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Boundedness of Solutions for a Class of Liénard Equation with Multiple Deviating Arguments*

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Abstract. Consider the Liénard equation with multiple deviating arguments

$$x''(t) + f_1(x(t))(x'(t))^2 + f_2(x(t))x'(t) + g_0(t, x(t)) + \sum_{j=1}^{m} g_j(t, x(t - \tau_j(t))) = p(t),$$

where $f_1,\ f_2 : \mathbb{R} \to \mathbb{R}$, and $g_1,\ g_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, $\mathbb{R} = (-\infty, +\infty)$, $\tau_j(t) \geq 0,\ j = 1, 2, \ldots, m$ are bounded continuous functions on $\mathbb{R}$, and $p(t)$ is a bounded continuous function on $\mathbb{R}^+ = [0, +\infty)$. We obtain some new sufficient conditions for all solutions and their derivatives to be bounded, which substantially extend and improve important results in the literature.

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Key words: Liénard equation, deviating argument, boundedness.

1. Introduction

Consider the Liénard type equation with multiple deviating arguments

$$x''(t) + f_1(x(t))(x'(t))^2 + f_2(x(t))x'(t) + g_0(t, x(t)) + \sum_{j=1}^{m} g_j(t, x(t - \tau_j(t))) = p(t), \tag{1}$$

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where $f_1,\ f_2 : \mathbb{R} \to \mathbb{R}$, and $g_1,\ g_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, $\mathbb{R} = (-\infty, +\infty)$, $\tau_j(t) \geq 0,\ j = 1, 2, \ldots, m$ are bounded continuous functions on $\mathbb{R}$, and $p(t)$ is a bounded continuous function on $\mathbb{R}^+ = [0, +\infty)$.

Define

$$\begin{aligned} a(x) &= \exp\left(\int_0^x f_1(u)du\right), \\ \varphi(x) &= \int_0^x a(u)[f_2(u) - a(u)]du, \\ y &= a(x)\frac{dx}{dt} + \varphi(x), \end{aligned} \tag{2}$$

then we can transform (1) into the following system

$$\begin{cases} \dfrac{dx(t)}{dt} = & \frac{1}{a(x(t))}\left[-\varphi(x(t)) + y(t)\right], \\ \dfrac{dy(t)}{dt} = & a(x(t))\times \\ & \times\left\{-y(t) - [g_0(t, x(t)) - \varphi(x(t))] - \sum\limits_{j=1}^{m} g_j(t, x(t - \tau_j(t))) + p(t)\right\}. \end{cases} \tag{3}$$

In applied science some practical problems concerning physics, mechanics and the engineering technique fields associated with Liénard equation can be found in [1, 2, 5, 10]. Hence, it has been the object of intensive analysis by numerous authors. In particular, there have been extensive results on boundedness of solutions of Liénard equation with constant delays in the literature. Some of these results can be found in [1]-[11]. However, to the best of our knowledge, few authors have considered boundedness of solutions of Liénard equation with multiple deviating arguments. Thus, it is worth while to continue to investigate the boundedness of solutions of Equation (1).

A primary purpose of this paper is to study the boundedness of solutions of (3). We will establish some sufficient conditions for all solutions of (3) to be bounded. Applying our results to (1), one can see that our results are different from those in [1]-[11]. An illustrative example is given.

2. Definitions and Assumptions

We suppose that $h = \max\limits_{1\leq j\leq m}\{\sup\limits_{t\in\mathbb{R}} \tau_j(t)\} \geq 0$. Let $C([-h, 0], \mathbb{R})$ denote the space of continuous functions $\phi : [-h, 0] \to \mathbb{R}$ with the supremum norm $\|\cdot\|$. It is known in [1, 2, 5, 10] that for $g_1, g_2, \varphi, \tau_j$ and p continuous, given a continuous initial function $\phi \in C([-h, 0], \mathbb{R})$ and a number y_0, then there exists a solution of (3) on an interval $[0, T)$ satisfying the initial condition and satisfying (3) on

$[0, T)$. If the solution remains bounded, then $T = +\infty$. We denote such a solution by $(x(t), y(t)) = (x(t, \phi, y_0), y(t, \phi, y_0))$.

Definition 2.1. Solutions of (3) are *uniformly bounded* (UB) if for each $B_1 > 0$ there is a $B_2 > 0$ such that

$$(\phi, y_0) \in C([-h, 0], \mathbb{R}) \times \mathbb{R} \text{ and } \|\phi\| + |y_0| \leq B_1$$

imply that $|x(t, \phi, y_0)| + |y(t, \phi, y_0)| \leq B_2$ for all $t \in \mathbb{R}^+$.

Throughout this paper, we will assume that the following conditions (C_1) and (C_2) hold.

(C_1) There exists a constant $\underline{d} > 1$ such that

$$\underline{d}|u| \leq \text{sign}(u)\varphi(u), \text{ for all } u \in \mathbb{R}.$$

(C_2) For $j = 0, 1, 2, \ldots, m$, there exist nonnegative constants L_j and q_j such that

$$\sum_{j=0}^{m} L_j < 1,$$
$$|(g_0(t, u) - \varphi(u))| \leq L_0|u| + q_0,$$
$$|g_1(t, u)| \leq L_1|u| + q_1, \cdots, |g_m(t, u)| \leq L_m|u| + q_m,$$

for all $t, u \in \mathbb{R}$.

3. Main Results

Theorem 3.1. *Suppose that* (C_1) *and* (C_2) *hold. Then solutions of* (3) *are UB.*

Proof. Let $(x(t), y(t)) = (x(t, \phi, y_0), y(t, \phi, y_0))$ be a solution of (3) defined on $[0, T)$. We may assume that $T = +\infty$ since the estimates which follow give an a priori bound on $(x(t), y(t))$.

Calculating the upper right derivative of $|x(s)|$ and $|y(s)|$ along (3), in view of (C_1) and (C_2), we have

$$\begin{aligned} D^+(|x(s)|)|_{s=t} &= \text{sign}(x(t)) \left\{ \frac{1}{a(x(t))} \left[-\varphi(x(t)) + y(t) \right] \right\} \\ &\leq \frac{1}{a(x(t))} \left[-\underline{d}|x(t)| + |y(t)| \right], \end{aligned} \tag{4}$$

and

$$\begin{aligned} D^+(|y(s)|)|_{s=t} = & \ \mathrm{sign}(y(t))a(x(t))\Big\{ -y(t) - [g_0(t,x(t)) - \varphi(x(t))] - \\ & -\sum_{j=1}^{m} g_j(t,x(t-\tau_j(t))) + p(t)\Big\} \\ \leq & \ a(x(t))\Big\{ -|y(t)| + L_0|x(t)| + \sum_{j=1}^{m} L_j|x(t-\tau_j(t))| + \\ & + \sum_{j=0}^{m} q_j + |p(t)|\Big\}. \end{aligned} \tag{5}$$

Let

$$M(t) = \max_{-h\le s\le t} \{\max\{|x(s)|,\ |y(s)|\}\}, \tag{6}$$

where $y(s) = y(0)$, for all $-h \le s \le 0$. It is obvious that $\max\{|x(t)|,\ |y(t)|\} \le M(t)$, and $M(t)$ is non-decreasing, for $t \ge -h$.

Now, we consider two cases.

Case (i). If

$$M(t) > \max\{|x(t)|,\ |y(t)|\} \text{ for all } t \ge 0, \tag{7}$$

then, we claim that

$$M(t) \equiv M(0) \text{ is a constant for all } t \ge 0. \tag{8}$$

Arguing by contradiction, assume that (8) does not hold. Then, there exists $t_1 > 0$ such that $M(t_1) > M(0)$. Since

$$\max\{|x(t)|,\ |y(t)|\} \le M(0) \text{ for all } -h \le t \le 0,$$

there exists $\beta \in (0,\ t_1)$ such that

$$\max\{|x(\beta)|,\ |y(\beta)|\} = M(t_1) \ge M(\beta),$$

which contradicts (7). This contradiction implies that (8) holds. It follows that

$$\max\{|x(t)|,\ |y(t)|\} \le M(t) = M(0) \text{ for all } t \ge 0. \tag{9}$$

Case (ii). If there is such a point $t_0 \ge 0$ that $M(t_0) = \max\{|x(t_0)|,\ |y(t_0)|\}$. Let

$$\eta = \min\Big\{\underline{d} - 1,\ 1 - \sum_{j=0}^{m} L_j\Big\},\quad \theta = \sum_{j=0}^{m} q_j + \sup_{t\in R^+} |p(t)| + 1.$$

Then, if $M(t_0) = \max\{|x(t_0)|,\ |y(t_0)|\} = |x(t_0)|$, in view of (4), we get

$$\begin{aligned} D^+(|x(s)|)|_{s=t_0} &\le \frac{1}{a(x(t_0))}\left[-\underline{d}|x(t_0)| + |y(t_0)|\right] \\ &\le \frac{1}{a(x(t_0))}(-\underline{d} + 1)M(t_0) \end{aligned}$$

$$< \frac{1}{a(x(t_0))}[-\eta M(t_0) + \theta]. \quad (10)$$

If $M(t_0) = \max\{|x(t_0)|,\ |y(t_0)|\} = |y(t_0)|$, in view of (5), we get

$$\begin{aligned} & D^+(|y(s)|)|_{s=t_0} \\ & \leq a(x(t_0)) \left\{ -|y(t_0)| + L_0|x(t_0)| + \sum_{j=1}^{m} L_j|x(t_0 - \tau_j(t_0))| + \sum_{j=0}^{m} q_j + |p(t_0)| \right\} \\ & < a(x(t_0)) \left[\left(-1 + \sum_{j=0}^{m} L_j \right) M(t_0) + \theta \right] \\ & \leq a(x(t_0))[-\eta M(t_0) + \theta]. \end{aligned} \quad (11)$$

In addition, if $M(t_0) \geq \frac{\theta}{\eta}$, it follows from (10) and (11) that $M(t)$ is strictly decreasing in a small neighborhood $(t_0,\ t_0 + \delta_0)$. This contradicts that $M(t)$ is non-decreasing. Hence,

$$\max\{|x(t_0)|,\ |y(t_0)|\} = M(t_0) < \frac{\theta}{\eta}. \quad (12)$$

For all $t > t_0$, by the same approach used in the proof of (12), we have

$$\max\{|x(t)|,\ |y(t)|\} < \frac{\theta}{\eta}, \quad \text{if} \ \ M(t) = \max\{|x(t)|,\ |y(t)|\}. \quad (13)$$

On the other hand, if $M(t) > \max\{|x(t)|,\ |y(t)|\}$, $t > t_0$. We can choose $t_0 \leq t_2 < t$ such that

$$M(t_2) = \max\{|x(t_2)|,\ |y(t_2)|\} < \frac{\theta}{\eta},$$

and

$$M(s) > \max\{|x(s)|,\ |y(s)|\} \text{ for all } s \in (t_2,\ t].$$

Using a similar argument as in the proof of Case (i), we can show that

$$M(s) \equiv M(t_2) \text{ is a constant for all } s \in (t_2,\ t], \quad (14)$$

which implies that

$$\max\{|x(t)|,\ |y(t)|\} < M(t) = M(t_2) = \max\{|x(t_2)|,\ |y(t_2)|\} < \frac{\theta}{\eta}.$$

In summary, the solutions of (3) are UB. The proof of Theorem 3.1 is complete. ∎

4. An Example

Example 4.1. All solutions and their derivatives of the Liénard equation with two deviating arguments

$$x''(t) + (x'(t))^2 + \left[e^{-x(t)}(3x^2(t)+2) + e^{x(t)}\right] x'(t) + \frac{1}{2}\sin x(t) + x^3(t) + 2x(t) + \frac{1}{6}|x(t - |\sin t|)| + \frac{\sin t}{16}\arctan x(t - \frac{1}{1+t^2}) = e^{\frac{1}{t^2+1}} \quad (15)$$

are bounded.

Indeed, set

$$a(x) = e^x,\ \varphi(x) = \int_0^x (3u^2+2)du,\quad y = e^x\frac{dx}{dt} + x^3 + 2x, \quad (16)$$

then we can transform (1) into the following system

$$\begin{cases} \dfrac{dx(t)}{dt} = e^{-x(t)}[-(x^3(t)+2x(t)) + y(t)], \\ \dfrac{dy(t)}{dt} = e^{x(t)}\left[-y(t) - \dfrac{1}{2}\sin x(t) - \dfrac{1}{6}|x(t-|\sin t|)| - \dfrac{\sin t}{16}\arctan x\left(t - \dfrac{1}{1+t^2}\right) + e^{\frac{1}{t^2+1}}\right]. \end{cases} \quad (17)$$

It is straight forward to check that all assumptions needed in Theorem 3.1 are satisfied. Therefore, solutions of system (17) are UB. This implies that all solutions and their derivatives of (15) are bounded.

Remark 4.2. Equation (15) is a very simple Liénard equation with two deviating arguments. Since $\tau_1(t) = |\sin t|$ and $\tau_2(t) = \frac{1}{1+t^2}$ are not constants, it is clear that the results obtained in [1]-[11] can not be applicable to system (15). Moreover, we propose a totally new approach to proving the boundedness of solutions of Liénard equation, which is different from that of [1]-[11] and the references therein. This implies that the results of this paper are essentially new.

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