Vietnam Journal of Mathematics **37**:2&3(2009) 387-397

Vietnam Journal
of
MATHEMATICS

Recent Results in the Theory of Semilinear Elliptic Degenerate Differential Equations

Nguyen Minh Tri

Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam

Received January 9, 2009
Revised May 8, 2009

Abstract. In this paper we give a survey on recent study of semilinear elliptic degenerate differential equations. Here we will discuss the critical exponents phenomenon for boundary value problems and interior regularities of solutions of various classes of such equations. Similar problems for nonlinear elliptic equations were studied in [2, 3, 10, 12, 13, 19, 24, 25, 28, 30-32, 44].

2000 Mathematics Subject Classification: Primary 35-02; Secondary 32V99, 33C05, 33C65, 35A08, 35B65, 35C15, 35H10, 35J70.

Key words: Elliptic degenerate equations, semilinear equations, critical exponents, regularity of solutions.

1. Critical Exponents of a Boundary Value Problem for some Classes of Semilinear Elliptic Degenerate Differential Equations

Boundary value problems for nonlinear elliptic degenerate differential equations were treated in [21] and then subsequently in [7, 11, 22]. In [39, 40] the critical exponents phenomenon was observed for a model of the Grushin type operators. In [37] the result was extended for the following boundary value problem

$$L_{\alpha,\beta}u + g(u) := \Delta_x u + |x|^{2\alpha}\Delta_y u + |x|^{2\beta}\Delta_z u + g(u) = 0 \qquad \text{in } \Omega, \tag{1}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{2}$$

where $g(0) = 0, g(u) \not\equiv 0, g(u) \in C(\mathbb{R}), \alpha, \beta \geq 0, \alpha + \beta > 0$ and

$$\Delta_x = \sum_{i=1}^{n_1} \frac{\partial^2}{\partial x_i^2}, \quad \Delta_y = \sum_{j=1}^{n_2} \frac{\partial^2}{\partial y_j^2}, \quad \Delta_z = \sum_{l=1}^{n_3} \frac{\partial^2}{\partial z_l^2}, \quad |x| = \left(\sum_{i=1}^{n_1} x_i^2\right)^{\frac{1}{2}}.$$

Here $(x_1, ..., x_{n_1}, y_1, ..., y_{n_2}, z_1, ..., z_{n_3}) = (x, y, z) \in \mathbb{R}^{n_1+n_2+n_3} =: \mathbb{R}^N$ and Ω is a bounded domain with a smooth boundary in $\mathbb{R}^N$ containing the origin. Let us put $G(u) = \int_0^u g(s)\,ds$ and $\nu = (\nu_{x_1}, ..., \nu_{x_{n_1}}, \nu_{y_1}, ..., \nu_{y_{n_2}}, \nu_{z_1}, ..., \nu_{z_{n_3}})$ be the unit outward normal on $\partial\Omega$. Put $\tilde{N} = n_1 + n_2(\alpha+1) + n_3(\beta+1)$.

Lemma 1.1. *Let $u\,(x, y, z)$ be a solution of the boundary value problem* (1)-(2)*, which belongs to the class $H^2(\Omega)$. Then the function $u\,(x, y, z)$ satisfies the identity*

$$\int_{\Omega} \left[\frac{\tilde{N}}{n_1} G\,(u) - \frac{\tilde{N}-2}{2n_1} g\,(u)\,u\right] \mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{2n_1} \int_{\partial\Omega} \tilde{\nu}_{\alpha,\beta} \tilde{\nu}^{\alpha,\beta} \left(\frac{\partial u}{\partial \nu}\right)^2 \mathrm{d}S,$$

where

$$\begin{aligned}
&\mathrm{d}x = \mathrm{d}x_1...\mathrm{d}x_{n_1}, \quad \mathrm{d}y = \mathrm{d}y_1...\mathrm{d}y_{n_2}, \quad \mathrm{d}z = \mathrm{d}z_1...\mathrm{d}z_{n_3},\\
&\tilde{\nu}_{\alpha,\beta} = |\nu_x|^2 + |x|^{2\alpha}.|\nu_y|^2 + |x|^{2\beta}.|\nu_z|^2,\\
&\tilde{\nu}^{\alpha,\beta} = (x, \nu_x) + (\alpha+1)\,(y, \nu_y) + (\beta+1)(z, \nu_z),\\
&(x, \nu_x) = \sum_{i=1}^{n_1} x_i \nu_{x_i}, \quad (y, \nu_y) = \sum_{j=1}^{n_2} y_j \nu_{y_j}, \quad (z, \nu_z) = \sum_{l=1}^{n_3} z_l \nu_{z_l},\\
&|\nu_x|^2 = \sum_{i=1}^{n_1} |\nu_{x_i}|^2, \quad |\nu_y|^2 = \sum_{j=1}^{n_2} |\nu_{y_j}|^2, \quad |\nu_z|^2 = \sum_{l=1}^{n_3} |\nu_{z_l}|^2.
\end{aligned}$$

Theorem 1.2. *Let Ω be a $L_{\alpha,\beta}$-star-shaped domain with respect to the point $\{0\}$ (i.e. the inequality $\tilde{\nu}^{\alpha,\beta} > 0$ holds almost everywhere on $\partial\Omega$) and $g(u) = \lambda u + |u|^\gamma u$ with $\lambda \leq 0, \gamma \geq \frac{4}{\tilde{N}-2}$. Then the problem* (1)-(2) *has no nontrivial solution $u \in H^2(\Omega)$.*

Remark 1.3. If $\{0\} \notin \Omega$ Theorem 1.2 may not be true. In the case when $\Omega \cap \{0 \leq |x| < \varepsilon\} = \{\emptyset\}$ one can prove an existence theorem for any function $g(u)$ with growth order less than $\frac{N+2}{N-2}$ (note that $\frac{\tilde{N}+2}{\tilde{N}-2} < \frac{N+2}{N-2}$) by applying the classical Sobolev imbedding theorem.

The situation changes drastically if the growth rate of $g(u)$ is less than $\frac{\tilde{N}+2}{\tilde{N}-2}$ as we shall show in the rest of the paragraph. In order to formulate an existence theorem for BVP (1)-(2) we need some auxiliary results.

Definition 1.4. By $S_1^p(\Omega), 1 \leq p < \infty$, we will denote the set of all functions $u \in L^p(\Omega)$ such that

$$\frac{\partial u}{\partial x_i}, |x|^\alpha \frac{\partial u}{\partial y_j}, |x|^\beta \frac{\partial u}{\partial z_l} \in L^p(\Omega)$$

for all $i = 1, \dots, n_1$, $j = 1, \dots, n_2$, $l = 1, \dots, n_3$. For the norm in $S_1^p(\Omega)$ we take

$$\|u\|^p_{S_1^p(\Omega)} = \int\limits_\Omega \left(|u|^p + \sum_{i=1}^{n_1} \left| \frac{\partial u}{\partial x_i} \right|^p + \sum_{j=1}^{n_2} \left| |x|^\alpha \frac{\partial u}{\partial y_j} \right|^p + \sum_{l=1}^{n_3} \left| |x|^\beta \frac{\partial u}{\partial z_l} \right|^p \right) \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

If $p = 2$ we can also define the scalar product in $S_1^2(\Omega)$ as follows

$$\begin{aligned}(u, v)_{S_1^2(\Omega)} &= (u, v)_{L^2(\Omega)} + \sum_{i=1}^{n_1} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)} \\ &\quad + \sum_{j=1}^{n_2} \left(|x|^\alpha \frac{\partial u}{\partial y_j}, |x|^\alpha \frac{\partial v}{\partial y_j} \right)_{L^2(\Omega)} + \sum_{l=1}^{n_3} \left(|x|^\beta \frac{\partial u}{\partial z_l}, |x|^\beta \frac{\partial v}{\partial z_l} \right)_{L^2(\Omega)}.\end{aligned}$$

The space $S_{1,0}^p(\Omega)$ is defined as the closure of $C_0^1(\Omega)$ in the space $S_1^p(\Omega)$.

Proposition 1.5. *Assume that $1 \le p < \tilde{N}$. Then $S_{1,0}^p(\Omega) \subset L^{\frac{\tilde{N}p}{\tilde{N}-p} - \tau}(\Omega)$ for every positive small τ.*

Now set $k = \max\{[\alpha], [\beta]\} + 1$, where $[\cdot]$ stands for the integral part of the argument. The following proposition is due to Rothschild and Stein [34].

Proposition 1.6. *Assume that $1 \le p < \infty$. Then $S_{1,0}^p(\Omega) \subset L^p_{\frac{1}{k+1}}(\Omega)$.*

By Propositions 1.5, 1.6 we can easily obtain the following two propositions:

Proposition 1.7. *Assume that $1 \le p < \tilde{N}$. Then the imbedding map $S_{1,0}^p(\Omega)$ into $L^{\frac{\tilde{N}p}{\tilde{N}-p} - \tau}(\Omega)$ is compact for every positive small τ.*

Proposition 1.8. *Assume that $p > \tilde{N}$. Then $S_{1,0}^p(\Omega) \subset C^0(\bar{\Omega})$.*

Definition 1.9. A function $u \in S_{1,0}^2(\Omega)$ is called a weak solution of the problem (1)-(2) if the identity

$$\begin{aligned}&\sum_{i=1}^{n_1} \int\limits_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial \varphi}{\partial x_i} \mathrm{d}x\mathrm{d}y\mathrm{d}z + \sum_{j=1}^{n_2} \int\limits_\Omega |x|^{2\alpha} \frac{\partial u}{\partial y_j} \cdot \frac{\partial \varphi}{\partial y_j} \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &\quad + \sum_{l=1}^{n_3} \int\limits_\Omega |x|^{2\beta} \frac{\partial u}{\partial z_l} \cdot \frac{\partial \varphi}{\partial z_l} \mathrm{d}x\mathrm{d}y\mathrm{d}z - \int\limits_\Omega g(u)\varphi \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= 0\end{aligned}$$

holds for every $\varphi \in C_0^\infty(\Omega)$.

Now we can state our existence theorem.

Theorem 1.10. *Assume that $g(u)$ satisfies the following conditions*

- $g \in C^{0,\gamma}_{\mathrm{loc}}(\mathbb{R}), \ \gamma \in (0,1]$,
- $|g(u)| \leq C(1+|u|^m)$ *with* $1 < m < \frac{\tilde{N}+2}{\tilde{N}-2}$,
- $g(u) = \bar{\bar{o}}(u)$ *as* $u \longrightarrow 0$,
- *There exists a constant A such that for $|u| \geq A, G(u) \leq \mu g(u)u$ where $\mu \in [0, \frac{1}{2})$.*

Then the problem (1)-(2) *always has a weak nontrivial solution.*

Recently in [36] a generalized Pohozaev identity for a boundary value problem of a more complicated nonlinear elliptic degenerate differential operator was obtained. Namely, consider the problem

$$Pu + g(u) := \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + x_1^2 x_2^2 \frac{\partial^2 u}{\partial x_1^2} + g(u) = 0 \qquad \text{in } \Omega \tag{3}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{4}$$

Lemma 1.11. (Generalized Pohozaev's Identity) *Let $u(x_1, x_2, x_3)$ be a solution of the boundary value problem* (3)-(4)*, which belongs to the class $H^2(\Omega)$. Then the function $u(x_1, x_2, x_3)$ satisfies the identity*

$$\int\limits_{\Omega} \left[5G(u) - \frac{3}{2} g(u) u \right] \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$

$$= \frac{1}{2} \int\limits_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} \right)^2 (\nu_1^2 + \nu_2^2 + x_1^2 x_2^2 \nu_3^2)(x_1\nu_1 + x_2\nu_2 + 3x_3\nu_3) \,\mathrm{d}S.$$

With the help of Lemma 1.11 it is not difficult to establish:

Theorem 1.12. *Let Ω be a P-star-shaped domain with respect to the point* $\{0\}$ *(i. e. the inequality $(\nu_1^2 + \nu_2^2 + x_1^2 x_2^2 \nu_3^2)(x_1\nu_1 + x_2\nu_2 + 3x_3\nu_3) > 0$ holds almost everywhere on $\partial\Omega$) and $g(u) = \lambda u + |u|^\gamma u$ with $\lambda \leq 0, \gamma \geq \frac{4}{3}$. Then the problem* (3)-(4) *has no nontrivial solution $u \in H^2(\Omega)$.*

An existence theorem for the problem (3)-(4) in relevant Sobolev spaces is expected to obtain in the near future.

Note that apart from papers cited above, many works have devoted to the study of boundary value problems for semilinear elliptic degenerate operators or related equations. We refer the interested readers to [1, 4-6, 8, 9, 26] and references therein.

2. C^∞-smoothness of Solutions of Semilinear Polynomial Type Elliptic Degenerate Differential Equations

In this section we study the C^∞-regularity of solutions of the equation

$$\sum_{j=1}^{k} X_j^{2m} u + \Phi(x, X^\iota u)_{|\iota|\leqslant 2m-1} = 0, \tag{5}$$

where $X_1, \ldots, X_k$ are real vector fields in a domain $\Omega \subset \mathbb{R}^n$. Linear polynomial type operators were investigated in [16, 27, 33]. Regularity of solutions of linear second order differential equations were studied in [20, 29]. In [20] the following condition was introduced:

Condition $(\mathbf{H})_1$: there is a natural number l such that commutators $\{X_\iota\}_{|\iota|\leqslant l}$ span the whole space $\mathbb{R}^n$ at every point in Ω.

Results concerning smoothness of solutions of second order semilinear elliptic degenerate equations were obtained in [47, 46]. The approaches in [47, 46] are quite different. The following theorem was obtained in [46]:

Theorem 2.1. *Let $\{X_j\}_{j=1}^k$ satisfy the condition* $(\mathrm{H})_1$ *in Ω and $\Phi(x, u, \tau_1, \ldots, \tau_k)$ be an infinitely differentiable function. If u is a $C^{nl+2}(\Omega)$-solution of the equation*

$$\sum_{j=1}^{k} X_j^2 u + \Phi(x, u, X_1 u, \ldots, X_k u) = 0,$$

then $u \in C^\infty(\Omega)$.

Remark 2.2. The condition of semilinearity in Theorem 2.3 may be weakened (see [47]) but it cannot be discarded completely. Indeed, consider in $\mathbb{R}^2$

$$X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial x_2}.$$

It is easy to see that

$$X_1^2(|x_2|^{2m+1}) + (X_2^2(|x_2|^{2m+1}))^2 = (2m+1)^2(2m)^2 x_2^{4m-2} \in C^\infty(\mathbb{R}^2)$$

for all $m \in \mathbb{Z}_+$. However $|x_2|^{2m+1} \in C^{2m}(\mathbb{R}^2)\backslash C^\infty(\mathbb{R}^2)$.

The situation for higher order operators ($m \geq 2$) becomes much more complicated. We need the following condition:

Condition $(\mathbf{K})_1$: for every i and any K in Ω there exists a constant C such that

$$\|X_\iota u\|_s \leqslant C \left(\sum_{j=1}^{k} \|X_j u\|_{s+\frac{i-1}{l}} + \|u\|_s \right),$$

for all $|\iota| = i, u \in C_0^\infty(K)$.

Theorem 2.3. *Let the vector fields* $\{X_j\}_{j=1}^k$ *satisfy the conditions* $(\mathrm{H})_l$ *and* $(\mathrm{K})_l$ *in* Ω*. Assume that the function* $\Phi(x,\tau_\iota)_{|\iota|\leqslant 2m-1}$ *is infinitely differentiable. If* u *is a* $C^{nl+2m}(\Omega)$*-solution of the Equation* (5) *then* $u \in C^\infty(\Omega)$.

In practice it may not be easy to verify the conditions $(\mathrm{K})_l$. Here we give some examples of systems that satisfy this condition. The following two systems

a) Complete system of vector fields degenerate on a submanifold:

$$X_1 = \frac{\partial}{\partial x_1}, \dots, X_{n_1} = \frac{\partial}{\partial x_{n_1}}, X_{1,1} = x_1^l \frac{\partial}{\partial y_1}, \dots, X_{1,n_2} = x_1^l \frac{\partial}{\partial y_{n_2}}, \dots,$$
$$X_{n_1,1} = x_{n_1}^l \frac{\partial}{\partial y_1}, \dots, X_{n_1,n_2} = x_{n_1}^l \frac{\partial}{\partial y_{n_2}}.$$

b) Noncomplete system of vector fields degenerate on a submanifold:

$$X_1 = \frac{\partial}{\partial x_1}, \dots, X_{n_1} = \frac{\partial}{\partial x_{n_1}}, X_{1,1} = x_1^l \frac{\partial}{\partial y_1}, \dots, X_{1,n_2} = x_1^l \frac{\partial}{\partial y_{n_2}}, \dots,$$
$$X_{n_1,1} = x_{n_1}^l \frac{\partial}{\partial y_1}, \dots, X_{n_1,n_2} = x_{n_1}^l \frac{\partial}{\partial y_{n_2}}, Z_1 = \frac{\partial}{\partial z_1}, \dots, Z_{n_3} = \frac{\partial}{\partial z_{n_3}}.$$

both satisfy the $(\mathrm{H})_l$ and $(\mathrm{K})_l$ conditions.

Example 2.4. Assume that $u \in C^{3k+2m}(\mathbb{R}^3)$ and

$$\frac{\partial^{2m} u}{\partial x^{2m}} + x^{2mk}\Big(\frac{\partial^{2m} u}{\partial y^{2m}} + \frac{\partial^{2m} u}{\partial z^{2m}}\Big) + \cos\Big(x^{(2m-1)k}\frac{\partial^{2m-1} u}{\partial y^{2m-1}}\Big) e^{\frac{\partial^{2m-1} u}{\partial x^{2m-1}}} \in C^\infty(\mathbb{R}^3).$$

Then $u \in C^\infty(\mathbb{R}^3)$.

Example 2.5. Assume that $u \in C^{3k+2m}(\mathbb{R}^3)$ and

$$\frac{\partial^{2m} u}{\partial x^{2m}} + x^{2mk}\frac{\partial^{2m} u}{\partial y^{2m}} + \frac{\partial^{2m} u}{\partial z^{2m}} + \Big(\frac{\partial^{2m-1} u}{\partial x^{2m-1}}\Big)^5 \Big(x^k \frac{\partial u}{\partial y}\Big)^2 \in C^\infty(\mathbb{R}^3).$$

Then $u \in C^\infty(\mathbb{R}^3)$.

When $m = 2$ and $l = 3$ the condition $(\mathrm{K})_l$ is not necessary as shown in the following theorem.

Theorem 2.6. *Let* $\{X_j\}_{j=1}^k$ *satisfy the condition* $(\mathrm{H})_3$ *in* Ω*. Assume that* $\Phi(x,\tau_\iota)_{|\iota|\leqslant 3}$ *is infinitely differentiable. If* u *is a* $C^{3n+4}(\Omega)$*-solution of the equation*

$$\sum_{j=1}^k X_j^4 u + \Phi(x, X^\iota u)_{|\iota|\leqslant 3} = 0,$$

then $u \in C^\infty(\Omega)$.

Example 2.7. Assume that $u \in C^{13}(\mathbb{R}^3)$ and

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + x^4 y^4 \frac{\partial^4 u}{\partial z^4} + e^{x^2 y^2 \frac{\partial^2 u}{\partial z^2}} \frac{\partial u}{\partial x} \in C^\infty(\mathbb{R}^3).$$

Then $u \in C^\infty(\mathbb{R}^3)$.

Note that other related aspects of semilinear differential equations are also investigated in many other works (see for example the papers [14, 48] and the references therein).

3. Analyticity and Gevrey Regularity of Solutions of Semilinear Elliptic Degenerate Differential Equations

A) Semilinear Grushin's type differential equations. Analyticity and Gevrey regularity of solutions of nonlinear perturbations of powers of the Mizohata operator, a model of the Grushin type operators, the Gilioli-Treves operator were investigated in [17, 18, 41-43]. Recently, a result concerning analyticity of solutions of general semilinear Grushin type operators was obtained in [23]. Here we will give main statements and sketches of proofs from [23]. The details will appear elsewhere. Let $z = (x, y) \in \mathbb{R}^N, x \in \mathbb{R}^n, y \in \mathbb{R}^k, k + n = N$; and let there be given an integer $m > 0$ and a rational positive number δ such that $m\delta$ is an integer. For every non-negative integer t we set

$$\begin{aligned}
\mathcal{M}_t &= \{(\alpha, \beta, \gamma) \in \bar{\mathbb{Z}}^n_+ \times \bar{\mathbb{Z}}^k_+ \times \bar{\mathbb{R}}^n_+ : |\alpha| + |\beta| \le t; \\
&\qquad\qquad\qquad m\delta \ge |\gamma| \ge |\alpha| + (1+\delta)|\beta| - t\}, \\
\tilde{\mathcal{M}}_t &= \{(\alpha, \beta, \gamma) \in \mathcal{M}_t : \gamma \in \bar{\mathbb{Z}}^n_+\}, \\
\tilde{\mathcal{M}}^0_t &= \{(\alpha, \beta, \gamma) \in \tilde{\mathcal{M}}_t : |\gamma| = |\alpha| + (1+\delta)|\beta| - t\}, \\
\tilde{\mathcal{M}}^{00}_t &= \{(\alpha, \beta, \gamma) \in \tilde{\mathcal{M}}^0_t : |\alpha| + |\beta| = t\}.
\end{aligned}$$

We will consider the following equation

$$Pu + \Psi(z, x^\gamma \partial_x^\alpha \partial_y^\beta u)_{(\alpha,\beta,\gamma)\in\tilde{\mathcal{M}}_{m-1}} = 0 \ \text{ in } \Omega, \tag{6}$$

where

$$Pu = \sum_{(\alpha,\beta,\gamma)\in\tilde{\mathcal{M}}^0_m} a_{\alpha\beta\gamma} x^\gamma \partial_x^\alpha \partial_y^\beta u,$$

$a_{\alpha\beta\gamma}$ are complex constants and Ω is a bounded domain containing the origin in $\mathbb{R}^N$ with smooth boundary. We consider the following conditions (see [15]):

Condition 1:

$$P(x, \xi, \eta) = \sum_{(\alpha,\beta,\gamma)\in\tilde{\mathcal{M}}^{00}_m} a_{\alpha\beta\gamma} x^\gamma \xi^\alpha \eta^\beta \ne 0, \ \ \forall (\xi, \eta) \in \mathbb{R}^N \backslash \{0\}, \ \ \forall\, x \ne 0.$$

Condition 2: For every $|\eta| = 1$ the equation

$$L(x, \partial_x, \eta)v(x) = \sum_{(\alpha,\beta,\gamma)\in\tilde{\mathcal{M}}_m^0} a_{\alpha\beta\gamma} x^\gamma (\mathrm{i}\eta)^\beta \partial_x^\alpha v(x) = 0$$

has no nontrivial solution in $S(\mathbb{R}^n)$.

The main theorem in [23] is

Theorem 3.1. *Assume that* $t \geq (N+2)(1+\delta)+m+3$, *Conditions 1, 2 hold. Suppose that* u *is a* $\mathcal{C}^t(\Omega)$*-solution of Equation* (6) *and* Ψ *is an analytic function of its arguments. Then* u *is analytic, too.*

The proof of Theorem 3.1 consists of the following two theorems:

Theorem 3.2. *Assume that Conditions 1, 2 hold,* Ψ *is a* $C^\infty-$*function of its arguments and* $t \geq (N+2)(1+\delta)+m+3$. *If* $u \in C^t(\Omega)$ *is a solution of Equation* (6) *then* $u \in C^\infty(\Omega)$.

Theorem 3.3. *Assume that Conditions 1, 2 hold and* Ψ *is an analytic function. If* u *is a* C^∞ *solution of Equation* (6) *then* u *is analytic.*

B) Semilinear Kohn-Laplacian operator on the Heisenberg group. The Heisenberg group

$$\mathbb{H}^n := (z,t) = (z_1, \ldots, z_n, t) = (x, y, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$$

is equipped with the multiplication:

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(yx' - xy')).$$

On $\mathbb{H}^n$ there are left invariant complex vector fields

$$\mathbb{Z}_j = \frac{1}{2}(\mathbb{X}_j - i\mathbb{Y}_j) = \frac{\partial}{\partial z_j} + i\bar{z}_j\frac{\partial}{\partial t}, \bar{\mathbb{Z}}_j = \frac{1}{2}(\mathbb{X}_j + i\mathbb{Y}_j) = \frac{\partial}{\partial \bar{z}_j} - iz_j\frac{\partial}{\partial t},$$

where

$$\mathbb{X}_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \mathbb{Y}_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, T = \frac{\partial}{\partial t}; j = 1, \ldots, n.$$

$T_{1,0}$ of $\mathbb{C}T\mathbb{H}^n$ spanned by $Z_1, \ldots, Z_n$ defines a CR structure on $\mathbb{H}^n$. Define the $\bar{\partial}_b-$complex: $\bar{\partial}_b : C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q+1})$ and its formal adjoint $\vartheta_b : C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q-1})$, where $\Lambda^{p,q} = (\Lambda^p T^*_{1,0}) \otimes (\Lambda^q \bar{T}^*_{1,0})$. The Kohn-Laplacian is defined as

$$\square_b = \bar{\partial}_b\vartheta_b + \vartheta_b\bar{\partial}_b : C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q}).$$

In a specific basis $\square_b$ can be diagonalized with elements $\mathcal{L}_{n,\lambda}$ on the diagonal

$$\mathcal{L}_{n,\lambda} = -\frac{1}{2}\sum_{j=1}^{n}(\mathbb{Z}_j\bar{\mathbb{Z}}_j + \bar{\mathbb{Z}}_j\mathbb{Z}_j) + i\lambda T = -\frac{1}{4}\sum_{j=1}^{n}(\mathbb{X}_j^2 + \mathbb{Y}_j^2) + i\lambda T; \lambda \in \mathbb{C}.$$

In [45] we proved:

Theorem 3.4. *Let $s \geq 2, l \geq 2n+4$ and $\pm\lambda \neq n, n+2, n+4, \ldots$. Assume that $\Psi(x, y, t, u, \tau_1, \ldots, \tau_{2n}) \in G^s$ and u is a $C^l(\Omega)$-solution of the equation*

$$\mathcal{L}_{n,\lambda}u + \Psi(x, y, t, u, \mathbb{X}_1 u, ..., \mathbb{X}_n u, \mathbb{Y}_1 u, ..., \mathbb{Y}_n u) = 0.$$

Then $u \in G^s(\Omega)$.

The condition $s \geq 2$ in Theorem 3.4 seems to be redundant comparing with the results on analyticity of solutions of the linear Kohn-Laplacian on the Heisenberg group in [38, 35]. This condition is imposed due to some technical difficulties arisen in the nonlinear problem. One should expect some improvements in the near future.

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