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Report on the Proof of some Conjectures on Orbital Integrals in Langlands' Program

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Abstract. Robert Langlands has formulated a series of conjectures in local harmonic analysis known as the fundamental lemma and the transfer conjectures. Though the statements are complicated, these statements are entirely elementary and of combinatorial nature. They become more notorious because of their difficulty and also of some deep theorems in representation theory, number theory and arithmetic algebraic geometry that rely thereon. The proof we have now of their conjecture, due to the effort of many mathematicians, is based on local harmonic analysis, Arthur-Selberg's trace formula but surprisingly enough also on rather involved algebraic geometry of certain moduli space which has origin from mathematical physics.
In this report, I will recall the basics about orbital integrals, the natural places in mathematics where we encounter with them, the fundamental lemma and of the transfer conjecture which is stated in a precise form only in certain cases. After surveying different contributions to the solution of these conjectures, I will focus on certain algebraic varieties that play a central role in the understanding of non-archimedean orbital integrals.

1. Orbital Integrals over Non-archimedean Local Fields

1.1. First Example
Let V be an n-dimensional vector space over a non-archimedean local field F, for instance the field $\mathbb{Q}_p$ of p-adic numbers. Let $\gamma : V \to V$ be a regular semisimple endomorphism, i.e. having two by two disctinct eigenvalues in an algebraic closure of F. The centralizer I_γ of γ is then of the form

$$I_\gamma = E_1^\times \times \cdots \times E_r^\times ,$$

$E_1, \ldots, E_r$ being finite extensions of F. This centralizer is a commutative locally compact topological group.

Let $\mathcal{O}_F$ denote the ring of integers in F. We call lattices of V sub-$\mathcal{O}_F$-modules $\mathcal{V} \subset V$ of finite type and of maximal rank. We consider the set $\mathcal{M}_\gamma$ of lattices $\mathcal{V}$ of V such that $\gamma(\mathcal{V}) \subset \mathcal{V}$. The set $\mathcal{M}_\gamma$ is infinite in general but the set of equivalence classes of element of $\mathcal{M}_\gamma$ with respect to the equivalence relation defined by the action of I_γ is finite. The most basic example of orbital integrals consists in counting the number of equivalence classes of lattices weighted by inverse the measure of the stabilizers. Fix a Haar measure dt on the locally compact group I_γ. The sum

$$\sum_{x \in \mathcal{M}_\gamma / I_\gamma} \frac{1}{\mathrm{vol}(I_{\gamma,x}, dt)} \tag{1}$$

is a typical example of orbital integrals. Here x runs over a set of representatives of orbits of I_γ on $\mathcal{M}_\gamma$ and $I_{\gamma,x}$ is the subgroup of I_γ of elements stabilizing x that is a compact open subgroup of I_γ.

1.2. General Orbital Integrals

Let G be a reductive group over F. Let $\mathfrak{g}$ denote its Lie algebra. Let γ be an element of $G(F)$ or $\mathfrak{g}(F)$ which is strongly regular semi-simple in the sense that its centralizer I_γ if a F-torus. Choose a Haar measure dg on $G(F)$ and a Haar measure dt on $I_\gamma(F)$.

For $\gamma \in G(F)$ and for any compactly supported and locally constant function $f \in C_c^\infty(G(F))$, we set

$$\mathbf{O}_\gamma(f) = \int_{I_\gamma(F) \backslash G(F)} f(g^{-1}\gamma g) \frac{dg}{dt}.$$

We have the same formula in the case $\gamma \in \mathfrak{g}(F)$ and $f \in C_c^\infty(\mathfrak{g}(F))$. By definition, orbital integral $\mathbf{O}_\gamma$ does not depend on γ but only on its conjugacy class. We also notice the obvious dependence of $\mathbf{O}_\gamma$ on the choice of Haar measures dg and dt.

We are mostly interested in the "unramified" case, i.e. G has a reductive model over $\mathcal{O}_F$ which is the case for any split reductive group for instance. The subgroup $K = G(\mathcal{O}_F)$ is then a maximal compact subgroup of $G(F)$. We can fix the Haar measure dg on $G(F)$ by assigning to K the volume one. Consider the set

$$\mathcal{M}_\gamma = \{x \in G(F)/K \mid gx = x\}, \tag{2}$$

acted on by $I_g(F)$. Then we have

$$\mathbf{O}_\gamma(1_K) = \sum_{x \in I_\gamma(F) \backslash \mathcal{M}_\gamma} \frac{1}{\mathrm{vol}(I_\gamma(F)_x, dt)}, \tag{3}$$

where 1_K is the characteristic function of K, x runs over a set of representatives of orbits of $I_\gamma(F)$ in $\mathcal{M}_\gamma$ and $I_\gamma(F)_x$ the stabilizer subgroup of $I_\gamma(F)$ at x that is a compact open subgroup.

In the case $G = \mathrm{GL}(n)$, the space of cosets $G(F)/K$ can be identified with the set of lattices in F^n so that we recover the lattice counting problem as in the First Example. For classical groups like symplectic and orthogonal groups, orbital integrals for the unit function can also be expressed as a number of self-dual lattices fixed under an automorphism or stable under an endomorphism in Lie algebra case counted in an appropriate way.

1.3. Another Example

It is one of the basic problem in arithmetic geometry to count the number of abelian varieties equipped with a principal polarization defined over a finite field $\mathbb{F}_p$. The isogeny classes of abelian varieties over finite fields are described by Honda-Tate theory. The usual strategy consists in counting the principally polarized abelian varieties equipped with a polarization compatible isogeny to a fixed one. We will be concerned here with the ℓ-polarizations for fixed prime ℓ that is different from p.

Let A be an n-dimensional abelian variety over a finite field $\mathbb{F}_p$ equipped with a principal polarization. The $\mathbb{Q}_\ell$-Tate module of A

$$T_{\mathbb{Q}_\ell}(A) = \mathrm{H}_1(A \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$$

is a $2n$-dimensional $\mathbb{Q}_\ell$-vector space equipped with

- a non-degenerate alternating form coming from the polarization,
- a Frobenius operator σ_p as A is defined over $\mathbb{F}_p$,
- a self-dual lattice $T_{\mathbb{Z}_\ell}(A) = \mathrm{H}_1(A \otimes \bar{\mathbb{F}}_p, \mathbb{Z}_\ell)$ which is stable under σ_p.

Let A' be a principally polarized abelian variety equipped with a ℓ-isogeny to A which is defined over $\mathbb{F}_p$ and compatible with polarizations. This isogeny defines an isomorphism between the $\mathbb{Q}_\ell$-vector spaces $T_{\mathbb{Q}_\ell}(A)$ and $T_{\mathbb{Q}_\ell}(A')$ equipped with their symplectic forms and Frobenius operators. Giving such an ℓ-isogeny is therefore equivalent to a self-dual lattice $\mathrm{H}_1(A', \mathbb{Z}_\ell)$ of $\mathrm{H}_1(A, \mathbb{Q}_\ell)$ which is stable under σ_p.

For this reason, orbital integral for symplectic group enters in the counting the number of principally polarized abelian varieties over finite field within a fixed isogeny class. This is a general phenomenon which does not happen only for the Siegel moduli space of principally polarized abelian varieties but a large class of Shimura varieties, [10].

1.4. Global Motivations

We are considering now a reductive group G defined over a global fields F which can be either a number field or field of rational functions on a curve defined over a finite field. It is of interest to understand the traces of Hecke operator on automorphic representations of G. Arthur-Selberg's trace formula is a poweful tool for this quest; it has the following forms

$$\sum_{\gamma\in G(F)/\sim} \mathbf{O}_\gamma(f) + \cdots = \sum_{\pi} \mathrm{tr}_\pi(f) + \cdots , \tag{4}$$

where γ runs over the set of elliptic conjugacy classes of $G(F)$ and π over the set of discrete automorphic representations. The test functions f are usually of the form $f = \otimes f_v$ with f_v being the unit function in Hecke algebra of $G(F_v)$ for almost all finite places v of F. The global orbital integrals

$$\mathbf{O}_\gamma(f) = \int_{I_\gamma(F)\backslash G(\mathbb{A})} f(g^{-1}\gamma g)dg$$

are convergent for elliptic conjugacy classes $\gamma \in G(F)/\sim$. Arthur's monumental works allow us to understand the dots that hide much more complicated terms.

In some cases, it is possible to compare the formulas (4) for different groups, possibly in a twisted form, by equating terms that appears in the orbital side and allow a transfer principle of automorphic representations in the spectral side. One of Arthur's great achievements consists in establishing such a comparison from which follows a transfer of automorphic representations from classical groups to linear group via the standard representation of Langlands' dual group. Among of the many difficulties he encountered in this program is the need of proving identities of local orbital integrals called the fundamental lemma.

Similar strategy has been used for a seemingly very different problem namely the calculation of Hasse-Weil zeta function of Shimura varieties. For a particular but large class of Shimura varieties $\mathcal{S}$ classifying polarized abelian varieties with additional structures, Kottwitz established a general formula for the number of points of $\mathcal{S}$ with values in a finite field $\mathbb{F}_q$. The formula he obtained is close to the orbital side of (4) for the reductive group G entering in the definition of $\mathcal{S}$. Again local identities of orbital integrals are needed to establish an equality of $\mathcal{S}(\mathbb{F}_q)$ with a combination of the orbital sides of (4) for G and a collection of smaller groups called endoscopic groups of G. Eventually, this strategy allows us to attach Galois representation to general auto-dual automorphic representations that is so far the best result we have at hand in direction of Langlands' correspondence for number fields.

2. Stable Conjugacy

2.1. An Annoying Problem

While working with orbital integrals for other groups for $\mathrm{GL}(n)$, one notices quickly an annoying problem with conjugacy classes. For $\mathrm{GL}(n)$, two regular semisimple elements in $\mathrm{GL}(n, F)$ are conjugate if and only if they are conjugate in the larger group $\mathrm{GL}(n, \bar{F})$ where $\bar{F}$ is an algebraic closure of F and this latter condition is equivalent to γ and γ' having the same characteristic polynomial. For a general reductive group G, we have a characteristic polynomial map $\chi : G \to T/W$ where T is a maximal torus and W is its Weyl group. Strongly regular semisimple elements $\gamma, \gamma' \in G(\bar{F})$ with the same characterictic polynomial if and only if they are $G(\bar{F})$-conjugate. But in $G(F)$ there are in general more than one $G(F)$-conjugacy classes within the set of strongly regular semisimple elements having the same characteristic polynomial. These conjugacy classes are called "stably conjugate". It is probably more accurate to call it "geometric conjugacy" instead but the tradition is now so well established to be uprooted, [11].

For a fixed $\gamma \in G(F)$, assumed strongly regular semisimple, the set of $G(F)$-conjugacy classes in the stable conjugacy of γ is a subset, denoted by A_γ of the group $\mathrm{H}^1(F, I_\gamma)$. In general, in particular for a global field F, A_γ is not a finite set. This has an unpleasant effect on the orbital side of (4) since we see infinitely many different $\gamma \in G(F)/\sim$ having the same characteristic polynomial though only finitely many of them have a non-zero contribution. The process of stabilization of the trace formula allows one to transform the orbital side of (4) into a more satisfying form both from philosophical or computational points of views. This process consists in rewriting the contribution of a given characteristic polynomial $a \in (T/W)(F)$ as a finite sum of terms each of which is a product of linear combination of local orbital integrals called stable orbital integrals and κ-orbital integrals. We will not review here the stabilization based in an involved analysis of Galois cohomology and class field theory but focus ourselves on local harmonic analysis.

2.2. Stable Orbital Integral and its κ-brothers

For a local non-archimedean field F, $\mathrm{H}^1(F, I_\gamma)$ is a finite abelian group, usually killed by 2, and A_γ is one of its subgroup. It is relevant for local harmonic analysis purpose to form certain linear combinations of orbital integrals within a stable conjugacy class. In particular, the stable orbital integral is the sum

$$\mathbf{SO}_\gamma(f) = \sum_{\gamma'} \mathbf{O}_{\gamma'}(f)$$

over a set of representatives γ' of conjugacy classes within the stable conjugacy class of γ. In forming this sum we need to choose in a consistent way Haar measures on different centralizers $I_{\gamma'}(F)$. For strongly regular semisimple elements, the tori $I_{\gamma'}$ for γ' in the stable conjugacy class of γ, are canonically isomorphic so that we can just export the chosen Haar measure from $I_\gamma(F)$ to $I_{\gamma'}(F)$. Obviously, the stable orbital integral $\mathbf{SO}_\gamma$ depends only on the stable conjugacy which is to say on the characteristic polynomial of γ. Let a be the characteristic polynomial of a strongly regular semisimple element γ, then it makes sense to put $\mathbf{SO}_a = \mathbf{SO}_\gamma$ as a definition of the distribution $\mathbf{SO}_a$.

For any character $\kappa : A_\gamma \to \mathbb{C}^\times$ of the finite group A_γ we can form the κ-orbital integral

$$\mathbf{O}^\kappa_\gamma(f) = \sum_{\gamma'} \kappa(\mathrm{cl}(\gamma'))\mathbf{O}_{\gamma'}(f)$$

over a set of representatives γ' of conjugacy classes within the stable conjugacy class of γ and $\mathrm{cl}(\gamma')$ is the class of γ' in A_γ. As above, for γ' in the stable conjugacy class of γ, we have a canonical isomorphism $A_\gamma = A_{\gamma'}$ so that we can transport the character κ of A_γ on a character of A'_γ. Now, $\mathbf{O}^\kappa_\gamma$ and $\mathbf{O}^\kappa_{\gamma'}$ will differ by the scalar $\kappa(\mathrm{cl}(\gamma'))$ where $\mathrm{cl}(\gamma')$ is the class of γ' in A_γ. Thus, we can not a priori talk about κ-orbital $\mathbf{O}^\kappa_a$ for a characteristic polynomial a.

At least in the case of Lie algebra, there exists a section $\iota : \mathfrak{t}/W \to \mathfrak{g}$ due to Kostant of the characteristic polynomial map $\chi : \mathfrak{g} \to \mathfrak{t}/W$ and we put

$$\mathbf{O}^\kappa_a = \mathbf{O}^\kappa_{\iota(a)}$$

as the definition of the distribution $\mathbf{O}^\kappa_a$. Thanks to Kottwitz' calculation of transfer factor, we know that this naively looking definition is the "right" one for the statement of the fundamental lemma and the transfer conjecture for Lie algebra [9].

If the derived group of G is simply connected, Steinberg has constructed a section $\iota : T/W \to G$ of the characteristic polynomial map $\chi : G \to T/W$. It is tempting to adopt the above definition for $\mathbf{O}^\kappa_a$ by using Steinberg's section. We don't know yet if this is the "right" definition lacking a calculation similar to Kottwitz' in Lie algebra case.

2.3. Endoscopy

Langlands conjectured that we can transfer stable orbital integrals and its κ-sisters from one group to another following a recipe known as endoscopy. Roughly speaking, one can compare κ-orbital integral with stable orbital integral on an endoscopic group attached to κ. To make this statement more precise, we will need to recall the definition of an endoscopic group.

Let $\hat{G}$ be the dual group of a split reductive group G which is a connected reductive group over $\mathbb{C}$ whose root datum is obtained from the one of G by exchanging roots and coroots. In the non-split case, the definition is a little bit more complicated by the need of taking into account of the action of the Galois group of F on the Dynkin diagram attached to G. For instance, if $G = \mathrm{Sp}(2n)$ then $\hat{G} = \mathrm{SO}(2n+1)$ and conversely. The group $\mathrm{SO}(2n)$ is selfdual.

By Tate-Nakayama duality, a character κ of $\mathrm{H}^1(F, I_\gamma)$ corresponds to semisimple element $\hat{G}$ well-defined up to conjugacy. Let $\hat{H}$ be the neutral component $\hat{G}^0_\kappa$ of the centralizer of κ in $\hat{G}$. We have then the endoscopic group H defined as the quasi-split dual of $\hat{H}$ equipped with anouter action of the Galois group of F through $\pi_0(\hat{G}^0_\kappa)$.

For the purpose of this expository paper, it is good to assume that G is split and has a connected center, in which case $\hat{G}$ has a derived group simply connected. This implies that the centralizer $\hat{G}_\kappa$ is connected and therefore H is split.

For instance if $G = \mathrm{Sp}(2n)$, then H can be $\mathrm{SO}(2n)$ which has no direct ties to G.

2.4. Transfer of Stable Conjugacy Classes
According to the above definition of endocospic group, one can check there exists a morphism $T/W_H \to T/W$ which allows us to transfer stable conjugacy classes of H to G. This is due to the fact that dual groups have the same Weyl groups so that at least in the split cases, W_H is a subgroup of W. Let $\gamma_H \in H(F)$ with characteristic polynomial a_H mapping to the characteristic polynomial a of $\gamma \in G(F)$, we will say abusively that γ and γ_H have the same characteristic polynomial.

We can do the same for Lie algebras. For Lie alebras, we can transfer stable conjugacy class between two groups with isogenous root systems for instance $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$.

3. Conjectures on Orbital Integrals

3.1. Transfer Conjecture
The first conjecture concerns the possibility of transfer of smooth functions:

Conjecture 3.1. For every $f \in C_c^\infty(G(F))$ there exists $f^H \in C_c^\infty(H(F))$ such that

$$\mathbf{SO}_{\gamma_H}(f^H) = \Delta(\gamma_H, \gamma)\mathbf{O}_\gamma^\kappa(f) \tag{5}$$

for all strongly regular semisimple elements γ_H and γ having the same characteristic polynomial, $\Delta(\gamma_H, \gamma)$ being a factor which is independent of f.

Under the assumption γ_H and γ strongly regular semisimple with the same characteristic polynomial, their centralizers in H and G respectively are canonically isomorphic. It is then obvious how to transfer Haar measures between those locally compact groups.

The "tranfer" factor $\Delta(\gamma_H, \gamma)$, defined by Langlands and Shelstad in [12], is a power of the number q which is the cardinal of the residue field and a root unity which is in most of the cases is a sign. This sign takes into account the fact that $\mathbf{O}_\gamma^\kappa$ depends on the choice of γ in its stable conjugacy class. In the case of Lie algebra, if we pick $\gamma = \iota(a)$ where ι is the Kostant section to the characteristic polynomial map, this sign equals one, according to Kottwitz in [9]. According to Kottwitz again, if the derived group of G is simply connected, Steinberg's section would do the same for Lie group instead of Lie algebra.

3.2. Fundamental Lemma
Assume that we are in unramified situation, i.e. both G and H have reductive models over $\mathcal{O}_F$. Let $1_{G(\mathcal{O}_F)}$ be the characteristic function of $G(\mathcal{O}_F)$ and $1_{H(\mathcal{O}_F)}$ the characteristic function of $H(\mathcal{O}_F)$.

Conjecture 3.2. The equality (5) holds for $f = 1_{G(\mathcal{O}_F)}$ and $f^H = 1_{H(\mathcal{O}_F)}$.

There is a more general version of the fundamental lemma. Let $\mathcal{H}_G$ be the algebra of $G(\mathcal{O}_F)$-biinvariant functions with compact support of $G(F)$ and $\mathcal{H}_H$ the similar algebra for G. Using Satake isomorphism we have a canonical homomorphism $b : \mathcal{H}_G \to \mathcal{H}_H$. Here is the more general version of the fundamental lemma for Hecke algebra.

Conjecture 3.3. The equality (5) holds for any $f \in \mathcal{H}_G$ and for $f^H = b(f)$.

3.3. Lie Algebras

There are similar conjectures for Lie algebras. The transfer conjecture can be stated in the same way with $f \in C_c^\infty(\mathfrak{g}(F))$ and $f^H \in C_c^\infty(\mathfrak{h}(F))$. Idem for the fundamental lemma with $f = 1_{\mathfrak{g}(\mathcal{O}_F)}$ and $f^H = 1_{\mathfrak{h}(\mathcal{O}_F)}$.

Waldspurger made a cousin conjecture which is however much simpler to state. Let G_1 and G_2 be two semisimple groups with isogenous root systems, i.e. there exists an isomorphism between their maximal tori which maps a root of G_1 on a scalar multiple of a root of G_2 and conversely. In this case, there is an isomorphism $\mathfrak{t}_1/W_1 \simeq \mathfrak{t}_2/W_2$. We can therefore transfer regular semisimple stable conjugacy classes from $\mathfrak{g}_1(F)$ to $\mathfrak{g}_2(F)$ and back.

Conjecture 3.4. Let $\gamma_1 \in \mathfrak{g}_1(F)$ and $\gamma_2 \in \mathfrak{g}_2(F)$ be regular semisimple elements having the same characteristic polynomial. Then we have

$$\mathbf{SO}_{\gamma_1}(1_{\mathfrak{g}_1(\mathcal{O}_F)}) = \mathbf{SO}_{\gamma_2}(1_{\mathfrak{g}_2(\mathcal{O}_F)}). \tag{6}$$

This conjecture is particularly agreeable as it involves no complicated transfer factors.

3.4. Applications

The above conjectures on orbital integrals are necessary ingredients in the stabilization of the trace formulas. The upshot is a complete understanding of the structures of L-packets. Also, promised work of Arthur would provide the transfer of automorphic representations from a classical group to linear group via the standard representation of the dual group. It follows in particular the analytic properties of standard L-functions for classical groups.

Another important application is the study of cohomology of Shimura varities. The upshot is the construction of Galois representation attached to selfdual automorphic representations.

The above applications of the transfer conjecture and the fundamental lemma are usually referred by the strange name "theory of endoscopy". Though it had been very effective in the above questions, it also has obvious limitations which were pointed out by Langlands. Endoscopy is concerned with certain very small cases of a much broader conjectural principle called functoriality.

3.5. History of the Proof

All the above conjectures are now theorems. Let me sketch the contribution of different peoples coming into its proof.

In a landmark paper, Waldspurger proved that the fundamental lemma implies the transfer conjectures. Due to his and Hales' works, we can go from Lie algebra to Lie group.

Particular cases of the fundamental lemma were proved by different peoples: Labesse-Langlands for SL(2), Kottwitz for SL(3), Kazhdan and Waldspurger for SL(n), Rogawski for U(3), Laumon and Ngô for U(n), Hales, Schroder and Weissauer for Sp(4).

Waldspurger and independently Cluckers, Hales and Loeser, proved that for proving the fundamental lemma for a p-adic fields, it is enough to prove it for a local field in characteristic p like $\mathbb{F}_q(\!(\pi)\!)$, see [16, 2].

For local fields of Laurent series, there is an approach using algebraic geometry which was proved eventually successful. This was first introduced by Goresky, Kottwitz and MacPherson using affine Springer fibers. The Hitchin fibration was introduced in this context in [14]. Laumon and I used it to prove the fundamental lemma for unitary group in [13]. The general case was proved in [15] with essentially the same strategy as in [13] except in the determination of the support of simple perverse sheaves occurring in the cohomology of Hitchin fibration.

4. Geometric Method

4.1. Affine Springer Fibers
Let $k = \mathbb{F}_q$ be a finite field with q elements. Let G be a reductive group over k and $\mathfrak{g}$ its Lie algebra. Let denote $F = k((\pi))$ and $\mathcal{O}_F = k[[\pi]]$. Let $\gamma \in \mathfrak{g}(F)$ be a regular semisimple element. According to Kazhdan and Lusztig [8], there exists a k-scheme $\mathcal{M}_\gamma$ whose set of k points is

$$\mathcal{M}_\gamma(k) = \{g \in G(F)/G(\mathcal{O}_F) \mid \mathrm{ad}(g)^{-1}(\gamma) \in \mathfrak{g}(\mathcal{O}_F)\}.$$

They proved that the affine Springer fiber $\mathcal{M}_\gamma$ is finite dimensional and locally of finite type.

There exists a finite dimensional k-group scheme $\mathcal{P}_\gamma$ acting on $\mathcal{M}_\gamma$. We know that $\mathcal{M}_\gamma$ admits a dense open subset $\mathcal{M}_\gamma^{\mathrm{reg}}$ which is a principal homogeneous space of $\mathcal{P}_\gamma$. The group of connected components $\pi_0(\mathcal{P}_\gamma)$ of $\mathcal{P}_\gamma$ might be infinite which is precisely what prevents $\mathcal{M}_\gamma$ from being of finite type. The group of k-points $\mathcal{P}_\gamma(k)$ is a quotient of the group of F-points $I_\gamma(F)$ of the centralizer of γ

$$I_\gamma(F) \to \mathcal{P}_\gamma(k)$$

and its action of $\mathcal{M}_\gamma(k)$ is that of $I_\gamma(F)$.

Consider the simplest nontrivial example. Let $G = \mathrm{SL}_2$ and let g be the diagonal matrix

$$\gamma = \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix}.$$

In this case $\mathcal{M}_\gamma$ is an infinite chain of projective lines with the point ∞ in one copy identified with the point 0 of the next. The group $\mathcal{P}_\gamma$ is $\mathbb{G}_m \times \mathbb{Z}$ with $\mathbb{G}_m$ acts on each copy of $\mathbb{P}^1$ by rescaling and $\mathbb{Z}$ acts by translation from one copy to another. The dense open orbit is obtained by removing $\mathcal{M}_\gamma$ all its double points. The centralizer of γ is $\mathbb{G}_m$ over F. The surjective homomorphism

$$I_\gamma(F) = F^\times \to k^\times \times \mathbb{Z} = \mathcal{P}_\gamma(k)$$

attaches to a nonzero Laurent series its first non-zero coefficient and its degree.

In general there is not such an explicit description of the affine Springer fiber at our disposal. The group $\mathcal{P}_\gamma$ is nevertheless rather explicit. Thus it is convenient to keep in mind that $\mathcal{M}_\gamma$ is a kind of equivariant compactification of $\mathcal{P}_\gamma$.

4.2. Counting Points over Finite Fields

The terms which are inverse the volume in the definition of orbital integrals suggest we should count the number of points of the quotient $[\mathcal{M}_\gamma/\mathcal{P}_\gamma]$ as an algebraic stack. In that sense $[\mathcal{M}_\gamma/\mathcal{P}_\gamma](k)$ is not a set but a groupoid. The cardinal of a groupoid $\mathcal{C}$ is by definition the number

$$\sharp\mathcal{C} = \sum_x \frac{1}{\sharp\mathrm{Aut}(x)}$$

for x in a set of representative of its isomorphism classes and $\sharp\mathrm{Aut}(x)$ being the order of the group of automorphisms of x. In our case, it can be proved that

$$\sharp[\mathcal{M}_\gamma/\mathcal{P}_\gamma](k) = \mathbf{SO}_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)})$$

for an appropriate choice of Haar measure on the centralizer. Roughly speaking, this Haar measure gives the volume one to the kernel of the homomorphism $I_\gamma(F) \to \mathcal{P}_\gamma(k)$ while the correct definition is a little bit more subtle.

The group $\pi_0(\mathcal{P}_\gamma)$ of geometric connected components of $\mathcal{P}_\gamma$ is an abelian group of finite type equipped with an action of Frobenius σ_q. For every character of finite order $\kappa : \pi_0(\mathcal{P}_\kappa) \to \mathbb{C}^\times$ fixed by σ_ℓ, we consider the finite sum

$$\sharp[\mathcal{M}_\gamma/\mathcal{P}_\gamma](k)_\kappa = \sum_x \frac{\kappa(\mathrm{cl}(x))}{\sharp\mathrm{Aut}(x)},$$

where $\mathrm{cl}(x) \in \mathrm{H}^1(k, \mathcal{P}_\gamma)$ is the class of the $\mathcal{P}_\gamma$-torsor $\pi^{-1}(x)$ where $\pi : \mathcal{M}_\gamma \to [\mathcal{M}_\gamma/\mathcal{P}_\gamma]$ is the quotient map. By a similar counting argument as in the stable case, we have

$$\sharp[\mathcal{M}_\gamma/\mathcal{P}_\gamma](k)_\kappa = \mathbf{O}^\kappa_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}).$$

This provides a cohomological interpretation for κ-orbital integrals. Let fix an isomorphism $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$ so that κ can be seen as taking values in $\mathbb{Q}_\ell$. Then we have the formula

$$\mathbf{O}^\kappa_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}) = \sharp\mathcal{P}^0_\gamma(k)^{-1}\mathrm{tr}(\sigma_q, \mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)_\kappa),$$

where assuming $\pi_0(\mathcal{O}_\gamma)$ finite, $\mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)_\kappa$ is the biggest direct summand of $\mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)$ on which $\mathcal{P}_\gamma$ acts through the character κ. When $\pi_0(\mathcal{O}_\gamma)$ is infinite, the definition of $\mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)_\kappa$ is a little bit more complicated.

By taking $\kappa = 1$, we obtained a cohomological interpretation of the stable orbital integral

$$\mathbf{SO}_\gamma(1_{\mathfrak{g}(\mathcal{O}_F)}) = \sharp\mathcal{P}^0_\gamma(k)^{-1}\mathrm{tr}(\sigma_q, \mathrm{H}^*(\mathcal{M}_\gamma, \bar{\mathbb{Q}}_\ell)_{st}),$$

where index *st* means the direct summand where $\mathcal{P}_\gamma$ acts trivially at least in the case $\pi_0(\mathcal{P}_\gamma)$ is finite.

This cohomological interpretation is essentially the same as the one given by Goresky, Kottwitz and MacPherson [5]. It allows us to shift focus from a combinatorial problem of counting lattices to a geometric problem of computing ℓ-adic cohomology. Those are however different formulation of the same problem and are at the same level of difficulty. There does not seem to be a good way to compute neither orbital integral nor cohomology of affine Springer fibers.

This stems from the basic fact that we don't know much about $\mathcal{M}_\gamma$. The only information which is available in general is that $\mathcal{M}_\gamma$ is a kind of equivariant compactification of a group $\mathcal{P}_\gamma$ that we know better. It is not true that $\mathcal{P}_\gamma$ determines $\mathrm{H}^*(\mathcal{M}_\gamma)$ but the similar statement is true if we shift from the local context to the global context of Hitchin fibration.

4.3. Hitchin Fibration: the Stable Part

For setting up the Hitchin fibration [7], we need a smooth projective curve X over the field k, a semisimple group G and a line bundle D of large degree. We consider the moduli stack of Higgs bundles (E, ϕ) where E is a G-principal bundle on X and ϕ is a global section of $\mathrm{ad}(E) \otimes_{\mathcal{O}_X} D$. By taking the characteristic polynomial of the Higgs fields ϕ, we get the Hitchin fibration

$$f : \mathcal{M} \to \mathcal{A},$$

where $\mathcal{A}$ is a finite dimensional vector space over k. In general, $\mathcal{M}$ is an algebraic stack which is not of finite type and so is f. There is a good way to use GIT to truncate $\mathcal{M}$ to get a scheme of finite type which is proper over $\mathcal{A}$. For our purpose, there exists a dense open subset $\mathcal{A}^{ani}$ of $\mathcal{A}$ over which f is already a proper map before any truncation. In the sequel, we will write $\mathcal{A}$ for $\mathcal{A}^{ani}$.

Since $\mathcal{M}$ is smooth over k, we know by Deligne that the complex $f_*\mathbb{Q}_\ell$ of direct image is pure, see [4].

The fibers of f can be seen in many way as global analog of affine Springer fibers. If we count the number of points of $\mathcal{M}_a = f^{-1}(a)$ we get a sum of global orbital integral

$$\sharp\mathcal{M}_a(k) = \sum_\gamma \int_{I_\gamma(F)\backslash G(\mathbb{A}_F)} 1_D(\mathrm{ad}(g)^{-1}\gamma)dg, \qquad (7)$$

where γ runs over the set of conjugacy classes of $\mathfrak{g}(F)$, F being the field of rational functions on X, within the stable class a, $\mathbb{A}_F$ is the ring of adeles of F, 1_D is a very simple function on $\mathfrak{g}(\mathbb{A}_F)$ associated with a choice of divisor within the linear equivalence class D. In that way, if we sum over $a \in \mathcal{A}(k)$, we get an expression which looks like the geometric side of the trace formula for Lie algebra.

If we want to convert the sum (7) into local orbital integrals, we face the following basic problem: there exists a Galois cohomological obstruction which prevents a collection of local conjugacy classes $(\gamma_x)_{x\in|X|}$ within stable class a from being patched to a global conjugacy class γ.

There exists a group $\mathcal{P}_a$ acting on the fiber $\mathcal{M}_a$ with a dense open subset such that the quotient $[\mathcal{M}_a/\mathcal{P}_a]$ can be expressed as a product of local quotients $[\mathcal{M}_{x,a}/\mathcal{P}_{x,a}]$ which are trivial at almost all places $x \in |X|$. Here the only difference between $\mathcal{M}_{x,a}$ and the affine Springer fiber $\mathcal{M}_\gamma$ for the local field F_x, is that $\mathcal{M}_{x,a}$ only depends on the characterictic polynomial and not on the choice of an element γ. In the way $\mathcal{M}_{x,a}$ is constructed, it is just the affine Springer fiber $\mathcal{M}_\gamma$ for the local field F_x for the particular element $\gamma = \iota(x)$ given by the Kostant section. At any rate, the Galois cohomological obstruction above mentioned now lies naturally in $\mathrm{H}^1(k, \mathcal{P}_a)$ which can be computed easily from the action of σ_q on the group of geometric connected components $\pi_0(\mathcal{P}_a)$ of $\mathcal{P}_a$.

For a generic parameter a, $\mathcal{P}_a$ is an abelian variety up to connected components and finite automorphism group and $\mathcal{P}_A$ acts simply transitively on $\mathcal{M}_a$. In this case the quotient $[\mathcal{M}_a/\mathcal{P}_a]$ is trivial and so are all the local quotients $[\mathcal{M}_{x,a}/\mathcal{P}_{x,a}]$. We can ask ourselves in what extent, the cohomology of generic $\mathcal{M}_a$ determines the cohomology of special $\mathcal{M}_a$. This question can be answered very precisely in using the action of $\mathcal{P}$ on the cohomology.

The groups $\mathcal{P}_a$ are organized in a smooth group scheme over $\mathcal{A}$ or more precisely a Picard Deligne-Mumford algebraic stack $g : \mathcal{P} \to \mathcal{A}$. The group of connected components $\pi_0(\mathcal{P}_a)$ are also organized in a sheaf of abelian groups $\pi_0(\mathcal{P})$ for the etale topology of $\mathcal{A}$.

The action of $\mathcal{P}$ on the higher direct image $\mathrm{R}^i f_* \mathbb{Q}_\ell = \mathrm{H}^i(f_* \mathbb{Q}_\ell)$ factors through the discrete sheaf $\pi_0(\mathcal{P})$. In order to exploit the purity of $f_*\mathbb{Q}_\ell$, it is better to consider the cohomology ${}^p\mathrm{H}^i(f_*\mathbb{Q}_\ell)$ for the perverse t-structure instead since those are pure perverse sheaves. Let consider the direct summand

$$ {}^p\mathrm{H}^i(f_*\mathbb{Q}_\ell)_{st}, $$

where $\pi_0(\mathcal{P})$ acts trivially.

Theorem 4.1. *The perverse sheaf ${}^p\mathrm{H}^i(f_*\mathbb{Q}_\ell)_{st}$ is completely determined by its restriction to any nonempty open subset of $\mathcal{A}$ via the functor of intermediate extension.*

For some technical reason, this theorem is only proved so far for $k = \mathbb{C}$. When k is a finite field, we proved a weaker variant enough for local applications which is too complicated to be stated in this expository note.

Let G_a and G_2 be two semisimple groups with isogenous root systems like $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$. We have two corrsponding Hitchin fibration $f_\alpha : \mathcal{M}_\alpha \to \mathcal{A}$ for $\alpha \in \{1, 2\}$ over the same base. For a generic a, $\mathcal{P}_{1,a}$, and $\mathcal{P}_{2,a}$ are essentially isogenous abelian varieties in the following sense

$$0 \to A_{\alpha,a}/\mathrm{Aut}(\alpha) \to \mathcal{P}_{\alpha,a} \to \pi_0(\alpha) \to 0,$$

where $A_{\alpha,a}$ is an abelian variety, $\mathrm{Aut}(\alpha)$ is a finite abelian group acting trivially on $A_{\alpha,a}$ and $\pi_0(\alpha)$ is also a finite abelian group. We have $A_{1,a}$ is the dual abelian variety of $A_{2,a}$. It follows that ${}^p\mathrm{H}^i(f_{1,*}\mathbb{Q}_\ell)_{st}$ and ${}^p\mathrm{H}^i(f_{2,*}\mathbb{Q}_\ell)_{st}$ restricited to a nonempty open subset of $\mathcal{A}$ are isomorphic local systems. By the intermediate extension, we obtain an isomorphism between the perverse sheaves ${}^p\mathrm{H}^i(f_{1,*}\mathbb{Q}_\ell)_{st}$ and ${}^p\mathrm{H}^i(f_{2,*}\mathbb{Q}_\ell)_{st}$. We derive from this isomorphism Waldspurger's Conjecture 6.

4.4. Hitchin Fibration: the Endoscopic Part

The fundamental lemma can be proved by similar argument by a close examination of the other summand of ${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell)$. Here one needs a precise description of the group of connected components $\pi_0(\mathcal{P}_a)$. By a kind of global Tate-Nakayama duality, there exists a surjective map

$$\mathbf{X}_*(T) \to \pi_0(\mathcal{P}_a)$$

well defined up to W-conjugation, where $\mathbf{X}_*(T)$ is the group of cocharacter of a maximal torus T and W is its Weyl group for every geometric point $a \in \mathcal{A}$. Since we are considering only the anisotropic locus of Hitchin fibration, $\pi_0(\mathcal{P}_a)$ is an abelian finite group, quotient of $\mathbf{X}_*(T)$ up to W-conjugation.

Since $\mathcal{P}$ is a smooth commutative group scheme over $\mathcal{A}$, there exists a sheaf $\pi_0(\mathcal{P})$ of abelian groups that interpolates $\pi_0(\mathcal{P}_a)$ with respect to the parameter a. It is agreeable to consider to construct an étale covering $\tilde{\mathcal{A}} \to \mathcal{A}$ which is galoisian with Galois group W, such that over $\tilde{\mathcal{A}}$, we have a canonical surjective map

$$\mathbf{X}_*(T) \to \pi_0(\mathcal{P}).$$

This allows us to decompose

$${}^p\mathrm{H}^j(f_*\mathbb{Q}_\ell) = \bigoplus_{[\kappa]}^{p} \mathrm{H}^j(f_*\mathbb{Q}_\ell)_{[\kappa]}$$

over W-conjugacy classes of the dual torus $\hat{T}$ which amount to the same as semisimple conjugacy classes $[\kappa]$ of the dual group $\hat{G}$. Obviously, only a finite number of $[\kappa]$ have a nontrivial contribution.

It happens that those $[\kappa]$ correspond to equivalence classes of endoscopic groups at least if G is adjoint. For an endoscopic group H corresponding to $[\kappa]$, we can form the Hitchin fibration $f_H : \mathcal{M}_H \to \mathcal{A}_H$ for H. There is no direct link between $\mathcal{M}_H$ and $\mathcal{M}$ but we do have a canonical morphism $\mathcal{A}_H \to \mathcal{A}$ which is (locally) a closed embedding. Let denote $\mathcal{B}_H$ the image of $\mathcal{A}_H$ in $\mathcal{A}$.

Theorem 4.2. *Assume G is adjoint. The perverse sheaf ${}^p\mathrm{H}^i(f_*\mathbb{Q}_\ell)_{[\kappa]}$ is supported by $\mathcal{B}_H$ and is determined by its restriction to any nonempty local subset of $\mathcal{B}_H$ by the middle extension functor.*

Again, this theorem is only proved for $k = \mathbb{C}$. Over a finite field, we proved a weaker variant which is good enough for local applications. When G is not adjoint there are more than one endoscopic group corresponding to $[\kappa]$. The theorem remains valid if we take into account all such endoscopic groups.

Again, the perverse sheaf ${}^p\mathrm{H}^i(f_*\mathbb{Q}_\ell)_{[\kappa]}$ can be computed explicitly on an open subset $\mathcal{B}_H$. For a generic $a \in \mathcal{B}_H$, we only need to do computations with the infinite chain of projective lines as in SL(2) example.

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