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Maximal Stability Bound for Generalized Singularly Perturbed Systems

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Abstract. In this paper, we propose computable formulas for the maximal stability bound for implicit systems of linear differential equations which contain an uncertain small parameter in the leading term. It is supposed that the systems are previously transformed into an appropriate sparse form and they are robustly stable. To find the maximal stability bound of the systems, the time-domain method is used. This leads to generalized eigenvalue problems for matrix pencils. Details of the numerical algorithms and some illustrative examples are given.

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1. Introduction

The maximal stability bound (i.e., the so-called ϵ-bound) problem for singularly perturbed systems has been studied for decades. Discussions and results on this problem may be found in [3, 4, 7, 14, 15] and references therein. However, all of these results are restricted to a special case, concretely, the classical singularly perturbed system [9]

$$\begin{pmatrix} I_{n_1} & 0 \\ 0 & \epsilon I_{n_2} \end{pmatrix} \dot{x}(t) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x(t), \tag{1}$$

where I_{n_k} is the $n_k \times n_k$ identity matrix, $A_{kl} \in \mathbb{R}^{n_k \times n_l}$, $(k, l = 1, 2)$. Further, $x(t) \in \mathbb{R}^n$, $t \geq 0$, $n = n_1 + n_2$. The singular perturbation parameter ϵ is in general small and positive. In many real-life applications, this parameter is usually uncertain. Suppose that the reduced system, i.e. when $\epsilon = 0$, is asymptotically stable. Then, the ϵ-bound problem is formulated as follows: *Find the maximal value of the singular perturbation parameter, denoted by ϵ^*, such that for all $0 \leq \epsilon < \epsilon^*$, system* (1) *is still asymptotically stable.*

Various approaches have been investigated to find this bound. In [7], Feng developed a frequency-domain technique that gives a solution for the calculation of ϵ^* via the generalized Nyquist plot. Based on a condition for matrices that have only characteristic roots with negative real parts, Sen and Datta [15] derived a formula for the maximal bound ϵ^*. Chen et al. [4] used two different approaches including time- and frequency-domain techniques to solve the stability bound problem.

In this paper, we consider the ϵ-bound problem in a more general situation, namely the problem for the generalized singularly perturbed system of the form

$$(E + \epsilon F)\dot{x}(t) = Ax(t), \tag{2}$$

with $E, F, A \in \mathbb{R}^{n \times n}; x(t) \in \mathbb{R}^n$. Such systems appear frequently in many applications, e.g., multibody mechanics simulation, electrical circuit design, chemical reactions, etc, see [9, 1]. The uncertainty of the small parameter makes the maximal stability bound problem important and interesting for the system science and engineering community.

We assume that the system (2) is asymptotically stable when $\epsilon = 0$. In [6], Du and Linh classified the parametrized perturbation and gave sufficient conditions to ensure the stability of system (2) for all suficiently small ϵ. In the current research, under these conditions, we investigate the ϵ-bound problem for the system (2) in three cases: the index-change, the rank-change, and the structure-invariance cases. Using the time-domain method, we propose computable formulas for ϵ^*. Solving auxiliary generalized eigenvalue problems, the maximal stability bound ϵ^* can be determined. We also show that as a special case, our result coincides with Chen's result in [4]. We note that a closely related problem - the robust stability of uncertain implicit systems in which the parametrized perturbation appears on the right hand side - was solved in [12]. As far as we know, our present paper is the first one that investigates the maximal bound problem for the generalized singularly perturbed systems of the form (2).

The paper is organized as follows. In Sec. 2, some basic concepts and preliminary results are briefly reviewed. Main results for determining the maximal bounds are given in Sec. 3. Several numerical examples are given in Sec. 4 for illustrating the computational procedures. Finally, some conclusions will close the paper.

2. Preliminary

2.1. Matrix Pencils and Linear Differential-algebraic Equations

In this section, we give a brief summary of needed results on linear constant coefficient differential-algebraic equations (DAEs). We assume the reader is familiar with the basic theory of linear time invariant DAEs [1, 11, 10], such as

$$E\dot{x} = Ax. \tag{3}$$

The matrix pencil $\{E,\ A\}$ is said to be regular if there exists $\lambda \in \mathbb{C}$ such that the determinant of $(\lambda E - A)$, denoted by $\det(\lambda E - A)$, is nonzero. The system (3) is uniquely solvable if and only if $\{E,\ A\}$ is regular. If $\det(\lambda E - A) = 0 \quad \forall \lambda \in \mathbb{C}$, we say that $\{E,\ A\}$ is irregular or nonregular. If $\{E,\ A\}$ is regular, then λ is a (generalized finite) eigenvalue of $\{E,\ A\}$ if $\det(\lambda E - A) = 0$. The set of all eigenvalues is called the spectrum of the pencil $\{E,\ A\}$ and denoted by $\sigma\{E,\ A\}$. Thus, for a given matrix $A \in \mathbb{C}^{m\times m}$, the well-known spectrum $\sigma(A)$ is understood by $\sigma(I,\ A)$.

Suppose that E is singular and pencil $\{E, A\}$ is regular. Then there exist nonsingular matrices W, T such that

$$WET = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix}, \qquad WAT = \begin{pmatrix} B & 0 \\ 0 & I_{n-d} \end{pmatrix}, \tag{4}$$

where N is nilpotent of index k [1, 11, 10]. If N is a zero matrix, then $k = 1$. If $\{E,\ A\}$ is regular, the nilpotency index of N in (4) is called the index of matrix pencil $\{E,\ A\}$ and we write $\operatorname{index}\{E,\ A\} = k$. If E is nonsingular, we set $\operatorname{index}\{E,\ A\} = 0$.

Definition 2.1. Suppose that $\{E, A\}$ is regular. Let P be a projector onto the subspace of consistent initial conditions. We say that the zero solution of (3) is stable if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for an arbitrary vector $x_0 \in \mathbb{C}^m$ satisfying $\|x_0\| < \delta$, the solution of the initial value problem

$$\begin{cases} E\dot{x} = Ax, \quad t \in [0, \infty), \\ P(x(0) - x_0) = 0 \end{cases}$$

exists uniquely and the estimate $\|x(t)\| < \varepsilon$ holds for all $t \geq 0$. The zero solution is said to be asymptotically stable if it is stable and $\lim_{t\to\infty} \|x(t)\| = 0$ for solutions x of (3). If the zero solution of (3) is asymptotically stable, we say that system (3) is asymptotically stable.

If index $\{E,\ A\} = 1$ one may choose $P = I - Q$, where Q is a projector onto ker (A) [11]. A difference between ODE-s and DAE-s is that the equality

$x(0) = x_0$ is not expected here, in general. That is, for DAEs, we need *consistent* initial value x_0 such that (3) with the initial condition $x(0) = x_0$ holds for a smooth solution. We do not consider impulsive solutions in this paper and for that reason will frequently make an index one assumption. For linear time-invariant systems, the concepts of asymptotic stability and exponential stability are equivalent.

Proposition 2.2. *The system* (3) *is asymptotically stable if and only if the matrix pencil* $\{E,\ A\}$ *is* (*asymptotically*) *stable, i.e.,* $\sigma(E,\ A) \subset \mathbb{C}^-$, *where* $\mathbb{C}^-$ *denotes the open left-half complex plane.*

Clearly $\sigma(RES,\ RAS) = \sigma(E,\ A)$ for nonsingular R, S. Then, we say that the pencils $\{RES,\ RAS\}$ and $\{E,\ A\}$ are kinematically equivalent.

2.2. Sufficient Conditions for the Robust Stability of Perturbed Systems

In this section, we summarize some preliminary results on the robust stability of the system (2), see [6]. In the main part of this paper, we suppose the following assumption.

Assumption A1: Pencil $\{E, A\}$ is (asymptotically) stable and $\mathrm{index}\{\mathrm{E}, \mathrm{A}\} = 1$.

For the sake of simplicity, from now on the problem (2) is supposed to be given in a block form. Without loss of generality, we assume that the triplet $\{E, F, A\}$ has the form

$$E = \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{5}$$

where E_{11} is nonsingular. The condition $\mathrm{index}\{\mathrm{E}, \mathrm{A}\} = 1$ is equivalent to the nonsingularity of A_{22}. Here, all the submatrices are supposed to have appropriate sizes, e.g. A_{ij} have the size $n_i \times n_j, i, j = 1, 2$, where $n_1 + n_2 = n$. If the original matrix E is not of the sparse and block form as in (5), it is sufficient to use any decomposition of the form $E = P \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix} Q$, where matrices E_{11}, P, Q are nonsingular in order to transform the problem (2) into the form (5). In practice, this decomposition can be realized in a numerically stable way by using a singular value decomposition (SVD) $E = U^T \mathrm{diag}(\Sigma, 0)V$, where U, V are orthogonal matrices, $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_{n_1}), \{\sigma_i\}_{i=1}^{n_1}$ are the singular values of E in a non-increasing order. Then, one obtains a kinematically equivalent problem with the new data set

$$E_{new} = \mathrm{diag}(\Sigma, 0), \quad F_{new} = UFV^T, \quad A_{new} = UAV^T. \tag{6}$$

Now, we are interested in the following three cases: the index-change case, the rank-change case and the structure-invariance case.

a) The index-change case: In this case, the index of $\{E+\epsilon F, A\}$ changes from 0 to 1 when ϵ becomes 0, i.e. there exists $\epsilon_0 > 0$ such that $\mathrm{index}\{\mathrm{E} + \epsilon\mathrm{F}, \mathrm{A}\} = 0$

for all $\epsilon, 0 < \epsilon \leq \epsilon_0$, but $\text{index}\{\mathrm{E}, \mathrm{A}\} = 1$. Of course, the index-change immediately implies the rank-change of the leading term.
In this case, in addition to the assumptions A1, we suppose the following conditions:

Assumption A2: Matrix F_{22} is nonsingular.

Assumption A3: Matrix pencil $\{F_{22}, A_{22}\}$ is (asymptotically) stable, i.e. their spectra belong to the open left half plane $\mathbb{C}^-$.

An important result on the robust stability of system (2) given in [6] is as follows.

Proposition 2.3. *Assume that Assumptions A1-A3 hold true. Then the system (2) (or (5)) is asymptotically stable for all sufficiently small ϵ, i.e. there exists a number $\overline{\epsilon}$ such that $\sigma\{E + \epsilon F, A\} \subset \mathbb{C}^-$ for each parameter $\epsilon \in [0, \overline{\epsilon})$.*

b) The rank-change case: In this case, the index of $\{E + \epsilon F, A\}$ is equal to 1 for all sufficiently small ϵ and $\epsilon = 0$. However, the rank of $(E+\epsilon F)$ is constant for all sufficiently small and positive ϵ but changes when ϵ becomes 0. This means exactly that some generalized finite eigenvalues may be lost as ϵ reaches 0.
In this case, instead of $A2$, we suppose the following assumption.

Assumption A2$^{\#}$: Matrix F is of block-triangular form, i.e. F_{21} (or F_{12}) is a zero matrix and $\text{index}\{\mathrm{F}_{22}, \mathrm{A}_{22}\} = 1$.

We also have a statement analogous to Proposition 2.3, see [6].

Proposition 2.4. *Assume that $A1, A2^{\#}, A3$ hold true. For all sufficiently small ϵ, the pencil $\{E + \epsilon F, A\}$ remains of index-1 and stable, i.e. there exists $\overline{\epsilon}$ such that*

$$\text{index}\{\mathrm{E} + \epsilon\mathrm{F}, \mathrm{A}\} = 1 \quad \textit{and} \quad \sigma\{\mathrm{E} + \epsilon\mathrm{F}, \mathrm{A}\} \subset \mathbb{C}^-$$

hold for all $\epsilon \in [0, \overline{\epsilon})$.

c) The structure-invariance case: In this case, the index of the pencil $\{E + \epsilon F, A\}$ as well as the rank of the leading term do not change for all sufficiently small ϵ, i.e. there exists $\epsilon_0 > 0$ such that $\text{index}\{\mathrm{E} + \epsilon\mathrm{F}, \mathrm{A}\} = \text{index}\{\mathrm{E}, \mathrm{A}\}$ and $\text{rank}\,(E + \epsilon F) = \text{rank}\,E$ for all $\epsilon, 0 < \epsilon < \epsilon_0$. We split this case into two small subcases: either E is nonsingular or E is singular. Then, we have the following result, see [6].

Proposition 2.5. *Assume that E is nonsingular and $\{E, A\}$ is stable. Then the pencil $\{(E+\epsilon F), A\}$ is of index-0 and stable for all sufficiently small ϵ, i.e. there exists a positive number $\overline{\epsilon}$ such that $E+\epsilon F$ is nonsingular and $\sigma\{E+\epsilon F, A\} \subset \mathbb{C}^-$ for all $\epsilon \in [0, \overline{\epsilon})$.*

When E is singular, we can use the transformation

$$\hat{P}(E + \epsilon F)\hat{Q} = \begin{pmatrix} E_{11} + \epsilon F_{11} & 0 \\ 0 & \epsilon[F_{22} - \epsilon F_{21}(E_{11} + \epsilon F_{11})^{-1} F_{12}] \end{pmatrix}, \tag{7}$$

where

$$\hat{P} = \begin{pmatrix} I_{n_1} & 0 \\ -\epsilon F_{21}(E_{11} + \epsilon F_{11})^{-1} & I_{n_2} \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} I_{n_1} & -\epsilon(E_{11} + \epsilon F_{11})^{-1} F_{12} \\ 0 & I_{n_2} \end{pmatrix}.$$

The invariant-rank assumption implies that for sufficiently small ϵ, we have

$$\epsilon[F_{22} - \epsilon F_{21}(E_{11} + \epsilon F_{11})^{-1} F_{12}] = 0. \tag{8}$$

It holds true, for instance, when F_{21}, F_{22} are zero matrices. That is,

$$E = \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} F_{11} & F_{12} \\ 0 & 0 \end{pmatrix}. \tag{9}$$

Such a matrix F is called admissible perturbation, see [2]. And we also have the following proposition, see [6].

Proposition 2.6. *Let Assumption A_1 hold again and the perturbation direction F satisfy* (8). *Then the parameterized pencil $\{E + \epsilon F, A\}$ is of index-1 and stable for all sufficiently small (not necessarily positive) ϵ.*

Finally, we recall an important auxiliary result from linear algebra that is useful in the next section. Suppose that H and K are both square matrices of dimensions m and n, respectively. The Kronecker sum $H \oplus K$ is an $mn \times mn$ matrix defined by

$$H \oplus K := H \otimes I_n + I_m \otimes K,$$

where $\otimes$ denotes a Kronecker product and I_k denotes the $k \times k$ identity matrix.

Lemma 2.7. (see [8]) *Let the eigenvalues of matrices $H \in \mathbb{R}^{m\times m}$ and $K \in \mathbb{R}^{n\times n}$ be $\lambda_1, \lambda_2, \cdots, \lambda_m$ and $\mu_1, \mu_2, \cdots, \mu_n$, respectively. Then, the eigenvalues of matrix $H \otimes K$ are the mn products of the form $\lambda_j \mu_k$, and the eigenvalues of matrix $H \oplus K$ are the mn sums of the form $\lambda_j + \mu_k$ where $1 \le j \le m$ and $1 \le k \le n$.*

3. Computation of the Stability Bound

In this section, we propose computational methods for determining the maximal stability bound for system (2).

Definition 3.1. The stability bound of an asymptotically stable system of the form (2) is defined by the maximal positive number ϵ^* such that the system remains asymptotically stable and the pencil $\{E, A\}$ has the same index and the same number of finite eigenvalues for all $\epsilon \in (0, \epsilon^*)$.

3.1. The Index-change Case

We recall that in this case, assumptions A1-A3 hold. The assumption A2 implies that F_{22} is nonsingular. Using the formula (2) we have

$$\begin{pmatrix} E_{11}+\epsilon F_{11} & \epsilon F_{12} \\ \epsilon F_{21} & \epsilon F_{22} \end{pmatrix}$$
$$= \begin{pmatrix} I_{n_1} & F_{12}F_{22}^{-1} \\ 0 & I_{n_2} \end{pmatrix} \begin{pmatrix} E_{11}+\epsilon(F_{11}-F_{12}F_{22}^{-1}F_{21}) & 0 \\ 0 & \epsilon F_{22} \end{pmatrix} \begin{pmatrix} I_{n_1} & 0 \\ F_{22}^{-1}F_{21} & I_{n_2} \end{pmatrix}.$$

In addition, E_{11} is nonsingular, there exists ϵ_{inv} such that for all $0<\epsilon<\epsilon_{inv}$, $E_{11}+\epsilon(F_{11}-F_{12}F_{22}^{-1}F_{21})$ is nonsingular, that means $(E+\epsilon F)$ is nonsingular. On the other hand, stability of pencil $\{E,A\}$ implies that A is nonsingular. Denoting by ϵ_{sta} the largest value such that $\sigma\{E+\epsilon F,A\}\subset\mathbb{C}^-, \quad \forall\epsilon\in[0,\epsilon_{sta})$, the maximal bound ϵ^* can be expressed as

$$\epsilon^*=\min\{\epsilon_{inv},\epsilon_{sta}\}. \tag{10}$$

We have the following theorem.

Theorem 3.2. *Assume that Assumption* $A1-A3$ *hold true. Then the maximal stability bound* ϵ^* *of* (2) *is given by*

$$\epsilon^*=\min\bigl\{\lambda\in\mathbb{R}^+:\lambda\in\sigma(M,-N)\bigr\}, \tag{11}$$

where

$$N:=A\otimes E+E\otimes A; \qquad M:=A\otimes F+F\otimes A,$$

and $\sigma(M,-N)=\bigl\{\lambda\in\mathbb{C}:\det(\lambda M+N)=0\bigr\}$.

If the pencil $(M,-N)$ *has no positive real eigenvalue, then we set* $\epsilon^*=+\infty$.

Proof. When $\epsilon\in(0,\epsilon_{inv})$, both $E+\epsilon F$ and A are nonsingular. We have

$$\sigma\{E+\epsilon F,A\}\subset\mathbb{C}^-\Leftrightarrow\sigma\{A,E+\epsilon F\}\subset\mathbb{C}^-\Leftrightarrow\sigma(A^{-1}E+\epsilon A^{-1}F)\subset\mathbb{C}^-.$$

Denote $A(\epsilon):=A^{-1}E+\epsilon A^{-1}F$. Now, we can use the time-domain approach as in [4]. It follows from Proposition 2.3 that for sufficiently small $\epsilon>0$ the system (2) (or (5)) is asymptotically stable, i.e. all eigenvalues of the matrix $A(\epsilon)$ lie in the open left-half $\mathbb{C}^-$ of the complex plane. As a result, when the values of parameter ϵ sweep from $\epsilon=0^+$ to $\epsilon>\epsilon_{sta}$, either of the following two possibilities occurs:

(i) a real eigenvalue enters the right half-plane via the origin, or

(ii) a pair of conjugate complex eigenvalues passes the imaginary axis to the right half-plane.

Both cases make the system (2) unstable. For the critical value $\epsilon=\epsilon_{sta}$, either an eigenvalue of $A(\epsilon)$ becomes 0 or a pair of conjugate eigenvalues appears on the imaginary axis. Lemma 2.7 states that if all eigenvalues of matrix $A(\epsilon)$ lie in the open left half-plane, then all eigenvalues of the Kronecker sum $A(\epsilon)\oplus A(\epsilon)$ are located in the open left half-plane, too. This implies that the matrix $A(\epsilon)\oplus A(\epsilon)$ is nonsingular. Hence, we have $\sigma(A(\epsilon))\subset\mathbb{C}^-\Leftrightarrow 0\notin A(\epsilon)\oplus A(\epsilon)$ for $0<\epsilon<\epsilon_{sta}$.

In other words, the stability bound problem can be converted equivalently to a nonsingularity analysis problem. Similarly to [13], we obtain immediately a formula for ϵ_{sta}

$$\epsilon_{sta} = \min\Big\{\lambda \in \mathbb{R}^+ : \lambda \in \sigma\big(A^{-1}F \oplus A^{-1}F, -[A^{-1}E \oplus A^{-1}E]\big)\Big\}.$$

Moreover, since $A \otimes A$ is nonsingular and

$$\begin{aligned}&\det(A \otimes A)\det\Big(\lambda(A^{-1}F \oplus A^{-1}F) + (A^{-1}E \oplus A^{-1}E)\Big)\\ &= \det\Big(\lambda(A \otimes F + F\otimes) + (A \otimes E + E \otimes A)\Big),\end{aligned}$$

we have

$$\sigma\big(A^{-1}F \oplus A^{-1}F, -[A^{-1}E \oplus A^{-1}E]\big) = \sigma\big(A \otimes F + F \otimes A, -A \otimes E - E \otimes A\big).$$

Furthermore, based on the definition of ϵ_{inv}, it is clear that when $\epsilon = \epsilon_{inv}$, the $A(\epsilon)$ has at least a zero eigenvalue, i.e. the case (i) happens and $A(\epsilon)$ is unstable. This means $\epsilon_{inv} \geq \epsilon_{sta}$. Hence, we get

$$\epsilon^* = \epsilon_{sta} = \min\Big\{\lambda \in \mathbb{R}^+ : \lambda \in \sigma\big(A \otimes F + F \otimes A, -A \otimes E - E \otimes A\big)\Big\}.$$

The proof is complete. ■

Remark 3.3. Consider the special case $E = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 0 \\ 0 & I_{n_2} \end{pmatrix}$.
On one hand, we have

$$A^{-1}(\epsilon) = \Big(A^{-1}(E + \epsilon F)\Big)^{-1} = (E + \epsilon F)^{-1}A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21}/\epsilon & A_{22}/\epsilon \end{pmatrix}.$$

On the other hand, $A(\epsilon)$ is stable if and only if $A^{-1}(\epsilon)$ is stable. Hence, the result of Theorem 3.2 in this case coincides with the result of Chen et al. [4].

For clarity, we give the algorithm for computing ϵ^* in detail.

Algorithm 1 (Computation of the stability bound in the index-change case)

- `Input: Three matrices` E, F, A.
- `Ouput: The maximal stability bound` ϵ^*.
- `Main steps:`

1. `Transform the data into an equivalent form (5) by using SVD.`
2. `Check the assumptions` $A1, A2, A3$.
3. `Compute` $N := A \otimes E + E \otimes A; M := A \otimes F + F \otimes A$.
4. `Compute` ϵ^* `by (11).`

3.2. The Rank-change Case

In this case, the assumptions are $A1, A2^{\#}, A3$. First, the system will be transformed into an equivalent one of the form (5) analogously to the index-change case. By Proposition 2.4, there exists $\epsilon^* > 0$ such that $\{E+\epsilon F, A\}$ is still of index-1 and asymptotically stable for all $\epsilon \in [0, \epsilon^*)$. When the values of parameter ϵ sweep from $\epsilon = 0^+$ to $\epsilon = \widetilde{\epsilon} > \epsilon^*$, either of the following two cases happens:

(i) There exists $\epsilon \in (0, \widetilde{\epsilon})$ such that $E_{11} + \epsilon F_{11}$ is singular, i.e. the index-1 property of pencil $\{E+\epsilon F, A\}$ is lost, or

(ii) there exists $\epsilon \in (0, \widetilde{\epsilon})$ such that $\{E+\epsilon F, A\}$ is unstable.

Denote by ϵ_{inv} and ϵ_{sta} the largest bounds such that the cases (i) and (ii) will not happen for all ϵ in each of intervals $(0, \epsilon_{inv})$ and $(0, \epsilon_{sta})$, respectively.

First, we easily get the value

$$\epsilon_{inv} = \min\bigl\{\lambda \in \mathbb{R}^+ : \lambda \in \sigma(F_{11}, -E_{11})\bigr\}. \tag{12}$$

Under the assumptions that $F_{21} = 0$ (the case of $F_{12} = 0$ is treated similarly) and $\mathrm{index}\{\mathrm{F}_{22}, \mathrm{A}_{22}\} = 1$, an appropriate transformation will yield a kinematically equivalent system of the form

$$\begin{pmatrix} \overline{E}_{11} + \epsilon\overline{F}_{11} & \epsilon\overline{F}_{12} & \epsilon\overline{F}_{13} \\ 0 & \epsilon\overline{F}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} & \overline{A}_{13} \\ \overline{A}_{21} & \overline{A}_{22} & \overline{A}_{23} \\ \overline{A}_{31} & \overline{A}_{32} & \overline{A}_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \tag{13}$$

where $\overline{F}_{22}$ is nonsingular.
For example, we may use a singular-value decomposition

$$F_{22} = U_1^T \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} V_1,$$

where U_1, V_1 are orthogonal matrices, $\Sigma_1 = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$, $(\sigma_i)_{i=1}^r$ are non-zero singular values of F_{22} in a non-increasing order, $r = \mathrm{rank}\,(F_{22})$. By defining $y = \mathrm{diag}(I_{n_1},\ V_1)x$, we obtain a new system with the following data

$$\overline{E} = \begin{pmatrix} I_{n_1} & 0 \\ 0 & U_1 \end{pmatrix} \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_{n_1} & 0 \\ 0 & V_1^T \end{pmatrix} = \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\overline{F} = \begin{pmatrix} I_{n_1} & 0 \\ 0 & U_1 \end{pmatrix} \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix} \begin{pmatrix} I_{n_1} & 0 \\ 0 & V_1^T \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12}V_1^T \\ 0 & U_1F_{22}V_1^T \end{pmatrix} = \begin{pmatrix} \overline{F}_{11} & \overline{F}_{12} & \overline{F}_{13} \\ 0 & \overline{F}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\overline{A} = \begin{pmatrix} I_{n_1} & 0 \\ 0 & U_1 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} I_{n_1} & 0 \\ 0 & V_1^T \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12}V_1^T \\ U_1A_{21} & U_1F_{22}V_1^T \end{pmatrix} = \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} & \overline{A}_{13} \\ \overline{A}_{21} & \overline{A}_{22} & \overline{A}_{23} \\ \overline{A}_{31} & \overline{A}_{32} & \overline{A}_{33} \end{pmatrix}.$$

So we have

$$\overline{E}_{11} = E_{11}; \quad \overline{F}_{11} = F_{11}; \quad \bigl(\overline{F}_{12}\ \overline{F}_{13}\bigr) = F_{12}V_1^T; \quad \overline{A}_{11} = A_{11};$$

$$\begin{pmatrix} \overline{F}_{22} & 0 \\ 0 & 0 \end{pmatrix} = U_1 F_{22} V_1^T; \quad \begin{pmatrix} \overline{A}_{12} & \overline{A}_{13} \end{pmatrix} = A_{12} V_1^T; \tag{14}$$
$$\begin{pmatrix} \overline{A}_{21} \\ \overline{A}_{31} \end{pmatrix} = U_1 A_{21}; \quad \begin{pmatrix} \overline{A}_{22} & \overline{A}_{23} \\ \overline{A}_{32} & \overline{A}_{33} \end{pmatrix} = U_1 A_{22} V_1^T,$$

where all the matrices are supposed to have appropriate sizes.

Due to assumption index$\{\mathrm{F}_{22}, \mathrm{A}_{22}\} = 1$, the matrix $\overline{A}_{33}$ is nonsingular. From the algebraic equation $\overline{A}_{31} y_1 + \overline{A}_{32} y_2 + \overline{A}_{33} y_3 = 0$, we get $y_3 = -(\overline{A}_{33}^{-1} \overline{A}_{31} y_1 + \overline{A}_{33}^{-1} \overline{A}_{32} y_2)$. Therefore, the system (13) can be transformed into the form

$$\begin{pmatrix} \widetilde{E}_{11} + \epsilon \widetilde{F}_{11} & \epsilon \widetilde{F}_{12} & 0 \\ 0 & \epsilon \widetilde{F}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} & 0 \\ \widetilde{A}_{21} & \widetilde{A}_{22} & 0 \\ \widetilde{A}_{31} & \widetilde{A}_{32} & \widetilde{A}_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \tag{15}$$

where $\widetilde{E}_{11}, \widetilde{F}_{22}, \widetilde{A}_{33}$ are nonsingular and the matrices in (15) are defined by the following formulas

$$\begin{aligned} \widetilde{E}_{11} &= \overline{E}_{11}; \quad \widetilde{F}_{11} = \overline{F}_{11} - \overline{F}_{13} \overline{A}_{33}^{-1} \overline{A}_{31}; \\ \widetilde{F}_{12} &= \overline{F}_{12} - \overline{F}_{13} \overline{A}_{33}^{-1} \overline{A}_{32}; \quad \widetilde{F}_{22} = \overline{F}_{22}; \\ \widetilde{A}_{11} &= \overline{A}_{11} - \overline{A}_{13} \overline{A}_{33}^{-1} \overline{A}_{31}; \quad \widetilde{A}_{12} = \overline{A}_{12} - \overline{A}_{13} \overline{A}_{33}^{-1} \overline{A}_{32}; \\ \widetilde{A}_{21} &= \overline{A}_{21} - \overline{A}_{23} \overline{A}_{33}^{-1} \overline{A}_{31}; \quad \widetilde{A}_{22} = \overline{A}_{22} - \overline{A}_{23} \overline{A}_{33}^{-1} \overline{A}_{32}; \\ \widetilde{A}_{31} &= \overline{A}_{31}; \quad \widetilde{A}_{32} = \overline{A}_{32}; \quad \widetilde{A}_{33} = \overline{A}_{33}. \end{aligned} \tag{16}$$

Now, our problem changes into finding the stability bound for an $(n_1 + r)$-dimensional system with pencil

$$\left\{ \begin{pmatrix} \widetilde{E}_{11} + \epsilon \widetilde{F}_{11} & \epsilon \widetilde{F}_{12} \\ 0 & \epsilon \widetilde{F}_{22} \end{pmatrix}, \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix} \right\},$$

where $\widetilde{F}_{22}$ is nonsingular, i.e. we have a problem similar to that of the index-change case. An analogous result to that in Subsec. 3.1 yields

$$\epsilon_{sta} = \min\{\lambda \in \mathbb{R}^+ : \lambda \in \sigma(M, -N)\}, \tag{17}$$

where

$$\begin{aligned} N &:= \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix} \otimes \begin{pmatrix} \widetilde{E}_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \widetilde{E}_{11} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}, \\ M &:= \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix} \otimes \begin{pmatrix} \widetilde{F}_{11} & \widetilde{F}_{12} \\ 0 & \widetilde{F}_{22} \end{pmatrix} + \begin{pmatrix} \widetilde{F}_{11} & \widetilde{F}_{12} \\ 0 & \widetilde{F}_{22} \end{pmatrix} \otimes \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}. \end{aligned} \tag{18}$$

Hence, we get the following theorem.

Theorem 3.4. *Let Assumption* $A1, A2^{\#}$ *and* $A3$ *hold true. The stability bound for system* (2) *is given by*

$$\epsilon^* = \min\{\epsilon_{inv}, \epsilon_{sta}\}, \tag{19}$$

where ϵ_{inv} *and* ϵ_{sta} *are determined by* (12) *and* (17), *respectively.*

Algorithm 2 (Computation the maximal stability bound in the rank-change case)

- `Input: Three matrices` E, F, A `in the block form (5) for which Assumptions A1, A2#, and A3 hold true.`
- `Output: The maximal stability bound` ϵ^*.
- `Main steps:`

1. `Compute` $\epsilon_{inv} = \min\{\lambda \in \mathbb{R}^+ : \lambda \in \sigma(F_{11}, -E_{11})\}$.
2. `Decompose` $F_{22} = U_1^T \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} V_1$.
3. `Compute` $\overline{E}_{11}$; $\overline{F}_{11}$; $\overline{F}_{12}$; $\overline{F}_{13}$; $\overline{F}_{22}$; $\overline{A}_{11}; \overline{A}_{12}; \overline{A}_{13}$; $\overline{A}_{21}$; $\overline{A}_{22}$; $\overline{A}_{23}$; $\overline{A}_{31}$; $\overline{A}_{32}$; $\overline{A}_{33}$ `via (14).`
4. `Use (16) to determine` $\widetilde{E}_{11}$; $\widetilde{F}_{22}$; $\widetilde{A}_{11}; \widetilde{A}_{12}; \widetilde{A}_{21}; \widetilde{A}_{22}; \widetilde{A}_{31}$; $\widetilde{A}_{32}; \widetilde{A}_{33}$.
5. `Compute` M, N `by (18) and determine` ϵ_{sta} `by (17).`
6. `Evaluate` $\epsilon^* = \min\{\epsilon_{inv}; \epsilon_{sta}\}$.

Remark 3.5. In fact, an alternative approach that avoids the inverse calculation and can be carried out in a more (numerically) stable manner is as follows: first, use any decomposition of the form

$$U_1 F_{22} = \begin{pmatrix} \overline{F}_{22} & \overline{F}_{23} \\ 0 & 0 \end{pmatrix},$$

where U_1 is orthogonal, to transform the system into a form that is similar to (13). Next, we propose another decomposition

$$\begin{pmatrix} \overline{A}_{31} & \overline{A}_{32} & \overline{A}_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \widetilde{A}_{33} \end{pmatrix} V,$$

where V is orthogonal. In practice, these decompositions can be easily obtained by using QR factorization. Then, defining $y = Vx$ will again give us a sparse system that is kinematically equivalent to the original one. Finally, we compute the stability bound for the underlying $(n_1 + r)$-dimensional system.

3.3. The Structure-invariance Case

First, we consider the simplest subcase when E is nonsingular. Similarly to the index-change case, we have the formula for the maximal stability bound

$$\epsilon^* = \min\{\lambda \in \mathbb{R}^+ : \lambda \in \sigma(M, -N)\}, \tag{20}$$

where $N := A \otimes E + E \otimes A$; $\quad M := A \otimes F + F \otimes A$.

Now, for the other subcase when E is singular and the perturbation is admissible, we consider the system (5) when

$$E = \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} F_{11} & F_{12} \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{21}$$

At first, we determine

$$\epsilon_{inv} = \min\big\{\lambda \in \mathbb{R}^+ : \lambda \in \sigma(F_{11}, -E_{11})\big\}. \tag{22}$$

Second, Assumption $A1$ implies that A and A_{22} are nonsingular and pencil $\{E_{11}, A_{11} - A_{12}A_{22}^{-1}A_{21}\}$ is (asymptotically) stable. To find ϵ_{sta}, we transform the triplet (E, F, A) into the form

$$\left(\begin{pmatrix} \overline{E}_{11} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \overline{F}_{11} & \overline{F}_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & \overline{A}_{22} \end{pmatrix} \right). \tag{23}$$

Similarly to the second decomposition in Remark 3.5, we find a nonsingular matrix P such that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} P = \begin{pmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & \overline{A}_{22} \end{pmatrix},$$

where $\overline{A}_{22}$ is nonsingular. For example, we set

$$P = \begin{pmatrix} I_{n_1} & 0 \\ -A_{22}^{-1}A_{21} & A_{22}^{-1} \end{pmatrix}.$$

Then, the triplet (E, F, A) is transformed into the form

$$\left(\begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} F_{11} - F_{12}A_{22}^{-1}A_{21} & F_{12}A_{22}^{-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & A_{12}A_{22}^{-1} \\ 0 & I_{n_2} \end{pmatrix} \right). \tag{24}$$

Now, our problem turns into finding the maximal stability bound for the system with pencil

$$\left\{\overline{E}_{11} + \epsilon\overline{F}_{11}, \overline{A}_{11}\right\}, \tag{25}$$

where

$$\overline{E}_{11} = E_{11}; \overline{F}_{11} = F_{11} - F_{12}A_{22}^{-1}A_{21}; \overline{A}_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21}. \tag{26}$$

Similarly to the index-change case, we obtain

$$\epsilon_{sta} = \min\big\{\lambda \in \mathbb{R}^+ : \lambda \in \sigma(M, -N\big\}, \tag{27}$$

where

$$M = \overline{A}_{11} \otimes \overline{F}_{11} + \overline{F}_{11} \otimes \overline{A}_{11}$$
$$N = \overline{A}_{11} \otimes \overline{E}_{11} + \overline{E}_{11} \otimes \overline{A}_{11}. \tag{28}$$

We have the following result.

Theorem 3.6. *Let Assumption* A1 *hold and let the perturbation F be admissible. Then, the maximal stability bound for system* (2) *is given by*

$$\epsilon^* = \min\{\epsilon_{inv}, \epsilon_{sta}\}, \tag{29}$$

where ϵ_{inv} and ϵ_{sta} are determined by (22) *and* (27), *respectively.*

Algorithm 3 (Computation of the maximal stability bound in the admissible perturbation case).

- `Input: Three matrices` E, F, A `in the form (21) for which Assumption A1 holds true.`
- `Output: The maximal stability bound` ϵ^*.
- `Main steps:`

1. `Determine` ϵ_{inv} `by (22).`
2. `Compute` $\overline{E}_{11}; \overline{F}_{11}; \overline{A}_{11}$ `by (26).`
3. `Compute` $M; N$ `by (28) and determine` ϵ_{sta} `by (27).`
4. `Evaluate` $\epsilon^* = \min\{\epsilon_{inv}; \epsilon_{sta}\}$.

Finally, we remark that since in this case ϵ is not necessarily positive as in the previous two cases, a (negative) lower bound for ϵ can be analogously obtained, as well.

4. Numerical Examples

In this section, some examples are presented to illustrate the results and the computational procedures proposed in Sec. 3. All calculations are carried out using MATLAB 7.0. In particular, the generalized eigenvalue problems which occur in these examples are solved by the MATLAB function EIG.

Example 4.1. Let us consider the same example worked out by Chen and Lin [4] and Sen and Datta [15]. The system is given by

$$\begin{pmatrix} x_1' \\ x_2' \\ \epsilon x_3' \\ \epsilon x_4' \end{pmatrix} = \begin{pmatrix} -3 & 4 & -3 & 4 \\ 0 & 2 & -1 & -2 \\ 1 & 2 & -2 & 3 \\ 0 & 2 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

This means that

$$E = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}; \quad F = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}; \quad A = \begin{pmatrix} -3&4&-3&4\\ 0&2&-1&-2\\ 1&2&-2&3\\ 0&2&0&-3 \end{pmatrix}.$$

Compute $N := A \otimes E + E \otimes A; M := A \otimes F + F \otimes A$, we have $\sigma(M, -N) \cap \mathbb{R}^+ = \{4.947981; 0.980295\}$. Therefore, $\epsilon^* = 0.980295$ This result coincides with that in [4] and [15]. The real parts of eigenvalues of parametrized pencil $\{E + \epsilon F,\ A\}$ versus ϵ are displayed in Fig. 1. It is clearly seen that at $\epsilon = \epsilon^*$, (at least) one eigenvalue hits the imaginary axis.

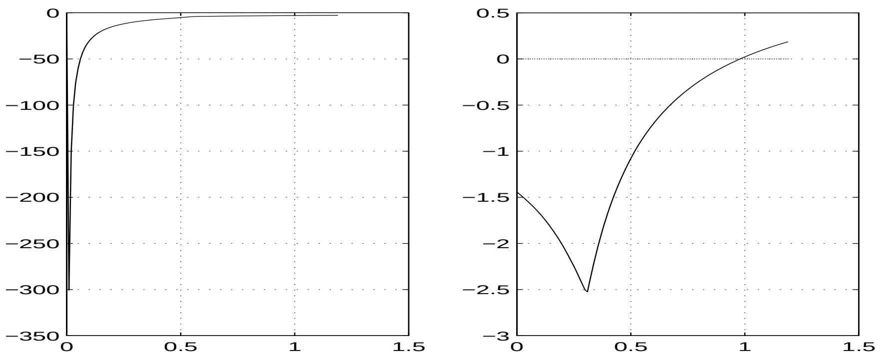

Fig. 1 The minimal and the maximal real parts of the eigenvalues in Example 1

Example 4.2. Now, we consider an example for the index-change case with the following data

$$E = \begin{pmatrix} 2&1&0\\ 1&2&0\\ 0&0&0 \end{pmatrix}; \qquad F = \begin{pmatrix} -3&-2&1\\ 1&-2&3\\ 0&1&-1 \end{pmatrix}; \qquad A = \begin{pmatrix} 1&1&1\\ 2&1&1\\ 0&2&1 \end{pmatrix}.$$

It is easy to check the assumptions A1, A2 and A3. We compute M and N and get $\sigma(M, -N) \cap \mathbb{R}^+ = \{0.581139; 0.117285\}$. Hence, $\epsilon^* = 0.117285$. The graph of the maximal real part of eigenvalues of parametrized pencil $\{E + \epsilon F,\ A\}$ versus ϵ is plotted in Fig. 2. It is also clearly observable that at the critical value $\epsilon = \epsilon^*$, an eigenvalue passes the imaginary axis.

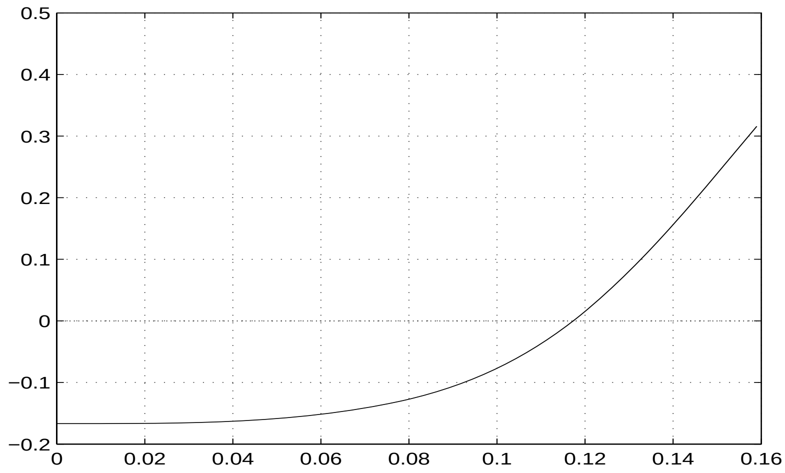

Fig. 2 The maximal real part of the eigenvalues in Example 2

Example 4.3. We consider an illustrative example for the rank-change case given by the following data

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad F = \begin{pmatrix} -1 & 1 & 5 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{pmatrix}; \qquad A = \begin{pmatrix} -3.5 & -1 & 3 \\ 1 & \frac{1}{2} & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

First, we immediately have $\epsilon_{inv} = 1$.
Decompose

$$F_{22} = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} -0.4472 & 0.8944 \\ 0.8944 & 0.4472 \end{pmatrix}^* \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.4472 & -0.8944 \\ -0.8944 & -0.4472 \end{pmatrix}.$$

So we transform $\{E, F, A\}$ into the triplet $\{\overline{E}, \overline{F}, \overline{A}\}$, where

$$\overline{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \overline{F} = \begin{pmatrix} -1 & -4.0249 & -3.1305 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$
$$\overline{A} = \begin{pmatrix} -3.5 & -3.1305 & -0.4472 \\ -1.3416 & -0.9000 & 0.8000 \\ 0.4472 & 0.8000 & 0.4000 \end{pmatrix}.$$

Then, using (16), we turn $\{\overline{E}, \overline{F}, \overline{A}\}$ into the form

$$\widetilde{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \widetilde{F} = \begin{pmatrix} 3.5 & 2.2361 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \widetilde{A} = \begin{pmatrix} -3 & -2.2361 & 0 \\ -2.2361 & -2.5 & 0 \\ 0.4472 & 0.8 & 0.4 \end{pmatrix}.$$

We evaluate

$$N = \begin{pmatrix} -6.0000 & -2.2361 & -2.2361 & 0 \\ -2.2361 & -2.5000 & 0 & 0 \\ -2.2361 & 0 & -2.5000 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$M = \begin{pmatrix} -15.0000 & -12.2984 & -12.2984 & -10.0000 \\ -5.5902 & -21.2500 & -5.0000 & -16.7705 \\ -5.5902 & -5.0000 & -21.2500 & -16.7705 \\ 0 & -11.1803 & -11.1803 & -25.0000 \end{pmatrix}$$

and get $\epsilon_{sta} = +\infty$. Hence, $\epsilon^* = 1$.

Example 4.4. We consider an example for the admissible perturbation case with the following data

$$E = \begin{pmatrix} -1 & 3 & 0 & 0 \\ -3 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad F = \begin{pmatrix} 2 & 3 & -2 & 1 \\ 1 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad A = \begin{pmatrix} 8 & -8 & 1 & 2 \\ 6 & -9 & 3 & -1 \\ -1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We have

$$E_{11} = \begin{pmatrix} -1 & 3 \\ -3 & 7 \end{pmatrix}, \quad F_{11} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}, \quad F_{12} = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}.$$

First, we calculate $\epsilon_{inv} = 4.29317$. Using (26), we have

$$\overline{E}_{11} = \begin{pmatrix} -1 & 3 \\ -3 & 7 \end{pmatrix}; \quad \overline{F}_{11} = \begin{pmatrix} -2 & 8 \\ 7 & -4 \end{pmatrix}; \quad \overline{A}_{11} = \begin{pmatrix} 5 & -8 \\ 11 & -16 \end{pmatrix}.$$

Then

$$M = \begin{pmatrix} -20 & 56 & 56 & -128 \\ 13 & 12 & 32 & -96 \\ 13 & 32 & 12 & -96 \\ 154 & -156 & -156 & 128 \end{pmatrix}; \; N = \begin{pmatrix} -10 & 23 & 23 & -48 \\ -26 & 51 & 57 & -104 \\ -26 & 57 & 51 & -104 \\ -66 & 125 & 125 & -224 \end{pmatrix}.$$

Solving the auxiliary eigenvalue problem, we obtain $\epsilon_{sta} = 0.14384$. Hence, $\epsilon^* = 0.14384$. The minimal and the maximal real parts of the eigenvalues of parametrized pencil $\{E + \epsilon F,\ A\}$ versus ϵ are plotted in Fig. 3.

It is very interesting to observe that, in this example, the system becomes unstable when the real part of an eigenvalue goes to the negative infinity and then appears again in the right half-plane at the positive infinity. Note that this phenomenon cannot happen in the classical singular perturbation problems.

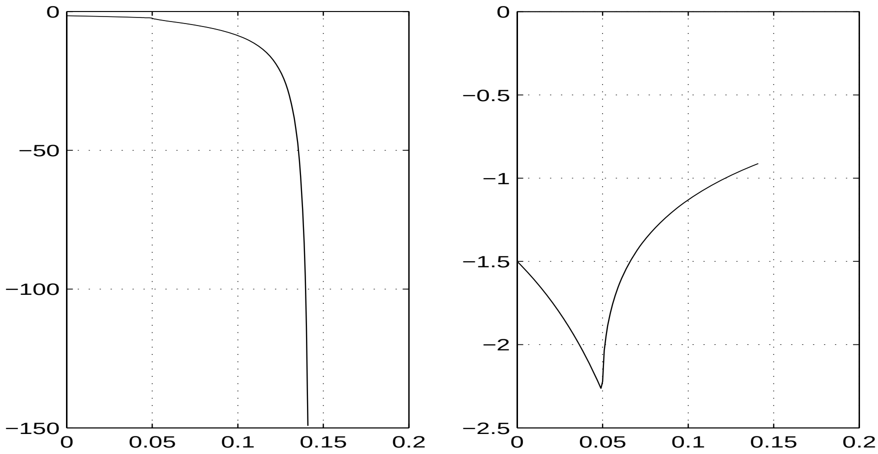

Fig. 3 The minimal and the maximal real parts of the eigenvalues in Example 4

5. Conclusion

In this paper, computable formulas for the maximal stability bound for generalized singularly perturbed systems have been proposed. By characterizing parametrized perturbations with respect to possible changes in the relevant properties of matrix pencils, three cases have been analyzed. In each case, details of numerical algorithms have been given. The theoretical results as well as the validity of the computational procedures have been confirmed by the illustrative examples.

A robust stability analysis as well as an extension of the results in this paper to singularly perturbed systems with higher-index pencils would be of interest and may be future works.

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