}{2} \leq \left|\frac{\partial r}{\partial x_1}(p_1)\right| \leq 2.$$

This proves (ii) for $j = 1$. Consider now the cones Γ_{p_k} for $k = 2, \ldots, n$. By (i) we have $\operatorname{Im}\partial r(p_k)p_k = 0$. Set

$$\alpha_k := \tau_1(q,\epsilon)\overline{\frac{\partial r}{\partial z_1}(p_k)}.$$

Then $|\alpha_k| \approx \tau_1(q,\epsilon)$ and $\operatorname{Im}\alpha_k \frac{\partial r}{\partial z_1}(p_k) = 0$. For a suitable (and independent of q, ϵ) constant $c > 0$, $w^k \in S_{q,\epsilon}$, where $w^k := (c\alpha_k, 0, \ldots, 0)$. By Lemma 2.4, we have $\gamma_{p_k}(w^k) \leq 0$, i.e.,

$$\operatorname{Re}\frac{\partial r}{\partial z_1}(p_k)c\alpha_k \leq \operatorname{Re}\frac{\partial r}{\partial z_k}(p_k)\tau_k(q,\epsilon). \tag{10}$$

Since $\operatorname{Re}\frac{\partial r}{\partial z_1}(p_k)c\alpha_k \approx |\frac{\partial r}{\partial z_1}(p_k)|^2\tau_1(q,\epsilon) \gtrsim \tau_1(q,\epsilon)$, this implies that (ii) holds for $k = 2, \ldots, n$. ∎

3. Scaling of $\Omega \cap U$

In this section we use the Gaussier's method in [10] to claim the convergence of scaled domains.

We assume that p_∞ is an accumulating point for a sequence of automorphisms of Ω. Then there exist a family $(h_\nu)_{\nu\geq 0}$ of automorphisms of Ω and a point q in Ω such that

$$\lim_{\nu\to\infty} h_\nu(q) = p_\infty.$$

For convenience we use the following notation

$$\begin{aligned} q^\nu &= h_\nu(q), \\ \epsilon_\nu &= -r(q^\nu). \end{aligned}$$

The new coordinates $(z_1^\nu, \ldots, z_n^\nu)$, the positive numbers $\tau_{\nu,1}, \ldots, \tau_{\nu,n}$ and the points $p_1^\nu, \ldots, p_n^\nu$ are the ones associated with q^ν and ϵ_ν.

The change of coordinates from the canonical system to the system $(z_1^\nu, \ldots, z_n^\nu)$ is the composition of a translation T_ν and of a unitary transform A_ν. $(A_\nu \circ T_\nu)^{-1}$ is defined in a fixed neighborhood of the origin. The corresponding defining function r_ν is defined by

$$r_\nu := r \circ (A_\nu \circ T_\nu)^{-1}.$$

It is given in a fixed neighborhood of 0 by

$$r_\nu(z) = -\epsilon_\nu + \mathrm{Re}\left(\sum_{j=1}^n a_j^\nu z_j\right) + \sum_{2\leq|\alpha|+|\beta|\leq 2m} C_{\alpha\beta}^\nu {z'}^\alpha \bar{z'}^\beta + O(|z|^{2m+1}),$$

where $\alpha = (\alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_2 + \ldots + \alpha_n$ and ${z'}^\alpha = z_2^{\alpha_2}.\ldots.z_n^{\alpha_n}$. We note that $O(|z|^{2m+1})$ is independent of ν.

Let $r \circ A$ be the limit of r_ν when ν goes to infinity. A is a unitary transform, and the convergence is C^∞ on a fixed compact neighborhood of p_∞. Then, for every j less than or equal to n and for every multi-indices α and β sayisfying $2 \leq |\alpha| + |\beta| \leq 2m$, there exist two complex numbers a_j and $C_{\alpha\beta}$ such that

$$\lim_{\nu\to\infty} a_j^\nu = a_j \text{ and } \lim_{\nu\to\infty} C_{\alpha\beta}^\nu = C_{\alpha\beta}.$$

Let us consider the dilation

$$\Lambda_\nu(z) := (\tau_{\nu,1} z_1, \ldots, \tau_{\nu,n} z_n)$$

and the function

$$\tilde{r}_\nu = \frac{1}{\epsilon_\nu} r_\nu \circ \Lambda_\nu.$$

The function $\tilde{r}_\nu$ has the following form

$$\tilde{r}_\nu(z) = -1 + \frac{1}{\epsilon_\nu}\mathrm{Re}\left(\sum_{j=1}^{n} a_j^\nu \tau_{\nu,j} z_j\right) + \frac{1}{\epsilon_\nu}\sum_{2\leq|\alpha|+|\beta|\leq 2m} C^\nu_{\alpha\beta}\tau_\nu^{\alpha+\beta} {z'}^\alpha {z'}^\beta + O((\epsilon_\nu)^{1/2m}|z|^{2m+1}),$$

where $\tau_\nu^{\alpha+\beta} = \tau_{\nu,2}^{\alpha_2+\beta_2}\dots\tau_{\nu,n}^{\alpha_n+\beta_n}$.

Proposition 3.1. *The functions* $\tilde{r}_\nu$ *are smooth and plurisubharmonic, and there exists a subsequence of* $(\tilde{r}_\nu)_\nu$ *that converges uniformly on compact subsets of* $\mathbb{C}^n$ *to a smooth plurisubharmonic function* $\tilde{r}$ *of the form*

$$\tilde{r}(z) = -1 + \mathrm{Re}\left(\sum_{j\geq 1} b_j z_j\right) + P(z'),$$

where P *is a plurisubharmonic polynomial of degree less than or equal to* $2m$*.*

Proof. The functions $\tilde{r}_\nu$ are smooth by the construction, and are obtained as affine transformations of the plurisubharmonic function r. Hence they are plurisubharmonic. This means that the function $\tilde{r}$ is also smooth and plurisubharmonic as the limit of smooth plurisubharmonic functions $\tilde{r}_\nu$.
Since $O((\epsilon_\nu)^{1/2m}|z|^{2m+1})$ converges to zero on compact subsets of $\mathbb{C}^n$ when ν goes to infinity, we must only study the convergence in the space of polynomials of degree less than or equal to $2m$. Since this space is of finite dimension, all the norms are equivalent and hence, there exists a positive constant d_1 such that, for every $\nu \geq 1$,

$$\sup_{j,\alpha,\beta}\left\{|a_j^\nu|\tau_{\nu,j}, |C^\nu_{\alpha,\beta}|\tau_\nu^{\alpha+\beta}\right\} \leq d_1 \sup_{|\omega|\leq C}\left|\mathrm{Re}\left(\sum_j a_j^\nu \tau_{\nu,j}\omega_j\right) + \sum_{\alpha,\beta} C^\nu_{\alpha,\beta}\tau_\nu^{\alpha+\beta}{\omega'}^\alpha \bar{\omega}'^\beta\right|,$$

where $C := \min\{c_1, c_2\}$; c_1, c_2 are given in Lemmas 2.3 and 2.4. This implies that there exists a positive constant d_2 such that, for every $\nu \geq 1$,

$$\sup_{j,\alpha,\beta}\left\{|a_j^\nu|\tau_{\nu,j}, |C^\nu_{\alpha,\beta}|\tau_\nu^{\alpha+\beta}\right\} \leq d_2 \sup_{z\in CP_{\epsilon_\nu}(q^\nu)}\left|\mathrm{Re}\left(\sum_j a_j^\nu z_j\right) + \sum_{\alpha,\beta} C^\nu_{\alpha,\beta}{z'}^\alpha \bar{z}'^\beta\right|.$$

Using Lemmas 2.2 and 2.3, we obtain

$$\sup_{z\in CP_{\epsilon_\nu}(q^\nu)} |r(z)| \leq 2\epsilon_\nu.$$

On the other hand, using Lemma 2.4 and the definition of polydiscs $P_{\epsilon_\nu}(q^\nu)$, we have

$$\sup_{z\in CP_{\epsilon_\nu}(q^\nu)} O(|z|^{2m+1}) \leq \epsilon_\nu.$$

All these estimates provide a constant d_3 independent of ν such that

$$\sup_{j,\alpha,\beta}\left\{|a_j^\nu|\tau_{\nu,j}, |C^\nu_{\alpha,\beta}|\tau_\nu^{\alpha+\beta}\right\} \le d_3\epsilon_\nu.$$

Consequently, we can extract from the sequence $(\tilde{r}_\nu)_\nu$ a subsequence that converges to the function $\tilde{r}$ given by Proposition 3.1, where b_j are complex numbers. ■

Let Ω_ν be the image of $\Omega \cap U$ under the change $\Lambda_\nu^{-1} \circ A_\nu \circ T_\nu$. Proposition 3.1 implies that the family Ω_ν converges to the domain $\tilde{D} = \{\tilde{r}(z) < 0\}$ in the sense of the Carathéodory kernel convergence.

4. Final Result

Let us cosider the mapping f_ν from $h_\nu^{-1}(\Omega \cap U)$ to Ω_ν defined by

$$f_\nu = \Lambda_\nu^{-1} \circ A_\nu \circ T_\nu \circ h_\nu.$$

We know that $\lim_{\nu\to\infty} h_\nu^{-1}(\Omega \cap U) = \Omega$ and we showed in Sec. 3 that $\lim_{\nu\to\infty} \Omega_\nu = \tilde{D}$.

Lemma 4.1. *The family $(f_\nu)_\nu$ is a normal family.*

Proof. Let $e_j = \Lambda_\nu^{-1}(p_j^\nu)$ for every $\nu \ge 1$ and $j \ge 1$. We now consider the coordinates which are defined in Sec. 1. and depend on ν. In such coordinates $(z_1, \ldots, z_n)$, $p_j^\nu = (0, \ldots, 0, \tau_{\nu,j}, 0, \ldots, 0)$ and so $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$. Moreover,

$$\frac{\partial \tilde{r}_\nu}{\partial z_j}(e_j) = \frac{\tau_{\nu,j}}{\epsilon_\nu}\frac{\partial r_\nu}{\partial z_j}(p_j^\nu).$$

Using part (ii) of Lemma 2.6, we obtain a positive constant d_4 such that

$$\left|\frac{\partial r_\nu}{\partial z_j}(p_j^\nu)\right| \ge d_4 \frac{\epsilon_\nu}{\tau_{\nu,j}}.$$

We conclude that there exists a positive constant d_5 such that , for all sufficiently large ν,

$$\left|\frac{\partial \tilde{r}_\nu}{\partial z_j}(e_j)\right| \ge d_5. \tag{11}$$

Part (iii) of Lemma 2.6 then implies that, for every $k > j$ and sufficiently large ν,

$$\frac{\partial \tilde{r}_\nu}{\partial z_k}(e_j) = 0.$$

Using estimates (11), we can show that the family $(f_\nu^1)_\nu$ of the first components of the mappings f_ν is a normal family. Indeed, for every ν, e_1 is a point in $\partial\Omega_\nu$. By Lemma 2.4, we have

$$\gamma_{e_1}(z) \le 0 \text{ for all } z \in \Omega_\nu, \text{i.e.,}$$

$$\left(\frac{\partial \tilde{r}_\nu}{\partial x_1}(e_1)\right)(\operatorname{Re} z_1 - 1) \le C.\left|\frac{\partial \tilde{r}_\nu}{\partial x_1}(e_1)\operatorname{Im} z_1\right|.$$

Let K be a compact subset of Ω. For each sufficiently large ν, K is a compact subset of $h_\nu^{-1}(\Omega \cap U)$ and so $f_\nu(K)$ is a compact subset of Ω_ν. Then every point ω in K satisfies the inequality

$$\left(\frac{\partial \tilde{r}_\nu}{\partial x_1}(e_1)\right)(\operatorname{Re} f_\nu^1(\omega) - 1) \le C.\left|\frac{\partial \tilde{r}_\nu}{\partial x_1}(e_1)\operatorname{Im} f_\nu^1(\omega)\right|.$$

Using condition (11), we may assume that

$$\operatorname{Re} f_\nu^1(\omega) - 1 \le C.|\operatorname{Im} f_\nu^1(\omega)|.$$

Consequently, the family $(f_\nu^1)_\nu$ is normal. However, for every $\nu \ge 1$ we have the equality $f_\nu^1(q) = 0$. Then we may extract from $(f_\nu^1)_\nu$ a subsequence, still called $(f_\nu^1)_\nu$, that converges to a holomorphic mapping $f^1 : \Omega \to \mathbb{C}$.

Let us show now that $(f_\nu^2)_\nu$ is a normal family. By Lemma 2.4, we have

$$\gamma_{e_2}(z) \le 0 \text{ for all } z \in \Omega_\nu, \text{i.e.,}$$

$$\begin{aligned} &\operatorname{Re}\left(\frac{\partial \tilde{r}_\nu}{\partial z_1}(e_2)z_1\right) + \frac{\partial \tilde{r}_\nu}{\partial x_2}(e_2)(\operatorname{Re} z_2 - 1) \\ &\le C.|\operatorname{Im}\left(\frac{\partial \tilde{r}_\nu}{\partial z_1}(e_2)z_1\right) + \frac{\partial \tilde{r}_\nu}{\partial x_2}(e_2)\operatorname{Im} z_2|. \end{aligned}$$

However,

$$\lim_{\nu\to\infty} \frac{\partial \tilde{r}_\nu}{\partial x_2}(e_2) = \frac{\partial \tilde{r}}{\partial x_2}(e_2);$$

$$\lim_{\nu\to\infty} \frac{\partial \tilde{r}_\nu}{\partial z_1}(e_2) f_\nu^1(\omega) = \frac{\partial \tilde{r}}{\partial z_1}(e_2) f^1(\omega).$$

Using condition (11) again, we may assume after a translation that, for all ω in K and sufficiently large ν,

$$\operatorname{Re} f_\nu^2(\omega) - 1 \le C.\left|\operatorname{Im} f_\nu^2(\omega)\right|.$$

Then the family $(f_\nu^2)_\nu$ is normal and there exists a subsequence which converges to a holomorphic mapping from Ω to $\mathbb{C}$. Repeating this process, we obtain, after extraction, the family $(f_\nu)_\nu$ converges to a mapping f from Ω to $\bar{\bar{D}}$. ■

In order to prove Theorem 1.1, we need the following proposition which is a generalization of the Greene-Krantz theorem [11].

Proposition 4.2. *Let* $\{A_i\}_{i=1}^\infty$ *and* $\{\Omega_i\}_{i=1}^\infty$ *be sequences of domains in a complex manifold* M *with* $\lim A_i = A_0$ *and* $\lim \Omega_i = \Omega_0$ *for some (uniquely determined) domains* A_0, Ω_0 *in* M. *Suppose that* $\{f_i : A_i \to \Omega_i\}$ *is a sequence* *of biholomorphic maps. Suppose also that the sequence* $\{f_i : A_i \to M\}$ *converges uniformly on compact subsets of* A_0 *to a holomorphic map* $F : A_0 \to M$ *and the sequence* $\{g_i := f_i^{-1} : \Omega_i \to M\}$ *converges uniformly on compact subsets of* Ω_0 *to a holomorphic map* $G : \Omega_0 \to M$*.Then one of the following two assertions holds.*

(i) *The sequence* $\{f_i\}$ *is compactly divergent, i.e., for each compact set* $K \subset A_0$ *and each compact set* $L \subset \Omega_0$*, there exists an integer* i_0 *such that* $f_i(K) \cap L = \emptyset$ *for* $i \geq i_0$*, or*

(ii) *There exists a subsequence* $\{f_{i_j}\} \subset \{f_i\}$ *such that the sequence* $\{f_{i_j}\}$ *converges uniformly on compact subsets of* A_0 *to a biholomorphic map* $F : A_0 \to \Omega_0$*.*

Proof. Assume that the sequence $\{f_i\}$ is not divergent. Then F maps some point p of A_0 into Ω_0. We will show that F is a biholomorphism of A_0 onto Ω_0. Let $q = F(p)$, we have

$$G(q) = G(F(p)) = \lim_{i\to\infty} g_i(F(p)) = \lim_{i\to\infty} g_i(f_i(p)) = p,$$

where the next to last identity is by uniform convergence. Take a neighborhood V of p in A_0 such that $F(V) \subset \Omega_0$. But then uniform convergence allows us to conclude that for all $z \in V$ it holds that $G(F(z)) = \lim_{i\to\infty} g_i(f_i(z)) = z$. Hence $F_{|V}$ is injective. By the Osgood's theorem, the mapping $F_{|V} : V \to F(V)$ is biholomorphic.

Consider the holomorphic functions $J_i : A_i \to \mathbb{C}$ and $J : A_0 \to \mathbb{C}$ given by $J_i(z) = \det((df_i)_z)$ and $J(z) = \det((dF)_z)$. Then $J(z) \neq 0$ $(z \in V)$ and, for each $i = 1, 2, \ldots$, the function J_i is non-vanishing on A_i. Moreover, the sequence $\{J_i\}_{i=0}^{\infty}$ converges uniformly on compact subsets of A_0 to J. By Hurwitz's theorem, it follows that J never vanishes. This implies that the mapping $F : A_0 \to M$ is open and any $z \in A_0$ is isolated in $F^{-1}(F(z))$. According to Proposition 5 in [16], we have $F(A_0) \subset \Omega_0$.

Of course this entire argument may be repeated to see that $G(\Omega_0) \subset A_0$. But then uniform convergence allows us to conclude that for all $z \in A_0$ it holds that $G \circ F(z) = \lim_{i\to\infty} g_i(f_i(z)) = z$ and likewise for all $w \in \Omega_0$ it holds that $F \circ G(w) = \lim_{i\to\infty} f_i(g_i(w)) = w$.

This proves that F and G are each one - to - one and onto, hence in particular that F is a biholomorphic mapping. ∎

Proof of Theorem 1.1. It is easy to see that after taking a subsequence, the following properties occur

(i) $(\Omega_\nu)_\nu$ is normally converging to $\tilde{D}$.
(ii) $(f_\nu)_\nu$ converges uniformly on compact subsets of Ω.
(iii) $(f_\nu)_\nu^{-1}$ converges uniformly on compact subsets of $\tilde{D}$.
(iv) If $f := \lim f_\nu$ then $f(\Omega) \subset \tilde{D}$.

Following Proposition 4.2, we claim that Ω is biholomorphically equivalent to $\tilde{D} = \{(z_1, z') \in \mathbb{C}^n : -1 + \mathrm{Re}\,(\sum_{j=1}^{n} b_j z_j) + P(z') < 0\}$. Using the condition (11), we see that the constant b_1 is different from 0. Then by an affine change of coordinates, $\tilde{D}$ is equivalent to the domain $D = \{(z_1, z') \in \mathbb{C}^n : \operatorname{Re} z_1 + P(z') < 0\}$. Since Ω is hyperbolic, D is hyperbolic and by a result of Barth [1], D contains no nontrivial complex affine line. Then there is no complex line in ∂D and, according to Theorem 2.1 in [7], D is of finite type and P is nondegenerate. ■

References

1. T. J. Barth, Convex domains and Kobayashi hyperbolicity, *Proc. Amer. Math. Soc* **79** (1980), 556–558.
2. E. Bedford and S. Pinchuk, Domains in $\mathbb{C}^{n+1}$ with noncompact automorphism group, *J. Geom. Anal.* **1** (1991), 165–191.
3. E. Bedford and S. Pinchuk, Domains in $\mathbb{C}^2$ with noncompact automorphism groups, *Indiana Univ. Math. J.* **47** (1998), 199–222.
4. F. Berteloot, Characterization of models in $\mathbb{C}^2$ by their automorphism groups, *Internat. J. Math.* **5** (1994), 619–634.
5. F. Berteloot, Principle de Bloch et Estimations de la Metrique de Kobayashi des Domains de $\mathbb{C}^2$, *J. Geom. Anal. Math.* **1** (2003), 29–37.
6. S. Cho, A lower bound on the Kobayashi metric near a point of finite type in $\mathbb{C}^n$, *J. Geom. Anal.* **2-4** (1992), 317-325.
7. M. Conrad, *Anisotrope optimale Pseudometriken für lineal konvexe Gebiete von endlichen typs (mit Anwendungen)*, Ph. D. thesis, Berg. Universiträt-GHS Wuppertal, 2002.
8. J. P. D'Angelo, Real hypersurfaces, orders of contact, and applications, *Ann. Math.* **115** (1982), 615-637.
9. P. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften 259, Springer-Verlag, 1983.
10. H. Gaussier, Characterization of convex domains with noncompact automorphism group, *Michigan Math. J.* **44** (1997), 375–388.
11. R. Greene and S. Krantz, Biholomorphic self-maps of domains, *Lecture Notes in Math.* **1276** (1987), 136–207.
12. A. Isaev and S. Krantz, Domains with non-compact automorphism group: A survey, *Adv. Math.* **146** (1999), 1–38.
13. S. Kobayashi, *Hyperbolic Complex Spaces*, **318**, Grundlehren der mathematischen Wissenschaften, Springer-Verlag, 1998.
14. J. D. McNeal, Convex domains of finite type, *J. Funct. Anal.* **108** (1992), 361–373.
15. J. D. McNeal, Estimates on the Bergman kernels of convex domains, *Adv. in Math.* **109** (1994), 108–139.
16. R. Narasimhan, *Several Complex Variables*, Chicago Lectures in Mathematics, University of Chicago Press, 1971.
17. S. Pinchuk, *The Scaling Method and Holomorphic Mappings*, Proc. Symp. Pure Math. **52**, Part 1, Amer. Math. Soc., 1991.
18. Do Duc Thai and Ninh Van Thu, Characterization of domains in $\mathbb{C}^n$ by their noncompact automorphism groups, *preprint.*
19. Do Duc Thai and Ninh Van Thu, Characterization of domains in $\mathbb{C}^n$ by their noncompact automorphism groups II, *preprint.*