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Characterizations of the Linear Topological Invariants and the Solutions to some Problems of Holomorphic Functions

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Abstract. The theory of holomorphic functions between locally convex spaces having linear topological invariants has attracted the attention of many researchers during the last thirty years. This report is a survey about recent results of the following problems from this theory:
1. Locally bounded holomorphic functions,
2. The regularity of spaces of germ of Fréchet-valued holomorphic functions,
3. Weakly holomorphic functions, δ- separately holomorphic functions,
4. $\sigma(\cdot, W)$-holomorphic functions and theorems of Vitali-type,
5. The exponential representation of holomorphic functions and absolute representation system,
6. Separately holomorphic functions,
7. Holomorphic functions of uniformly bounded type on tensor products.

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1. Introduction

In the 80's, Vogt successfully solved the problem of classifying Fréchet spaces by introducing and investigating the linear topological invariants of types (Ω) and

(DN). The works of Vogt [44, 46, 47], Vogt and Wagner [52, 53] and Wagner [55] have shown that they are related with power series spaces.

We recall the definitions of these properties. Let E be a Fréchet space with a fundamental system of semi-norms $\{\|.\|_k\}_{k\in\mathbb{N}}$. The following properties of E are called of (Ω)-types:

$$\begin{aligned}
\overline{\overline{\Omega}}: \text{if} \quad & \forall p\ \exists q\ \forall k,\ d>0\ \exists C>0 && \text{such that} && \|.\|_q^{*1+d} \leqslant C\|.\|_k^*\|.\|_p^{*d},\\
\overline{\Omega}: \text{if} \quad & \exists d>0\ \forall p\ \exists q\ \forall k\ \exists C>0 && \text{such that} && \|.\|_q^{*1+d} \leqslant C\|.\|_k^*\|.\|_p^{*d},\\
\widetilde{\Omega}: \text{if} \quad & \forall p\ \exists q,\ d>0\ \forall k\ \exists C>0 && \text{such that} && \|.\|_q^{*1+d} \leqslant C\|.\|_k^*\|.\|_p^{*d},\\
\Omega: \text{if} \quad & \forall p\ \exists q\ \forall k,\ \exists d, C>0 && \text{such that} && \|.\|_q^{*1+d} \leqslant C\|.\|_k^*\|.\|_p^{*d},\\
LB^\infty: \text{if} \quad & \forall \varrho_k \uparrow \infty, \forall p\, \exists q\, \forall k_0\, \exists \widehat{k}, C_{k_0}, && && \\
& \forall u\in E', \exists k: k_0 \leqslant k \leqslant \widehat{k} && \text{such that} && \|u\|_q^{*1+\varrho_k} \leqslant C_{k_0}\|u\|_k^*\|u\|_p^{*\varrho_k},
\end{aligned}$$

where $\|u\|_k^* = \|u\|_{U_k}^* = \sup\{|u(x)|:\ x\in U_k\}$ with $U_k=\{x\in E:\ \|x\|_k\leqslant 1\}$.

The following properties of E are called of (DN)-types:

$$\begin{aligned}
\overline{DN}: \text{if} \quad & \exists p\ \forall q\ \exists k\ \forall d\ \exists C>0 && \text{such that} && \|.\|_q^{1+d} \leqslant C\|.\|_k\|.\|_p^{d},\\
DN: \text{if} \quad & \exists p\ \exists d>0\ \forall q\ \exists k,\ C>0 && \text{such that} && \|.\|_q^{1+d} \leqslant C\|.\|_k\|.\|_p^{d},\\
\underline{DN}: \text{if} \quad & \exists p\ \forall q\ \exists k,\ d,\ C>0 && \text{such that} && \|.\|_q^{1+d} \leqslant C\|.\|_k\|.\|_p^{d},\\
LB_\infty: \text{if} \quad & \forall d_n \uparrow \infty, \exists p\ \forall q\, \exists k_q \geqslant q, C_q>0, && && \\
& \forall x\in E, \exists m: q\leqslant m\leqslant k_q && \text{such that} && \|x\|_q^{1+d_m} \leqslant C_q\|x\|_m\|x\|_p^{d_m}.
\end{aligned}$$

Roughly speaking, these invariants are convexity or concavity properties of the semi-norms or the dual semi-norms. It is easy to see that the properties of (Ω)-types are inherited by quotient spaces, while the ones of (DN)-types are inherited by linear subspaces. Moreover, the following implications hold:

$$(\overline{\overline{\Omega}}) \Rightarrow (\overline{\Omega}) \Rightarrow (\widetilde{\Omega}) \Rightarrow (LB^\infty) \Rightarrow (\Omega),$$

$$(\overline{DN}) \Rightarrow (LB_\infty) \Rightarrow (DN) \Rightarrow (\underline{DN}).$$

From now on, we write $E\in(\Omega)$ (resp. $(\widetilde{\Omega})$...) if E has the property (Ω) (resp. $(\widetilde{\Omega})$...)

During the last twenty years these properties have many applications in different fields of the modern mathematics, especially in the theory of Fréchet spaces. For example, they have been applied successfully to the complex analysis, in particular to investigate the structures of spaces of holomorphic functions, as well as of spaces of germs of holomorphic functions on compact subsets of Fréchet spaces. Base on them, one establishes the uniform boundedness for Fréchet-valued holomorphic/meromorphic functions and types of the Hahn-Banach extension for locally convex-valued holomorphic functions. Recently, they have become the main tool in investigating the implicit and inverse function theorems in Fréchet spaces. Hence, the continuation in looking for various applications of the linear topological invariants in complex analysis and functional analysis as well as studying them in the relation with various branches of mathematics, such as pluripotential theory,... is significant.

One of the trends in studying the problems on the holomorphy is to give characterizations of Fréchet spaces as well as dual Fréchet spaces in terms of types $(\Omega), (DN)$ in order to investigate the uniformity/ the possibility of expansion in the exponential form of holomorphic functions between them, the holomorphicity of weak/ separately holomorphic functions, or others.

Many results in the direction have been obtained by some authors. The local boundedness of holomorphic functions plays a very important part in studying the above problems. However, until now, the solutions to these problems have been obtained only if it is assumed that Fréchet spaces have one of the properties $(\overline{\overline{\Omega}}), (\overline{\Omega}), (\widetilde{\Omega})$ whereas the (Ω)-case has not been established by any authors. This explains why the solutions to many of the above problems (except some cases) were obtained only if Fréchet spaces are assumed to have one of the properties $(\overline{\overline{\Omega}}), (\overline{\Omega}), (\widetilde{\Omega}), (LB^{\infty})$.

We therefore are interesed in finding the answers to these problems. In the first section of the report we will introduce new characterizations of the properties $(\Omega), (LB_{\infty})$. Then we will go on to bring out solutions to these problems one by one.

2. Some Characterizations of the Properties $(\Omega), (LB_{\infty})$

In [38] we introduced a necessary condition for the property (Ω) as follows. If a Fréchet space $E \in (\Omega)$ then

$$\forall p\ \exists q\ \exists k_o\ \forall k \geqslant k_o\ \exists\, C(k) > 0 \ \text{ such that } \ \forall u \in E^* :$$

$$\|u\|_q^{*1+k} \leqslant C(k)\|u\|_k^*\|u\|_p^{*k}.$$

The following is an extension of the above, where the sequence of natural numbers is placed by a sequence $\{d_n\} \uparrow +\infty$.

Theorem 2.1. ([39]) *Let E be a Fréchet space. Then $E \in (\Omega)$ if and only if $E \in (\Omega^{\infty})$, which means*

$$\exists\ \{d_n\} \uparrow +\infty,\ \forall\ p\ \exists\ q\ \exists\ k_o\ \forall k \geqslant k_o\ \exists\ C(k) > 0 \ \textit{ such that } \ \forall\ u \in E^* :$$

$$\|u\|_q^{*1+d_k} \leqslant C(k)\|u\|_k^*\|u\|_p^{*d_k}.$$

The author is grateful to Professor D. Vogt (Wuppertal University) for having personally indicated this equivalence.

We say that a Fréchet space $E \in (\Omega_B)$, where $B \in \mathcal{B}(E)$, the family of all closed, bounded and absolutely convex subsets in E, if

$$\exists\, \{d_n\} \uparrow +\infty, \forall p\, \exists q\, \exists k_o\, \forall k \geqslant k_o\, \exists C(k) > 0 \text{ such that } \forall u \in E^*$$

$$\|u\|_q^{*1+d_k} \leqslant C(k)\|u\|_B^*\|u\|_p^{*d_k}. \tag{1}$$

From this theorem, by a suitable modification of the proof of Vogt [46], we get the following equivalent conditions of (Ω) :

Theorem 2.2. ([39]) *Let E be a Fréchet space. Then $E \in (\Omega_B)$ if and only if $\exists\{d_n\} \uparrow +\infty,\ \forall p\, \exists q\, \exists k_o\, \forall k \geqslant k_o\, \exists C(k) > 0$ such that*

$$U_q \subset r^{d_k}B + \frac{C(k)}{r}U_p \quad \text{for all } r > 0. \tag{2}$$

Theorem 2.3. ([39]) *Let E be a Fréchet space. Then $E \in (\Omega)$ if and only if there exists $B \in \mathcal{B}(E)$ such that $E \in (\Omega_B)$.*

Remark 2.4. As in [7] for the case where E is Fréchet-Schwartz, the set B can be chosen in $\mathcal{K}(E)$, the set of all closed, compact, absolutely convex subsets of E.

We continue this section with some characterizations of the property (LB_∞).

In [48] Vogt proved that a Fréchet space $F \in (LB_\infty)$ if and only if

$$L(\Lambda_\infty(\alpha), F) = LB(\Lambda_\infty(\alpha), F) \tag{LB}$$

or equivalently

$$L(F^*, \Lambda_\infty^*(\alpha)) = LB(F^*, \Lambda_\infty^*(\alpha)) \tag{LB*}$$

for every power series space of infinite type $\Lambda_\infty(\alpha)$, where $L(E,F)$ denotes the space of all continuous linear mappings between two locally convex spaces E, F while $LB(E,F)$ denotes the set of all $A \in L(E,F)$ for which there exists a zero neighborhood U in E such that $A(U)$ is bounded. Note that $\Lambda_\infty(\alpha) \in (\Omega)$ for every α.

The remaining part of this section deals with extension $(LB), (LB^*)$ to holomorphic and meromorphic versions.

Let E and F be locally convex spaces and let $D \subset E$ be open, $D \neq \emptyset$. A function $f : D \to F$ is called Gâteaux-holomorphic if for every $u \in F^*$, the topological dual space of F, the function $uf : D \to \mathbb{C}$ is holomorphic. This means that its restriction to each finite dimensional section of D is holomorphic as a function of several complex variables.

A function $f : D \to F$ is called holomorphic if f is continuous Gâteaux holomorphic on D. Denote by $H(D,F)$ the space of F-valued holomorphic functions on D equipped with the compact-open topology. Instead of $H(D,\mathbb{C})$ we write $H(D)$. Put

$$H_b(E,F) = \{f \in H(E,F) : f \text{ is of bounded type}\}.$$

Here $f \in H(E,F)$ is said to be of bounded type if it is bounded on every bounded set in E. The space $H_b(E,F)$ is considered with the topoplogy of uniform convergence on all bounded sets in E.

Let E, F be locally convex spaces. A holomorphic function f from E to F is called to be of uniform type if f can be holomorphically factorized through a Banach space. This means that there exists a continuous semi-norm ϱ on E and a holomorphic function g from E_ϱ, the canonical Banach space associated to ϱ, into F such that $f = g\omega_\varrho$, where $\omega_\varrho : E \to E_\varrho$ is the canonical map. By $H_u(E,F)$ we denote the subspace of $H(E,F)$ consisting of elements which are of uniform type.

A holomorphic function $f : D_o \to F$ on a dense open subset D_o of D with values in F is called meromorphic on D if, for every $z \in D$, there exist a neighborhood U of z and holomorphic functions $h : U \to F, \sigma : U \to \mathbb{C}$ with $\sigma \neq 0$ such that

$$f\Big|_{D_o \cap U} = \frac{h}{\sigma}\Big|_{D_o \cap U}.$$

By $M(D,F)$ we denote the space of F-valued meromorphic functions on D. The uniformity of meromorphic functions between locally convex spaces is defined similarly as for holomorphic functions. By $M_u(E,F)$ we denote the subspace of $M(E,F)$ consisting of elements which are of uniform type.

The research of uniformity of the holomorphic/meromorphic functions on locally convex spaces is of great significance, because through it we can examine them on Banach spaces.

The problem about characterizations of Fréchet spaces E and F satisfying one of the identities

$$H(E,F) = H_u(E,F) \tag{Hun}$$

$$M(E,F) = M_u(E,F) \tag{Mun}$$

has been investigated in many different treats. For example, Colombeau and Mujica [4] have shown that (Hun) holds in the case where E is a dual Fréchet-Montel space and F is a Fréchet space. Hai and Quang [18] proved that $H(E, \mathbb{CP}(F)) = H_u(E, \mathbb{CP}(F))$, where E is a (DFN)-space and $\mathbb{CP}(F)$ is a projective space associated with a Fréchet space F. They also showed that if nuclear locally convex space E is a subspace of a locally convex space F such that every holomorphic map on E with values in a complex Banach manifold X can be holomorphically extended to F, then every holomorphic map from E to X is of uniform type.

Some characterizations of spaces E and F concerning linear topological invariants for which the problems (Hun) / (Mun) for the cases where either E, F are Fréchet (we write (HUN) / (MUN) for short) or E, F are dual Fréchet (we write (HUN*) / (MUN*) for short) hold were inverstigated by many authors.

The problems (HUN), (MUN) were investigated first by Meise and Vogt [27]. Hai [12] proved that (HUN) holds in the case where E is a nuclear Fréchet space,

$E \in (\widetilde{\Omega})$ and F is a Fréchet space, $F \in (DN)$. However, this problem cannot be true for the case $E \in (\Omega), F \in (LB_\infty)$. Indeed, from [27] it easily follows that a locally convex space F which satisfies (HUN) does not exist. Specifically, from [27, 2.3] we conclude that (HUN) implies that E satisfies (LB^∞) which is stronger than (Ω). Therefore a result like the below table for the case $F \in (LB_\infty)$ could possibly be true only if one replaces the property (Ω) of E by something which implies $H(E) = H_{ub}(E)$. Whether (LB^∞) implies $H(E) = H_{ub}(E)$ has been an open problem for more than 20 years.

Thus we can recapitulate the results on this problem as follows.

Ref.	E	F	Conclusions
[12]	nuclear, Fréchet, $(\overline{\overline{\Omega}})$	Fréchet	$F \in (\underline{DN}) \Leftrightarrow$ HUN
	nuclear, Fréchet, $(\widetilde{\Omega})$	Fréchet	$F \in (DN) \Leftrightarrow$ HUN

For the dual Fréchet case, in [14] Hai proved that if E is Fréchet-Schwartz with an absolute basis and F is Fréchet such that $E^{**} \in (DN), F^{**} \in (\widetilde{\Omega})$ then (HUN*) and (MUN*) hold. In 2003, Quang [32] asserted that the (HUN*) is true in the cases where either $E \in (\underline{DN})$ and $F \in (\overline{\overline{\Omega}})$ or $E \in (DN)$ and $F \in (LB^\infty)$. For the meromorphic case, in 1997, Lan [26] showed that (MUN*) is correct if $E \in (DN)$ is Fréchet-Schwartz and $F \in (\widetilde{\Omega})$ is Fréchet. Quang extended in [33] the result of Lan for the cases where either $E \in (DN)$ is Fréchet-Montel and $F \in (H_{ub})$ is Fréchet or $E \in (DN)$ is Fréchet-Schwartz and $F \in (LB^\infty)$ is Fréchet.

Recently, when E is a Fréchet-Schwartz space with an absolutely Schauder basis, $E \in (\underline{DN})$ (resp. (DN)) and F is a Fréchet space with $F \in (\overline{\overline{\Omega}})$ (resp. (LB^∞)), the identities (HUN*), (MUN*) were proved by Quang [32, 33]. For the case where $E \in (\Omega), F \in (LB_\infty)$, the situation is similar as in the (HUN), (MUN) problems.

We now can summarize our recent results by the following for E, F *being Fréchet spaces*:

Ref.	F	E	Conclusions
[32]	$(\overline{\overline{\Omega}})$	Schwartz, abs. basis	$E \in (\underline{DN}) \Leftrightarrow$ HUN*
[33]	(H_{ub})	Montel	$E \in (DN) \Leftrightarrow$ HUN$^* \Leftrightarrow$ MUN*
	(LB^∞)	Schwartz	
[33]		Schwartz, abs. basis, (DN)	$F \in (LB^\infty) \Leftrightarrow$ HUN$^* \Leftrightarrow$ MUN*

3. Locally Bounded Holomorphic Functions

Let E, F be locally convex spaces and D an open set in E. A function $f \in H(D, F)$ is called locally bounded holomorphic on D if for every $z \in D$ there exists a neighborhood U_z of z such that $f(U_z)$ is bounded. Put

$$H_{LB}(D,F) = \{f \in H(D,F): \ f \text{ is locally bounded on } D\}.$$

Our aim here is to find conditions for E and F such that

$$H_{LB}(D,F) = H(D,F) \tag{HLB}$$

for every open set D on E. Observe that the equality (LB) is the (HLB) for continuous linear maps.

Along with items of (Hun), (Mun), the results of (HLB) are the effective tool in the proofs of the results that are presented in the sections below.

Khue and Thanh [24] showed that (HLB) holds in the case where E, F are Fréchet and $E \in (\Omega), F \in (\overline{DN})$. They also proved that the statement is true if either $E \in (\widetilde{\Omega})$ is Fréchet-Schwartz having an absolute basis, $F \in (DN)$ or $E \in (\widetilde{\Omega})$ is Fréchet having an absolute basis, $F \in (LB_\infty)$. In the case where E is nuclear Fréchet, $E \in (\widetilde{\Omega})$ and F is Fréchet, $F \in (DN)$ the equality (HLB) is proved by Lan [25].

The following result of Quang and Lam is an extension of the results of Khue and Thanh in [24] for the linear topological invariants $(\Omega), (LB_\infty)$.

Theorem 3.1. ([40]) *Let E, F be Fréchet spaces with $E \in (\Omega), F \in (LB_\infty)$. Assume that E has an absolute basis. Then* (HLB) *holds for every open set D in E.*

Now our results on the (HLB) are summarized as follows, where E, F *are Fréchet spaces and $D \subset E$ is open.*

Ref.	E	F	Conclusion
[24]	Schwartz, absolute basis, $(\widetilde{\Omega})$	(DN)	(HLB) holds
[25]	nuclear $(\widetilde{\Omega})$	(DN)	
[24]	absolute basis, $(\widetilde{\Omega})$	(LB_∞)	
[40]	absolute basis, (Ω)	(LB_∞)	

Note that it is impossible to drop the condition $F \in (LB_\infty)$ in the hypothesis of Theorem 3.1 because the following example shows that it is not true in the case $F \in (DN), F \notin (LB_\infty)$.

Example 3.2. ([24]) Let $\alpha = (\alpha_n)$ be an exponent sequence. Then there exists a non-locally bounded holomorphic function from $\ell_1 \in (\Omega)$ into $\Lambda_\infty(\alpha) \in (DN)$. Indeed, choose a sequence $j_k \uparrow \infty$ such that $[\alpha_{j,k}] < [\alpha_{j,k+1}]$ for $k \geqslant 1$. The function $\Lambda_\infty\big((\alpha_{j,k})\big) \ni (\xi_{j,k}) \mapsto (\xi_j) \in \Lambda_\infty(\alpha)$ with

$$\xi_j = \begin{cases} \xi_{j,k} & \text{if } j = j_k \\ 0 & \text{if } j \neq j_k \text{ for } k \geqslant 1 \end{cases}$$

identifies $\Lambda_\infty\big((\alpha_{j,k})\big)$ with a subspace of $\Lambda_\infty(\alpha)$. Note that

$$\Lambda_\infty\big((\alpha_{j,k})\big) \cong \overline{\operatorname{span}}\big(z^{[\alpha_{j,k}]}\big)_{k=1}^\infty \subset H(\mathbb{C}).$$

Define a holomorphic function $f : \ell_1 \to \Lambda_\infty\big((\alpha_{j,k})\big)$ given by

$$f\big((z_n)\big)(z) = \sum_{n\geqslant 0} z_n^n \sum_{k\geqslant 1} \frac{n^{[\alpha_{j,k}]}.z^{[\alpha_{j,k}]}}{[\alpha_{j,k}]!}.$$

It can be easily seen that $f\big(\varepsilon\big(\delta_n^{[\alpha_{j,m}]}\big)\big)(\varepsilon^{-1}) \to \infty$ as $m \to \infty$, for every $\varepsilon > 0$, hence, we deduce that f is non-locally bounded.

Note that, in particular, when $F = \mathcal{D}[a,b] \cong s \cong \Lambda_\infty(\log(n+1))$ where $\mathcal{D}[a,b]$ is the space of Whitney functions on $[a,b] \subset \mathbb{R}$ and s is the space of the rapidly decreasing sequences, we have $F \in (DN)$ but $F \notin (LB_\infty)$.

For the dual Fréchet case, in [14] Hai proved the following result:

Theorem 3.3. ([14]) *Let E, F be Fréchet spaces and E a Schwartz space with an absolutely Schauder basis, F a separable space and D an open subset of E^*. Then every F^*-valued holomorphic function on D is locally bounded if one of the following conditions holds:*

i) $E \in (\underline{DN}), F \in (\overline{\overline{\Omega}})$;

ii) $E \in (DN), F \in (\overline{\Omega})$;

iii) $E \in (\overline{DN}), F \in (\widetilde{\Omega})$.

Danh and Son [6] considered the problem for the case $(\widetilde{\Omega}, DN)$ when F is nuclear.

In [38] we proved the following and note that it is also true if the property "nuclear" of F is replaced by "Schwartz with an absolute basis" by using the same arguments. This is an extension of the (iii)-case in Theorem 3.3.

Proposition 3.4. ([38], Proposition 4.2) *Let E be a Fréchet space with $E \in (\Omega)$ and F a nuclear Fréchet space (or Fréchet-Schwartz space having an absolute basis) with $F \in (LB_\infty)$. Then every E^*-valued holomorphic function on an open set D in F^* is locally bounded.*

Thus, we get the following recapitulation, where E, F *are Fréchet,* $D \subset E^*$ *is open*:

Ref.	E	F	Conclusion
[14]	$(\underline{DN})$, Schwartz, abs. basis	$(\overline{\overline{\Omega}})$, separable	$H(D, F^*) = H_{LB}(D, F^*)$
[14]	(DN), Schwartz, abs. basis	$(\overline{\Omega})$, separable	
[6]	(DN), nuclear	$(\widetilde{\Omega})$	
[38]	(LB_∞), Schwartz, abs. basis	(Ω)	
[38]	(LB_∞), nuclear	(Ω)	

4. Regularity of the Space of Germs of Fréchet Valued Holomorphic Functions

A seminorm ϱ on $H(D, F)$ is said to be τ_ω-continuous if there exist a compact set K in D and a continuous seminorm α on F such that, for every neighborhood V of K in D, there exists $C(V) > 0$ such that

$$\varrho(f) \leqslant C(V) \sup_{z \in V} \alpha(f(z)), \quad \forall\, f \in H(D, F).$$

Recall that $f \in H(K, F)$ if there exist a neighborhood V of K in E and a holomorphic function $\hat{f} : V \to F$ whose germ on K is f.

We also recall that the space $H(K, F)$ of germs of F-valued holomorphic functions on a compact set K is regular if every bounded subset in

$$H(K, F) := \liminf_{K \subset U}(H(U, F); \tau_\omega)$$

is contained and bounded in some $[H(U, F); \tau_\omega]$.

The problem of the regularity of the space $H(K)$ was investigated by several authors. Chae [3] proved that $H(K)$ is regular for every compact subset K of a Banach space E. When E is a metrizable locally convex space, $H(K)$ is represented as an inductive limit of a sequence of (DF)-spaces. Using a theorem of Grothendieck on bounded subsets in an inductive limit of a sequence of (DF)-spaces, Mujica [30] generalized the result of Chae. In [51] Vogt gave a general characterization for the regularity of the inductive limit of a sequence of Fréchet spaces. In 1998, Hao and Tac [19] proved that $H(K, F)$ is regular for every set of uniqueness K of a quotient space of a power series space of infinity type and $F \in (LB_\infty)$ a reflexive Frechet space. They also proved the above assertion in the case where $F \in (DN)$ is Frechet and K is compact in $\mathbb{C}^n$. In 2005 [37], Quang extended this result to the case where $F \in (DN)$ is reflexive Fréchet, and K is a balanced convex compact set in a nuclear Fréchet space $E \in (\widetilde{\Omega}_K)$. In [38] Quang solved this problem for the case where F is a reflexive Fréchet space, $F \in (LB_\infty)$ and E a Fréchet-Schwartz space, $E \in (\Omega)$ and K is a compact set of uniqueness in E.

Here, we recall that a compact set K in a Fréchet space is called a set of uniqueness if for all $f \in H(K)$ and $f|_K = 0$ then $f = 0$ on a neighborhood of K in E.

The following table summarizes our results on this problem where *E is Fréchet, F is reflexive, Fréchet and K is compact in E.*

Ref.	E	F	K	Conclusion
[37]	nuclear, $(\widetilde{\Omega}_K)$	(DN)	balanced, convex	$H(K, F)$
[38]	Schwartz, (Ω)	(LB_∞)	of uniqueness	is regular

We will use these results as one of the important keys in resolving the problems of weakly holomorphic functions, separately holomorphic functions,...

5. Weakly Holomorphic Functions, δ-separately Holomorphic Functions

A F-valued function f on K is said to be weakly holomorphic on K if for every $x^* \in F^*_\beta$, the topological dual space of F equipped with the strong topology $\beta(F^*, F)$, $x^* f$ can be extended holomorphically to a neighborhood of K. By $H_w(K, F)$ we denote the space of F-valued weakly holomorphic functions on K.

A function $f : K \to H(F)$ is called δ-separately holomorphic (i.e. in the sense of Dirac) if $\delta_x \circ f \in H(K)$ for every $x \in F$, where δ_x is the Dirac delta function of x, i.e. $\delta_x : H(F) \to \mathbb{C}$ given by

$$\delta_x(\varphi) = \varphi(x) \quad \text{for each } \ \varphi \in H(F).$$

By $H_\delta(K, H(F))$ we denote the vector space of separately holomorphic functions on K with values in $H(F)$.

One of the aims of this section is to find some necessary and sufficient conditions by means of the property (DN) for which

$$H_w(X, F) = H(X, F) \tag{w}$$

$$H_\delta(X, H_b(F^*)) = H(X, H_b(F^*)) \tag{δ}$$

where E, F are Fréchet spaces and X is a compact set in E^*.

The statement (w) has been investigated by several authors. Siciak in [42] and Waelbroeck in [54] considered this problem for the case, where $\dim E < \infty$ and F^*_β is a Baire space. Then, in [23] Khue and Tac showed that (w) holds in the case where F^*_β is still Baire and either E is a nuclear metric space or F is nuclear. The Baireness of F^*_β plays a very important part in the works of the above authors. However, at present, when F^*_β is not Baire, in particular, that F is a Fréchet space which is not Banach, the identity (w) has not been established by any authors. In [12] Hai proved that (w) holds for every $\widetilde{L}$-regular compact set in any nuclear Fréchet space E if and only if $F \in (DN)$ is a Fréchet space. Here a compact subset X in a Fréchet space E is called $\widetilde{L}$-regular if $[H(X)]^* \in (\widetilde{\Omega})$. In [16] Hai and Khue advanced the result with a weaker condition "X is (LB^∞)-regular (that means $[H(X)]^* \in (LB^\infty)$) in a Fréchet space". In 2003, Hoan [20] considered the result of Hai in the case where X is a compact set of uniqueness in $\mathbb{C}^n$ and F is a Fréchet space, $F \in (LB_\infty)$. Then, as an extension, in [13], Hai proved this problem for the case where E is a Fréchet space, $E \in (\Omega)$.

Consideration for the (w) in the dual Fréchet case has been made. In [15], Hai and Bang proved that a nuclear Fréchet with basis space E having a continuous norm has the property (DN) if and only if either $H(X, F^*) = H_w(X, F^*)$ for every compact determining polydisc X in E^* and for every Fréchet space F, or $H(X, F) = H_w(X, F)$ and $H(X)$ is quasi-Montel for every compact determining polydisc X in E^* and for every Fréchet space $F \in (\overline{DN})$.

The first result of this section is the following, which is an extension of the previous result.

Theorem 5.1. ([39]) *Let E be a nuclear Fréchet space with a basis having a continuous norm. Then the following are equivalent*

a) $E \in (DN)$;

b) $H(X,F) = H_w(X,F)$ *and* $H(X)$ *is semi-reflexive for every compact determining polydisc* X *in* E^* *and for every Fréchet space* $F \in (LB_\infty)$.

Conversely, we have the following

Theorem 5.2. ([39]) *Let F be a Fréchet space. Then the following are equivalent*

a) $F \in (LB_\infty)$;

b) $H(X,F) = H_w(X,F)$ *for every compact determining polydisc* X *in the dual* E^* *of a nuclear Fréchet space* E *with* $E \in (DN)$.

In order to prove the following theorem about δ-separately holomorphic functions we recall a result of Hai and Khue [16].

Proposition 5.3. ([16, Theorem 2.1]) *Let F be a Fréchet space . Then*

$$H_w(K,F) = H(K,F)$$

holds for every LB^∞*-regular set* K *in a Fréchet space* E *if and only if* $F \in (DN)$.

Here a compact set K in a Fréchet space E is called LB^∞-regular if $[H(K)]^* \in (LB^\infty)$.

Using this theorem as one of the main tools we obtain

Theorem 5.4. ([39]) *Let E be a nuclear Fréchet space with a basis and a continuous norm. Then the following are equivalent*

a) $E \in (DN)$;

b) $H(X, H_b(F^*)) = H_\delta(X, H_b(F^*))$ *for every compact determining polydisc* X *in* E^* *and for every Fréchet-Schwartz space* $F \in (LB_\infty)$ *having an absolute basis.*

In [15], Hai and Bang also proved this result but for a relatively narrow class of Fréchet spaces: "Fréchet-Schwartz space $F \in (\overline{DN})$ having an absolute basis".

Corollary 5.5. *Let E be a nuclear Fréchet space with a basis and a continuous norm. Then the following are equivalent*

a) $E \in (DN)$;

b) $H(X,F^*) = H_w(X,F^*)$ *for every compact determining polydisc* X *in* E^* *and for every Fréchet space* F.

6. $\sigma(\cdot, W)$-holomorphic Functions and Theorems of Vitali Type

Note that when defining the "weak notion" for a Fréchet-valued function we require that for all $u \in F'$, the function uf should have the same property. Now we consider "weak notion" in a more general situation.

Let E, F be locally convex spaces, D an open subset of E and W a subset of F'. A function $f : D \to F$ is called $\sigma(F, W)$-holomorphic if uf are holomorphic for all $u \in W \subset F'$.

Recently in [1], Arendt and Nikolski have given the notion about $\sigma(F, W)$-holomorphic functions for the case where F is a Banach space, W is a subspace of F' defining the topology of F. They have established the relation between the $\sigma(F, W)$-holomorphicity and the holomorphicity of a function $f : D \to F$ in the case where $D \subset \mathbb{C}$ is an open subset and F is a Banach space ([1, 1.8]). In [13] Hai extended the above result for the Fréchet case when D is an open set either in a Schwartz-Fréchet space or in $\mathbb{C}$ and F is a Fréchet space. He proved that

Theorem 6.1. ([13]) *Let E, F be Schwartz-Fréchet spaces and D an open subset of E and $f : D \to F$. Assume that $E \in (\Omega), F \in (LB_\infty)$ and f is a $\sigma(F, W)$-holomorphic function for a subspace W of F' defining the topology of F. Then f is holomorphic on a dense open subset of D.*

Theorem 6.2. ([13]) *Let $f : D \to F$ be a $\sigma(F, W)$-holomorphic function where D is an open subset in $\mathbb{C}$ and F a Fréchet space, $F \in (LB_\infty)$. Then f is holomorphic on a dense open subset of D.*

In [13] the author posed the question: *"Is the above theorem true for the case $n \geqslant 2$?"* By using some knowledge of pluripotential theorem (namely, relative extremal plurisubharmonic function) we consider Theorem 6.2 for the case where D is a hyperconvex subset of $\mathbb{C}^n$ with $n \geqslant 2$.

Theorem 6.3. ([40]) *Let F be a Schwartz-Fréchet space, D a hyperconvex subset of $\mathbb{C}^n$ and $f : D \to F$. Assume that $F \in (DN)$ and f is a $\sigma(F, W)$-holomorphic function for a subspace W of F' defining the topology of F. Then f is holomorphic on a dense open subset of D.*

Here we say that an open bounded set $D \subset \mathbb{C}^n$ is hyperconvex if it is connected and there is a continuous plurisubharmonic function $\varrho : D \to (-\infty, 0)$ such that the set $\{z \in D : \varrho(z) < c\}$ is a relatively compact subset in D for each $c \in (-\infty, 0)$. We need the following result.

Lemma 6.4. ([43]) *Let K be a compact Stein set in $\mathbb{C}^n$, i.e. K has a Stein neighborhoods basis in $\mathbb{C}^n$. Then $[H(K)]' \in (\overline{\Omega})$ if and only if K is pluriregular in every neighborhood W of K.*

In the next part of this section we consider the above-mentioned results for Fréchet spaces and introduce the conditions under which the conclusion "holomorphic on a dense open subset of D" in Theorems 6.1, 6.2, 6.3 is substituted by "holomorphic on D".

First we need the following

Proposition 6.5. ([40]) *Let E, F be Fréchet spaces, and $D \subset E$ be an open set. Let $f : D \to F$ be a locally bounded function such that $\varphi \circ f$ is holomorphic for all $\varphi \in W \subset F'$, where W is separating. Then f is holomorphic.*

Here, a subset W of F' is called separating if for all $x \in F \setminus \{0\}$ there exists $\varphi \in F'$ such that $\varphi(x) \neq 0$.

Remark 6.6. This lemma is stated in the case where $F = \mathbb{C}$ by Arendt and Nikolski [1, Theorem 3.1].

In [1] Arendt and Nikolski showed that for $D \subset \mathbb{C}$ being an open set and W a norming subspace of X', where X is a Banach space, a $\sigma(X, W)$-holomorphic function $f : D \to X$ is holomorphic on a dense open subset of D if f is not assumed to be locally bounded (Theorem 1.8). For this reason, Hai also obtained similar conclusions in his Theorems 6.1 and 6.2. Our Theorem 3.1 has overcome that obstacle. Hence, from Lemma 6.5 and Theorem 3.1 together with the Baireness of a Fréchet space and new characterizations introduced in Sec. 2 we obtain

Theorem 6.7. ([40]) *Let E, F be Fréchet-Schwartz spaces and $f : E \to F$ be a $\sigma(F, W)$-holomorphic function such that f is bounded on all bounded sets in E for a subspace W of F' defining the topology of F. Assume that $E \in (\Omega)$, E has an absolutely Schauder basis and $F \in (LB_\infty)$. Then f is holomorphic.*

Remark 6.8. • In [13] Hai has shown that his result (hence, Theorem 6.7) is not true for Banach-valued analytic functions. Specifically, let $\{\xi_n\}$ be a dense sequence in $\mathbb{R}$. Consider the function $f : \mathbb{R} \to c_0$ given by

$$f(x) = \Big(\frac{1}{n(1 + n(x - \xi_n)^2)}\Big)_{n \geqslant 1}.$$

Let $W = \operatorname{span}\{e_n\} \subset c_0' = \ell_1$ where $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$. Then W defines the topology of c_0 and f is real analytic on $\mathbb{R}$ for every $u \in W$. But f is not real analytic at ξ_n for all $n \geqslant 1$, and hence f is not real analytic for every non-empty open subset of $\mathbb{R}$.

• Also in [13] Hai has shown that his result (hence, Theorem 6.7) is not true for every Fréchet space F. Namely, let B be an infinite dimension Banach space. There exists a subspace W of B' defining the topology of B but it does not determine boundedness. Take a dense subset $\{z_n\}$ of the unit disc Δ in $\mathbb{C}$. He constructed a sequence $\{h_n\}$ of $\sigma(B, W)$-holomorphic functions $h_n : \Delta \to B$ such that h_n is not holomorphic at z_n. Then $h = \{h_n\} : \Delta \to B^{\mathbb{N}}$ is a $\sigma(B^{\mathbb{N}}, \oplus W)$-holomorphic function which is not holomorphic at every z_n, hence h is not holomorphic on Δ.

In view of Theorem 6.2 and by the same arguments in the proof of Theorem 6.7 we obtain the result for the case where $E = \mathbb{C}$. However, the hypothesis of the Schwartz property for F is superfluous. Then, we have

Theorem 6.9. ([40]) *Let F be a Fréchet space having the property (LB_∞) and $f : \mathbb{C} \to F$ be a $\sigma(F, W)$-holomorphic function such that f is bounded on all bounded sets in $\mathbb{C}$ for a subspace W of F' defining the topology of F. Then f is holomorphic.*

We finish this section by a result on holomorphic extension of $\sigma(., W)$-holomorphic functions.

Theorem 6.10. ([40]) *Let E be a Fréchet space, D be an open, connected subset of E and $D_0 \subset D$ is not rare in D. Let F be a Fréchet-Montel space and W be a subset of F'_β defining the topology of F. Assume that $f : D_0 \to F$ is a function such that $\varphi \circ f$ has a holomorphic extension to D for all $\varphi \in W$. Then f has a holomorphic extension to D.*

In order to prove the theorem we need the following lemmas.

Lemma 6.11. (Uniqueness) ([40]) *Let D be a convex open set in a Fréchet space E and $f : D \to F$ be holomorphic, where F is a barrelled locally convex space. Assume that $D_0 = \{z \in D : \ f(z) \in G\}$ is not rare in D, where G is a closed subspace of F. Then $f(z) \in G$ for all $z \in D$.*

Lemma 6.12. ([40]) *Let E be a Fréchet space, D be an open, connected subset of E and F be a Fréchet-Montel space with the algebraic dual F^*. Assume that $h : D \to F^*$ is $\sigma(F^*, F)$-holomorphic and $D_0 = \{z \in D : h(z) \in F'_\beta\}$ is not rare in D. Then $h(z) \in F'_\beta$ for all $z \in D$ and $h : D \to F'_\beta$ is holomorphic.*

In the remaining part of this section we apply the above results to consider the sequences of the holomorphic functions which converge on a subset of a domain D. We look for additional properties which ensure convergence on the entire domain. Such results are of Tauberian type. An important example is Vitali's theorem, where subsets admitting a limit point in D are considered and local boundedness is a possible additional property. In the scalar case, it seems that, so far, the easiest proof of Vitali's theorem has been given with the help of Montel's theorem. However, Montel's theorem no longer holds in the vector-valued case if the underlying Banach space is infinite dimensional. It is surprising that Vitali's theorem is still valid. Here we consider theorem of Vitali-type in the Fréchet case, but subsets have to satisfy a stronger condition: "non-rare".

Theorem 6.13. (Vitali-type 1) ([40]) *Let E, F be Fréchet spaces and $D \subset E$ be an open convex set. Assume that $\{f_i\}_{i\in\mathbb{N}}$ is a locally bounded sequence of holomorphic functions on D with values in F. Then the following assertions are equivalent:*

i) *The sequence $\{f_i\}_{i\in\mathbb{N}}$ converges uniformly on all compact subsets of D to holomorphic function $f : D \to F$;*

ii) *The set $D_0 = \{z \in D : \lim\limits_{i\to\infty} f_i(z)$ exists$\}$ is not rare in D;*

iii) *There exists $z_0 \in D$ such that $\lim\limits_{i\to\infty} f_i(z_0)$ exists.*

Now we consider a theorem of Vitali-type for a sequence of holomorphic functions between Fréchet spaces having linear topological invariants. Here, instead of the local boundedness, the sequence is assumed to be bounded on all bounded sets in E. The proof of the theorem is based on Theorems 6.7, 6.13 and 3.1.

Theorem 6.14. (Vitali-type 2) ([40]) *Let E, F be Fréchet-Schwartz spaces and $\{f_i\}_{i\in\mathbb{N}}$ be a sequence of holomorphic functions from E into F such that $\{f_i\}_{i\in\mathbb{N}}$ is bounded on all bounded sets in E. Assume that E has an absolutely Schauder basis, $E \in (\Omega), F \in (LB_\infty)$. Then the following assertions are equivalent:*

i) *The sequence $\{f_i\}_{i\in\mathbb{N}}$ converges uniformly on all compact subsets of E to a holomorphic function $f : E \to F$;*

ii) *The set $D = \{z \in E : \lim_i f_i(z) \text{ exists}\}$ is not rare in E;*

iii) *There exists $z_0 \in E$ such that $\lim_i f_i(z_0)$ exists.*

Corollary 6.15. ([40]) *Let E, F be Fréchet-Schwartz spaces and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of holomorphic functions from E into F such that $\{f_i\}_{i\in\mathbb{N}}$ is bounded on all bounded sets in E. Assume that E has an absolutely Schauder basis, $E \in (\Omega), F \in (LB_\infty)$. Then there exists a subsequence which converges to a holomorphic function uniformly on all compact subsets of E if one of the following conditions is satisfied:*

i) *The set $D = \{z \in E : \{f_n(z) : n \in \mathbb{N}\}$ is relatively compact in $E\}$ is not rare in E;*

ii) *There exists $z_0 \in E$ such that the set $\{f_n(z_0) : n \in \mathbb{N}\}$ is relatively compact in F.*

As in the Theorem 6.9 we obtain the following for the case $E = \mathbb{C}$.

Theorem 6.16. (Vitali-type 3) ([40]) *Let F be a Fréchet space having the property (LB_∞) and $\{f_i\}_{i\in\mathbb{N}}$ be a sequence of holomorphic functions from $\mathbb{C}$ into F such that $\{f_i\}_{i\in\mathbb{N}}$ is bounded on all bounded sets in $\mathbb{C}$. Then the following assertions are equivalent:*

i) *The sequence $\{f_i\}_{i\in\mathbb{N}}$ converges uniformly on all compact subsets of $\mathbb{C}$ to a holomorphic function $f : \mathbb{C} \to F$;*

ii) *The set $D = \{z \in \mathbb{C} : \lim_i f_i(z) \text{ exists}\}$ has an accumulation point in $\mathbb{C}$;*

iii) *There exists $z_0 \in \mathbb{C}$ such that $\lim_i f_i(z_0)$ exists.*

From Therem 6.16, as in Corollary 6.15 we have

Corollary 6.17. ([40]) *Let F be a Fréchet space and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of holomorphic functions from $\mathbb{C}$ into F such that $\{f_i\}_{i\in\mathbb{N}}$ is bounded on all bounded sets in $\mathbb{C}$. Assume that $F \in (LB_\infty)$. Then there exists a subsequence which converges to a holomorphic function uniformly on all compact subsets of $\mathbb{C}$ if one of the following conditions is satisfied:*

i) *The set* $D = \{z \in \mathbb{C} : \{f_n(z) : n \in \mathbb{N}\}$ *is relatively compact in* $E\}$ *has an accumulation point in* $\mathbb{C}$;

ii) *There exists* $z_0 \in \mathbb{C}$ *such that the set* $\{f_n(z_0) : n \in \mathbb{N}\}$ *is relatively compact in* F.

7. The Exponential Representation of Holomorphic Functions and Absolute Representation System

Let E be a locally convex space and $\{x_k\}$ be a sequence in E. We say that $\{x_k\}$ is an absolute representation system (abbreviated ARS) if every element $x \in E$ can be written in the form

$$x = \sum_{k \geqslant 1} \xi_k(x) x_k$$

where the series is absolutely convergent. If ξ_k can be chosen as continuous linear functionals on E, the system $\{x_k\}$ is called a linearly absolute representation system which is denoted by LARS.

An ARS (resp. LARS) in $H(D)$ of the form $\{\exp u_k\}$, where u_k are continuous linear functionals on E, D is an open set in E, is said to be an absolutely (resp. linearly absolutely) exponential representation system of $H(D)$. It is denoted by AERS (resp. LAERS).

The ARS in $H(D)$, where D is a convex domain in $\mathbb{C}^n$, were investigated by many authors, for example Leontev, Korobeinik and others, for the case of one complex variable, and by Morzhakov, Napalkov and others for the case of several complex variables.

The author proved in [32] that "a nuclear Fréchet space $E \in (DN)$ (resp. $(\underline{DN})$)" is equivalent to "$H(E^*, F^*)$ has an E^*-AERS for every Fréchet space $F \in (LB^\infty)$ (resp. $(\overline{\overline{\Omega}})$)". Before that, this result was also proved by Ha and Khue [11] under a stronger assumption "$F \in (\overline{\Omega})$".

Theorem 7.1. ([32]) *Let E be a nuclear Fréchet space. Then the following conditions are equivalent*

i) $E \in (DN)$, *(resp.* $(\underline{DN})$*);*

ii) $H(E^*, F^*)$ *has an* F^**-AERS for every Fréchet space* F *having the property* (LB^∞) *(resp.* $(\overline{\overline{\Omega}})$*.) This means that every* $f \in H(E^*, F^*)$ *can be written in the form*

$$f(u) = \sum_{k \geqslant 1} \xi_k(u) \exp x_k(u),$$

where the series is absolutely convergent in $H(E^*, F^*), \{x_k\} \subset E$ *and* $\{\xi_k\} \subset F^*$.

We have thus investigated the exponential representation of holomorphic functions between duals of Fréchet spaces. We can summarize our results as follows, where E, F *are Fréchet.*

Ref.	F	$E \in (?) \Leftrightarrow H(E^*, F^*)$ has a E^*-AERS
[32]	$(\overline{\overline{\Omega}})$	$(\underline{DN})$
[32]	(LB^∞)	(DN)

Now let us mention this problem for the Fréchet case. In [10] Ha and Khue proved that a Fréchet space E is nuclear and $E \in (H_u)$ if and only if every holomorphic function on E with values in a Banach space B can be written in the form

$$f(x) = \sum_{k \geqslant 1} \xi_k \exp u_k(x), \tag{Exp}$$

where the series is absolutely convergent in the space $H(E, B)$ of holomorphic functions on E with values in B equipped with the compact-open topology.

In [35] Quang has considered the above result in another situation under the hypotheses $E \in (H_{ub})$ and $F \in (DN)$.

Theorem 7.2. ([35]) *Let E be a Fréchet space. Then E is nuclear and $E \in (H_{ub})$ if and only if every holomorphic function on E with values in a Fréchet space $F \in (DN)$ can be written in the form* (Exp) *where the sequences $(\xi_k) \subset F, (u_k) \subset E^*$, the dual space of E, and the series is absolutely convergent in the space $H_b(E, F)$.*

By the same reasons which affirm that there do not exist Fréchet spaces $E \in (\Omega), F \in (LB_\infty)$ satisfying (HUN) presented in Sec. 2, this theorem is not true for the case $E \in (\Omega), F \in (LB_\infty)$.

Thus, the results of the problem may be summarized as in the following table:

Ref.	E - Fréchet	relation		for F
[10]	(H_u), nuclear	$\Leftrightarrow$	$\forall f \in H(E, F)$	Banach
[35]	(H_{ub}), nuclear		$f(x) = \sum_{k \geqslant 1} \xi_k \exp u_k(x)$	Fréchet, (DN)

Using the above theorems we obtain the following examples:

Proposition 7.3. ([11, 35]) i) *$H(\mathbb{C}^\infty)$ has a LEARS.*

ii) *If $H(E^*)$ has a LEARS, where E is a nuclear Fréchet space, then $E \in (DN)$.*

iii) *If E and F are nuclear Fréchet spaces such that F^* has an absolute basis then:*

(a) *$H(E \times F^*)$ has a LEARS if $H(F^*)$ has a LEARS and $E \in (H_{ub})$.*

(b) *Conversely, $E \in (H_{ub})$ if $H(E \times F^*)$ has a LEARS and $F \in (DN)$.*

8. Separately Holomorphic Functions

The well-known Hartogs theorem on the holomorphy of separately holomorphic functions in $\mathbb{C}^n$ was extended to the infinite dimensional case by several authors. In particular, this theorem is true for the classes of Fréchet spaces and of dual Fréchet- Schwartz spaces. However, the problem is more complicated in the mixed case. In this section we will investigate the holomorphy of separately holomorphic functions in connection with their local Dirichlet representations and with the properties (Ω, LB_∞).

The regularity of the space $H(K, F)$, which was presented in Sec. 4, is one of the keys to prove the above result.

First, we introduce the notion of local Dirichlet representations. Let E be a locally convex space and D an open subset in E. A function $f : D \to \mathbb{C}$ is said to have a local Dirichlet representation on D if for every $x_0 \in D$ there exists a neighborhood U of x_0 and sequences $(\xi_k) \subset \mathbb{C}, (u_k) \subset E^*$, such that (Exp) holds for every $x \in U$ and $\sum_{k\geqslant 1} |\xi_k| \exp \|u_k\|_K^* < \infty$ for every compact set $K \subset U$.

The global Dirichlet representation of the entire functions was investigated in [5]. In [37, 38] Quang proved the results on this problem as follows.

Theorem 8.1. ([37, 38]) *Let F be a nuclear Fréchet space and K be a non-pluripolar, balanced, convex, compact set* (*resp. of uniqueness*) *in a nuclear Fréchet space E which has the bounded approximation property* (*resp.* $\in (\Omega)$)*. Then the following conditions are equivalent:*

a) *$F \in (DN)$* (*resp.* (LB_∞))*;*

b) *Every separately holomorphic function on $K \times F^*$ is holomorphic;*

c) *Every separately holomorphic function as in* b) *has a local Dirichlet representation.*

Theorem 8.2. ([37]) *Let F be a nuclear Fréchet space, $F \in (DN)$ and K be a compact set a Stein space X such that K is not pluripolar in every irreducible branch of all neighborhoods of K. Then every separately holomorphic function on $K \times E^*$ is holomorphic.*

Remark 8.3. In the case of $(\overline{\overline{\Omega}}, DN)$, the mixed Hartogs Theorem is proved by Khue and Thanh [24] for separately holomorphic functions on an open set $E \times D$ in $E \times F^*$.

By the same arguments as in the proof of the above theorem we obtain the following, which was proved by Danh and Son [6] for the case where $F \in (DN), E \in (\widetilde{\Omega})$.

Theorem 8.4. ([36]) *Let F be a nuclear Fréchet space. Then the following conditions are equivalent:*

a) $F \in (LB_\infty)$;

b) *Every separately holomorphic function on an open set $U \times V$ of $E \times F^*$, where $E \in (\Omega)$ is a nuclear Fréchet space having a basis, is holomorphic;*

c) *Every separately holomorphic function as in* b) *has a local Dirichlet representation.*

9. Holomorphic Functions of Uniformly Bounded Type on Tensor Products

A holomorphic function f from E to F is called of uniformly bounded type if there exists a neighborhood U of $0 \in E$ such that $f(rU)$ is bounded for all $r > 0$. By $H_{ub}(E, F)$ we denote the linear subspace of the space of holomorphic functions from E to F, consisting of all functions of uniformly bounded type. We write $H_{ub}(E)$ rather than $H_{ub}(E, \mathbb{C})$. A locally convex space E is called to have the property (H_{ub}), and write $E \in (H_{ub})$, if the identity $H(E) = H_{ub}(E)$ holds.

The problems about characterizations of locally convex spaces E and F satisfying the identities

$$H(E, F) = H_{ub}(E, F); \quad H_b(E, F) = H_{ub}(E, F)$$

have been investigated by many authors. Colombeau and Mujica [4] have shown $H(E) = H_{ub}(E)$ for each (DFM)-space. Galindo, Garcia and Maestre [8] extended it to (DFC)-spaces. For the class of (DF)-spaces three above authors have shown the equality $H_b(E) = H_{ub}(E)$ algebraically and topologically. In Fréchet spaces the situation is quite different. A classical example of Nachbin [31] shows $H_{ub}(E) \subsetneq H(E)$ for the nuclear Fréchet space $E = H(\mathbb{C})$. This example was extended to other power series spaces by Meise and Vogt [27]. In [29] Meise and Vogt have shown that a nuclear locally convex space E satisfies $H(E) = H_{ub}(E)$ if and only if the entire functions on E are universally extendable in the following sense: Whenever E is a topological linear subspace of a locally convex space F with a fundamental system of continuous semi-norms induced by semi-inner product, then each $f \in H(E)$ has a holomorphic extension to F. An important necessary and sufficient condition of $H(E) = H_{ub}(E)$ where E is nuclear Fréchet, was found by two above authors [27]. Galindo, Garcia and Maestre [9] have proved that for a Fréchet space E, the identity $H_b(E) = H_{ub}(E)$ is equivalent to

$$[H_b(E, F), \tau_b]_{bor} = (H_{ub}(E, F), \tau_{ub}) := \liminf_{p}(H_b(E_p, F), \tau_b)$$

which is regular for every Banach space F. Here τ_b is the topology of uniform convergence on all bounded sets and $[H_b(E, F), \tau_b]_{bor}$ denotes the bornological space associated to $H_b(E, F)$.

In recent years, various authors have investigated this problem in the interrelation to linear topological invariants. In [17] Hai and Quang have shown that $H_b(E, F) = H_{ub}(E, F)$ for Fréchet spaces E, F with $E \in (H_{ub}), F \in (\overline{DN})$. Recently [35], Quang extended this result to the case $F \in (DN)$.

Theorem 9.1. ([35]) *Let E be a Fréchet space. Then $E \in (H_{ub})$ if and only if*

$$H(E, F) = H_{ub}(EF)$$

for all Fréchet spaces $F \in (DN)$.

As for the properties $(\Omega), (H_u)$, the problem of the inheritability of the property (H_{ub}) by the Cartesian product and tensor product operations has been considered. In [21] Hoang and Quang gave the possible answer to the problem.

The example in [24] which has been quoted at Sec. 3 also shows that one cannot extend the result of Quang in [35] from the case where $E \in (H_{ub})$ to the case where $E \in (\Omega)$, as well as from the condition $F \in (LB_\infty)$ to $F \in (DN)$. In addition, the identity map from $\mathcal{D}[a,b] \cong s$ (with $\mathcal{D}[a,b] \in (\Omega), \mathcal{D}[a,b] \notin (H_{ub})$) into itseft (with $\mathcal{D}[a,b] \in (DN), \mathcal{D}[a,b] \notin (LB_\infty)$) also demonstrates the truth of this statement.

It is known that in the case where E is nuclear Fréchet, $E \in (\Omega)$ if $E \in (H_{ub})$. Then by Vogt [50] there exists a continuous linear map R from $\ell^1(I)\widehat{\otimes}_\pi s$ onto E, where s is the space of all rapidly decreasing sequences. Thus $g = fR \in H_b(\ell^1(I)\widehat{\otimes}_\pi s, F)$ for all $f \in H_b(E,F)$. Note that $s \in (DN)$. Dealing with the above problems, we would like to take a more general situation. Instead of $\ell^1(I)$ (resp. s, F) we shall consider a Banach space (resp. Fréchet spaces having the linear topological invariants). Then, we obtain the following

Theorem 9.2. ([34]) *Let E, F be Fréchet spaces and E a quasinormable space with an absolute basis. Then for every Banach space G we have*

$$H_b(G\widehat{\otimes}_\pi E, F) = H_{ub}(G\widehat{\otimes}_\pi E, F)$$

if one of the following conditions holds:

i) $E \in (\overline{\overline{\Omega}}), F \in (\underline{DN})$;

ii) $E \in (\widetilde{\Omega}), F \in (DN)$.

As in the problem (HUN), the theorem is not true in the case $E \in (\Omega), F \in (LB_\infty)$.

We now collect our results of the problem from [34], where *E, F, G are Fréchet, F^*, G have absolute bases:*

Ref.	E	F	G	Conclution
[34]	$(\overline{\overline{\Omega}})$, Schwartz	$(\underline{DN})$, Montel	$(\widetilde{\Omega})$	$H_b(G\widehat{\otimes}_\pi F^*, E^*) =$
	$(\widetilde{\Omega})$, Schwartz	(DN), Montel	$(\widetilde{\Omega})$	$= H_{ub}(G\widehat{\otimes}_\pi F^*, E^*)$
	(LB^∞), Schwartz	$(\overline{DN})$, Montel	$(\widetilde{\Omega})$	

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