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On the Hamiltonian and Classification Problems for some Families of Split Graphs*

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Abstract. A graph $G = (V, E)$ is called a split graph if there exists a partition $V = I \cup K$ such that the subgraphs $G[I]$ and $G[K]$ of G induced by I and K are empty and complete graphs, respectively. In this paper, we survey results on the hamiltonian and classification problems for split graphs G with the minimum degree $\delta(G) \geq |I| - 4$.

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Key words: Split graph, Burkard-Hammer condition, Burkard-Hammer graph, hamiltonian graph, maximal nonhamiltonian split graph.

1. Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ (or V and E in short) will denote its vertex-set and its edge-set, respectively. For a subset $W \subseteq V(G)$, the set of all neighbours of W is denoted by $N_G(W)$ or $N(W)$ in short. For a vertex $v \in V(G)$, the degree of v, denoted by $\deg(v)$, is the number $|N(v)|$. The minimum degree of G, denoted by $\delta(G)$, is the number $\min\{\deg(v) \mid v \in V(G)\}$. By $N_{G,W}(v)$ or $N_W(v)$ in short we denote the set $W \cap N_G(v)$. Then the degree of v with respect to W, denoted by $\deg_W(v)$ is the number $|N_W(v)|$. The subgraph of G induced by W is denoted by $G[W]$. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

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A graph $G = (V, E)$ is called a *split graph* if there exists a partition $V = I \cup K$ such that the subgraphs $G[I]$ and $G[K]$ of G induced by I and K are empty and complete graphs, respectively. We will denote such a graph by $S(I \cup K, E)$. Further, a split graph $G = S(I \cup K, E)$ is called a *complete split graph* if every $u \in I$ is adjacent to every $v \in K$. The notion of split graphs was introduced in 1977 by Földes and Hammer [4]. These graphs are interesting because they are related to many problems in combinatorics (see [3, 5, 9]) and in computer science (see [6, 7]).

In this paper, we survey results on the hamiltonian and classification problems for split graphs $G = S(I \cup K, E)$ with the minimum degree $\delta(G) \geq |I| - 4$.

2. A Polynomial Time Algorithm for Checking Hamiltonicity of Split Graphs G with $\delta(G) \geq |I| - 2$

In 1980, Burkard and Hammer gave in [2] a necessary but not sufficient condition for hamiltonian split graphs $G = S(I \cup K, E)$ with $|I| < |K|$. We will talk about this condition in more detail in the next section. Burkard and Hammer also asked in [2] if this condition can be sharpened to a necessary and sufficient one. This question was investigated by Peemöller [8] and Tan and Hung [10]. In [8], Peemöller gave some conditions equivalent to the condition which Burkard and Hammere gave in [2]. He also remarked there that the hamiltonian problem for split graphs is NP-complete. However, this does not exclude the availability of a polynomial time algorithm for checking hamiltonicity of graphs in some subclasses of split graphs.

In 2004, Tan and Hung [10] characterized hamiltonian split graphs $G = S(I \cup K, E)$ with $|I| \leq |K|$ and the minimum degree $\delta(G) = \min\{\deg(u) \mid u \in V(G)\} \geq |I| - 2$. For this purpose they define the graphs G_n^m, D_n^4 and F_n^5 as in Table 1. The following two theorems are main results proved in [10].

Theorem 2.1. [10] *Let $G = S(I \cup K, E)$ be a split graph with $|I| = m$, $|K| = n$ and $\delta(G) \geq m - 2$. Then G has a Hamilton cycle if and only if $m \leq n$ and $|N(I')| > |I'|$ for any $\emptyset \neq I' \subseteq I$ with $m - 2 \leq |I'| \leq \min\{m, n-1\}$, except the following graphs for which the sufficiency does not hold:*

(i) *$m = 3 < n$ and G is the graph G_n^3;*

(ii) *$m = 4 < n$ and G is a spanning subgraph of D_n^4 or G_n^4;*

(iii) *$m = 4 \leq n$ and $G - u$ is the graph G_n^3 for some $u \in I$;*

(iv) *$m = 5 < n$ and G is the graph F_n^5 or a spanning subgraph of G_n^5;*

(v) *$6 \leq m < n$ and G is a spanning subgraph of G_n^m.*

It is not difficult to see that the graphs G_n^m, D_n^4 and F_n^5 are split graphs $G = S(I \cup K, E)$ satisfying $|I| = m < n = |K|, \delta(G) \geq m - 2$ and $|N(I')| > |I'|$ for any $\emptyset \neq I' \subseteq I$ with $|I'| \geq m - 2$, but they have no Hamilton cycles. Every graph in (iii) also has no Hamilton cycles.

The graph $G=(V,E)$	The vertex-set $V=I\cup K$	The edge-set $E=E_1\cup E_2\cup E_3$.
G_n^m $(3\le m<n)$	$I=\{u_1,\dots,u_m\}$, $K=\{v_1,\dots,v_n\}$.	$E_1=\{u_1v_1,u_2v_2,u_3v_3\}$, $E_2=\{u_iv_j \mid i=1,\dots,m;\ j=4,\dots,m+1\}$, $E_3=\{v_iv_j \mid i\ne j; i,j=1,\dots,n\}$.
D_n^4 $(4<n)$	$I=\{u_1,u_2,u_3,u_4\}$, $K=\{v_1,\dots,v_n\}$.	$E_1=\{u_1v_2,u_2v_1,u_iv_i \mid i=1,2,3,4\}$, $E_2=\{u_iv_5 \mid i=1,2,3,4\}$, $E_3=\{v_iv_j \mid i\ne j; i,j=1,\dots,n\}$.
F_n^5 $(6<n)$	$I=\{u_1,\dots,u_5\}$, $K=\{v_1,\dots,v_n\}$.	$E_1=\{u_iv_i \mid i=1,\dots,5\}$, $E_2=\{u_iv_j \mid i=1,\dots,5; j=6,7\}$, $E_3=\{v_iv_j \mid i\ne j; i,j=1,\dots,n\}$.

Table 1 The graphs G_n^m, D_n^4 and F_n^5

Theorem 2.2. [10] *Let $G=B(I_1\cup I_2,E)$ be a bipartite graph with bipartition $V=I_1\cup I_2$, where $|I_1|=m\le n=|I_2|$ and $\delta(I_1)=\min\{deg(v)\mid v\in I_1\}\ge m-2$. Then G has a Hamilton cycle if and only if $m=n$ and $|N(I')|>|I'|$ for any $\emptyset\ne I'\subseteq I_1$ with $|I'|=m-2$ or $m-1$, unless $m=4$ and $G-u$ is the graph BG_4^3 for some $u\in I_1$, where BG_4^3 is obtained from G_4^3 by deleting all edges, the both endvertices of which are in I_2.*

Based on Theorem 2.1 we can develop a polynomial time algorithm to verify if a split graph $G=S(I\cup K,E)$ with $|I|=m$, $|K|=n$ and $\delta(G)\ge m-2$ has a Hamilton cycle.

An algorithm for checking hamiltonicity of split graphs $G=S(I\cup K,E)$ with $\delta(G)\ge|I|-2$

Input: A split graph $G=S(I\cup K,E)$ with $|I|=m, |K|=n$ and $\delta(G)\ge|I|-2$.

Output: The answer "Yes" if G is hamiltonian and the answer "No" if G is not hamiltonian.

Step 1. If $m>n$, answer "No" and stop. Otherwise continue.

Step 2. For each $\emptyset\ne I'\subseteq I$ with $m-2\le|I'|\le\min\{m,n-1\}$, compute $N(I')$ and check if $|N(I')|>|I'|$. If $|N(I')|>|I'|$ does not hold for some such a subset I', then answer "No" and stop. Otherwise continue.

Step 3. If G is a spanning subgraph of G_n^m, then answer "No" and stop. Otherwise continue.

Step 4. If $m\ne 4$ and 5, then answer "Yes" and stop. Otherwise continue.

Step 5. If $m = 4 < n$ and either G is a spanning subgraph of D_n^4 or $G - u$ is G_n^3 for some $u \in I$, then answer "No" and stop. If $m = 4 < n$ and neither G is a spanning subgraph of D_n^4 nor $G - u$ is G_n^3 for any $u \in I$, then answer "Yes" and stop. Otherwise continue.

Step 6. If $m = 4 = n$ and $G - u$ is G_4^3 for some $u \in I$, then answer "No" and stop. If $m = 4 = n$ and $G - u$ is not G_4^3 for any $u \in I$, then answer "Yes" and stop. Otherwise continue.

Step 7. If $m = 5 < n$ and G is F_n^5, then answer "No" and stop. Otherwise answer "Yes" and stop.

Now we consider the time that this algorithm requires. In below discussions, the graph G, the numbers $m, n, \ldots$ are as in Theorem 2.1. The number of subsets I' with $\emptyset \neq I' \subseteq I$ and $m-2 \leq |I'| \leq \min\{m, n-1\}$ is at most $\binom{m}{m-2} + \binom{m}{m-1} + 1 = \frac{m(m-1)}{2} + m + 1$ which is a polynomial in m. For every subset I', the computation $N(I')$ requires at most mn checking if $uv \in E$ where $u \in I'$ and $v \in K$. So the time that Step 2 requires is a polynomial in $m+n$, the number of vertices of G. Further, it is not difficult to show that a split graph $G = S(I \cup K, E)$ with $m < n$ and $|N(I')| > |I'|$ for any $\emptyset \neq I' \subseteq I$ with $m - 2 \leq |I'| \leq m$ is a spanning subgraph of G_n^m if and only if $|N(I)| = |I| + 1$ and G possesses vertices v_1, v_2 and v_3 in K such that $|N_I(v_1)| = |N_I(v_2)| = |N_I(v_3)| = 1$ and $N_I(v_1), N_I(v_2), N_I(v_3)$ are pairwise disjoint. So, the time that Step 3 requires is also a polynomial in $m+n$. By similar discussions we can see that the time that other steps of the algorithm require is a polynomial in the number of vertices of G. Thus, our algorithm is a polynomial time one.

3. The Burkard-Hammer Condition and Hamiltonicity of Split Graphs G with $\delta(G) \geq |I| - 3$

In this section, we consider the hamiltonian problem for split graphs $G = S(I \cup K, E)$ with $|I| < |K|$ and the minimum degree $\delta(G) \geq |I| - 3$. For split graphs $G = S(I \cup K, E)$ with $|I| < |K|$, Burkard and Hammer have given in [2] a necessary condition for them to be hamiltonian. We describe this condition now.

Let $G = S(I \cup K, E)$ be a split graph and $I' \subseteq I$, $K' \subseteq K$. Denote by $B_G(I' \cup K', E')$ the graph $G[I' \cup K'] - E(G[K'])$. It is clear that $G' = B_G(I' \cup K', E')$ is a bipartite graph with the bipartition subsets I' and K'. So we will call $B_G(I' \cup K', E')$ the *bipartite subgraph of G induced by I' and K'*. For a component $G'_j = B_G(I'_j \cup K'_j, E'_j)$ of $G' = B_G(I' \cup K', E')$ we define

$$k_G(G'_j) = k_G(I'_j, K'_j) = \begin{cases} |I'_j| - |K'_j| & \text{if } |I'_j| > |K'_j| \ , \\ 0, & \text{otherwise.} \end{cases}$$

If $G' = B_G(I' \cup K', E')$ has r components $G'_1 = B_G(I'_1 \cup K'_1, E'_1), \ldots, G'_r = B_G(I'_r \cup K'_r, E'_r)$, then we define

$$k_G(G') = k_G(I', K') = \sum_{j=1}^{r} k_G(G'_j).$$

A component $G'_j = B_G(I'_j \cup K'_j, E'_j)$ of $G' = B_G(I' \cup K', E')$ is called a *T-component* (resp., *H-component, L-component*) if $|I'_j| > |K'_j|$ (resp., $|I'_j| = |K'_j|$, $|I'_j| < |K'_j|$). Let $h_G(G') = h_G(I', K')$ denote the number of H-components of G'.

Now we can formulate the necessary but not sufficient condition for hamiltonian split graphs which Burkard and Hammer have proved in [2].

Theorem 3.1. [2] *Let $G = S(I \cup K, E)$ be a split graph with $|I| < |K|$. If G is hamiltonian, then*

$$k_G(I', K') + \max\left\{1, \frac{h_G(I', K')}{2}\right\} \le |N_G(I')| - |K'|$$

holds for all $\emptyset \ne I' \subseteq I, K' \subseteq N_G(I')$ with $(k_G(I', K'), h_G(I', K')) \ne (0, 0)$.

We will shortly call the condition in Theorem 3.1 *the Burkard-Hammer condition.* Also, we will call a split graph $G = S(I \cup K, E)$ with $|I| < |K|$, which satisfies the Burkard-Hammer condition, a *Burkard-Hammer graph.*

The Burkard-Hammer condition is a necessary but not sufficient condition for the existence of a Hamilton cycle in split graphs $G = S(I \cup K, E)$ with $|I| < |K|$. In [2], an example of a split graph satisfying the Burkard-Hammer condition but having no Hamilton cycles has been given. This graph is the graph $H^{1,6}$ in our Table 2.

Tan and Hung have proved in [11] the following results for split graphs $G = S(I \cup K, E)$.

Theorem 3.2. [11] *Let $G = S(I \cup K, E)$ be a split graph satisfying the Burkard-Hammer condition, then for any $u \in I$ and $v \in K$ with $uv \notin E$ the graph $G + uv$ is also a split graph satisfying the Burkard-Hammer condition.*

Theorem 3.3. [11] *Let $G = S(I \cup K, E)$ be a split graph with $|I| < |K|$ and the minimum degree $\delta(G) \ge |I| - 3$. Then*

(i) *If $|I| \ne 5$ then G has a Hamilton cycle if and only if G satisfies the Burkard-Hammer condition;*

(ii) *If $|I| = 5$ and G satisfies the Burkard-Hammer condition, then G has no Hamilton cycles if and only if G is isomorphic to one of the graphs $H^{1,n}$, $H^{2,n}$, $H^{3,n}$ or $H^{4,n}$ listed in Table 2.*

On Fig. 1 we show the graph $H^{4,n}$.

Theorem 3.3 shows that the Burkard-Hammer condition is almost a necessary and sufficient condition for split graphs $G = S(I \cup K, E)$ with $|I| < |K|$ and the minimum degree $\delta(G) \ge |I| - 3$ to be hamiltonian. There are only four kinds

The graph G	The vertex-set $V(G) = I^* \cup K^*$	The edge-set $E(G) = E_1^* \cup \ldots \cup E_5^* \cup E_{K^*}^*$
$H^{1,n}$ ($n > 5$)	$I^* = \{u_1^*, u_2^*, u_3^*, u_4^*, u_5^*\}$, $K^* = \{v_1^*, v_2^*, \ldots, v_n^*\}$.	$E_1^* = \{u_1^*v_1^*, u_1^*v_2^*\}$, $E_2^* = \{u_2^*v_2^*, u_2^*v_4^*\}$, $E_3^* = \{u_3^*v_2^*, u_3^*v_3^*, u_3^*v_6^*\}$, $E_4^* = \{u_4^*v_1^*, u_4^*v_4^*, u_4^*v_6^*\}$, $E_5^* = \{u_5^*v_5^*, u_5^*v_6^*\}$, $E_{K^*}^* = \{v_i^*v_j^* \mid i \neq j; i, j = 1, \ldots, n\}$.
$H^{2,n}$	$V(H^{2,n}) = V(H^{1,n})$	$E(H^{2,n}) = E(H^{1,n}) \cup \{u_4^*v_2^*\}$
$H^{3,n}$	$V(H^{3,n}) = V(H^{1,n})$	$E(H^{3,n}) = E(H^{1,n}) \cup \{u_5^*v_2^*\}$
$H^{4,n}$	$V(H^{4,n}) = V(H^{1,n})$	$E(H^{4,n}) = E(H^{1,n}) \cup \{u_4^*v_2^*, u_5^*v_2^*\}$

Table 2 The graphs $H^{1,n}$, $H^{2,n}$, $H^{3,n}$ and $H^{4,n}$

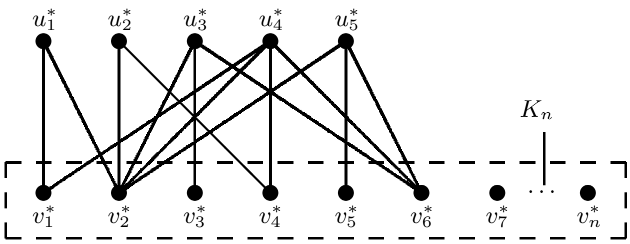

Fig. 1 The graph $H^{4,n}$

of exceptional graphs: $H^{1,n}$, $H^{2,n}$, $H^{3,n}$ and $H^{4,n}$. Since all these exceptional graphs are split graphs $G = S(I \cup K, E)$ with $|I| < |K|$ and $\delta(G) = |I| - 3$, we have immediately the following corollary.

Corollary 3.4. *Let $G = S(I \cup K, E)$ be a split graph with $|I| < |K|$ and $\delta(G) \geq |I| - 2$. Then G has a Hamilton cycle if and only if G satisfies the Burkard-Hammer condition.*

4. The Classification Problem for Nonhamiltonian Split Graphs G with $\delta(G) = |I| - 4$

A split graph $G = S(I \cup K, E)$ is called a *maximal nonhamiltonian split graph* if G is nonhamiltonian but the graph $G + uv$ is hamiltonian for every $uv \notin E$ where $u \in I$ and $v \in K$. It is known from Theorem 3.2 that any nonhamiltonian Burkard-Hammer graph is contained in a maximal nonhamiltonian Burkard-

Hammer graph. So knowledge about maximal nonhamiltonian Burkard-Hammer graphs provides us certain information about nonhamiltonian Burkard-Hammer graphs and also about hamiltonian split graphs. For example, if a split graph $G = S(I \cup K, E)$ with $|I| < |K|$ is not contained in any maximal nonhamiltonian Burkard-Hammer graphs, then it is not difficult to see that G has a Hamilton cycle if and only if G satisfies the Burkard-Hammer condition. Therefore, there is an interest in the classification problem for maximal nonhamiltonian Burkard-Hammer graphs.

Corollary 3.4 shows that there are no nonhamiltonian Burkard-Hammer graphs and therefore no maximal nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) \geq |I| - 2$. Further, Theorem 3.3 shows that for every integer $n > 5$ there exists up to isomorphisms exactly one maximal nonhamiltonian Burkard-Hammer graph $G = S(I \cup K, E)$ with $|K| = n$ and $\delta(G) = |I| - 3$ which is the graph $H^{4,n}$. In this section, we talk about the classification problem for maximal nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - 4$. We need the following construction of split graphs introduced by Tan and Iamjaroen in [12].

Let $G_1 = S(I_1 \cup K_1, E_1)$ and $G_2 = S(I_2 \cup K_2, E_2)$ be split graphs with

$$V(G_1) \cap V(G_2) = \emptyset$$

and v be a vertex of K_1. We say that a graph G is an *expansion of* G_1 *by* G_2 *at* v if G is the graph obtained from $(G_1 - v) \cup G_2$ by adding the set of edges

$$E_0 = \{x_i v_j \mid x_i \in V(G_1) \setminus \{v\}, v_j \in K_2 \text{ and } x_i v \in E_1\}.$$

It is clear that such a graph G is a split graph $S(I \cup K, E)$ with $I = I_1 \cup I_2$, $K = (K_1 \setminus \{v\}) \cup K_2$ and is uniquely determined by G_1, G_2 and $v \in K_1$. Because of this, we will denote this graph G by $G_1[G_2, v]$. Further, a graph G is called an *expansion of* G_1 *by* G_2 if it is an expansion of G_1 by G_2 at some vertex $v \in K_1$.

As an example, we show on Fig. 2 the expansion of the graph $H^{4,n}$ by the complete split graph $G_2 = S(I_2 \cup K_2, E_2)$ with $I_2 = \{u_1, u_2, u_3\}$ and $K_2 = \{v_1, v_2, v_3, v_4\}$ at the vertex v_2^* of $H^{4,n}$.

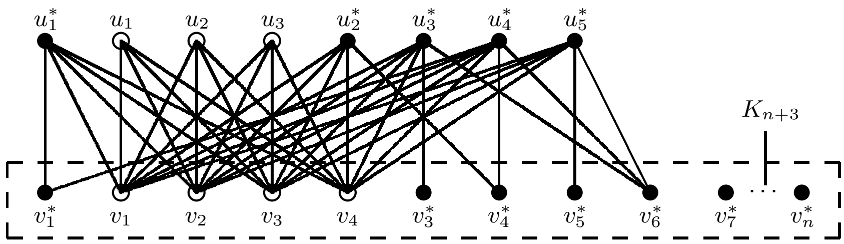

Fig. 2 The expansion $H^{4,n}[G_2, v_2^*]$

Recently, Tan and Iamjaroen have constructed in [13] a family of maximal nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - 4$.

These graphs are $H^{4,n}[G_2, v_2^*]$, where $G_2 = S(I_2 \cup K_2, E_2)$ is a complete split graph with $|I_2| = |K_2| - 1 \geq 1$. Later in [14] they have shown that if a maximal nonhamiltonian Burkard-Hammer graph $G = S(I \cup K, E)$ with $\delta(G) = |I| - 4$ has $|I| \neq 6, 7$, then G must be a graph in the family constructed by them in [13]. Namely, they have proved the following result.

Theorem 4.1. [14] *Let* $G = S(I \cup K, E)$ *be a split graph with* $|I| \neq 6, 7$ *and* $\delta(G) = |I| - 4$. *Then* G *is a maximal nonhamiltonian Burkard-Hammer graph if and only if* G *is isomorphic to the expansion* $H^{4,t}[G_2, v_2^*]$ *where* $t = |K| - |I| + 5$ *and* $G_2 = S(I_2 \cup K_2, E_2)$ *is a complete split graph with* $|K_2| - 1 = |I_2| = |I| - 5 \geq 3$.

Thus, we have got the classification of maximal nonhamiltonian Burkard-Hammer graphs $G = S(I \cup K, E)$ with $\delta(G) = |I| - 4$ for the case $|I| \neq 6, 7$.

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