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On Nonlinear Boundary Value Problems for Nonlinear Wave Equations

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Abstract. The paper is devoted to the study of existence, stability, regularity in time variable, asymptotic behavior and asymptotic expansion of solutions of nonlinear boundary value problems for nonlinear wave equations.

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1. Introduction

In this paper, we present recent results of our research group on nonlinear boundary value problems for nonlinear wave equations.

We will consider the initial-boundary value problem for the nonlinear wave equation

$$Lu \equiv u_{tt} + Au = f(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2,\|u_t\|^2),\ x \in \Omega \subset \mathbb{R}^N,\ 0 < t < T, \tag{1}$$

associated with the following boundary condition

$$Bu = g(x,t),\ x \in \partial\Omega,\ 0 < t < T, \tag{2}$$

and the following initial conditions

$$u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x), \tag{3}$$

where B is a linear or nonlinear operator and

$$Au = -\frac{\partial}{\partial x}\left[\mu\left(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2,\|u_t\|^2\right)u_x\right] + \int_0^t k(t-s)u_{xx}(x,s)ds, \tag{4}$$

with μ, k, f, $\widetilde{u}_0$, $\widetilde{u}_1$ being given functions. In which,

- The case $\Omega = (0,1)$, the nonlinear terms $f(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2,\|u_t\|^2)$ and $\mu(x,t,u,\ u_x,\ u_t,\ \|u\|^2,\ \|u_x\|^2,\ \|u_t\|^2)$ depend not only on the variables x,t,u,u_x,u_t, but also depend on the following integrals

$$\begin{aligned} \|u\|^2 &= \int_\Omega |u(x,t)|^2\,dx, \\ \|u_x\|^2 &= \int_\Omega |u_x(x,t)|^2\,dx, \text{ and} \\ \|u_t\|^2 &= \int_\Omega |u_t(x,t)|^2\,dx. \end{aligned} \tag{5}$$

- The case $\Omega \subset \mathbb{R}^N$, $N > 1$, the nonlinear terms f and μ are also defined as the above case, where the notations u_x, u_{xx} and $\|u_x\|^2 = \int_\Omega |u_x(x,t)|^2\,dx$ mean $\nabla u = (u_{x_1},...,u_{x_N})$, $\Delta u = \sum_{i=1}^N u_{x_i x_i}$ and $\|\nabla u\|^2 = \sum_{i=1}^N \int_\Omega |u_{x_i}(x,t)|^2\,dx$, respectively.

When $k \equiv 0$, $f \equiv 0$, $\mu = a^2$ is a positive constant, $\Omega = (0,L)$, (1) has the form $u_{tt} - a^2 u_{xx} = 0$ and describes the transverse vibrations of an elastic string with small amplitude without acting of external force. Then, we can see that the length of the string does not change.

When $k \equiv 0$, $\Omega = (0,L)$, (1) is related to the Kirchhoff equation

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\right)u_{xx}, \tag{6}$$

presented by Kirchhoff in 1876 (see [18]). This equation is an extension of the classical D'Alembert wave equation which considers the effects of the changes in the length of the string during the vibrations. The parameters in (6) have the following meanings: u is the lateral deflection, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In [7], Carrier has also established a model of the type

$$\rho h u_{tt} = \left(P_0 + P_1 \int_0^L u^2(y,t)dy \right) u_{xx}, \tag{7}$$

where ρ, h, P_0 and P_1 are given constants.

One of the early classical studies dedicated to Kirchhoff equations was given by Pohozaev [70]. After the work of Lions, for example see [21], the equation (6) received much attention where an abstract framework to the problem was proposed. We refer the reader to, e.g., Cavalcanti et al. [8] – [10], Ebihara, Medeiros and Miranda [15], Miranda et al. [59], Lasiecka and Ong [20], Clark [11], Frota [16], Hosoya, Yamada [17], Larkin [19], Medeiros [54], Menzala [60], Park et al. [71, 72], Rabello et al. [74], for many interesting results and further references. A survey of the results about the mathematical aspects of the Kirchhoff model can be found in Medeiros, Limaco and Menezes [55, 56].

Dmitriyeva [14] considered the following problem in two dimensions

$$u_{tt} + \lambda \varDelta^2 u - \|\nabla u\|^2 \varDelta u + \varepsilon u_t = F(x,t),\ x \in \varOmega = (0,\pi) \times (0,\pi),\ t > 0, \tag{8}$$

$$u = \sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i^2} \nu_i = 0,\ x \in \partial\varOmega,\ t > 0, \tag{9}$$

with the initial condition (3), where $\varepsilon > 0$, $\lambda = \frac{1}{6}\pi^2 h^2$ and $\nu_i = \cos(\nu, x_i)$. In this case, the problem (3), (8), (9) describes the nonlinear vibrations of a square plate with static load.

Based on the above, we have considered the problem (1)-(3) with some different forms of the nonlinear terms μ, f. By the fact that it is difficult to consider the problem (1)-(3) with some initial - boundary conditions in the case μ, f depending not only on the variables x, t, few works were done as far as we know.

In order to solve the ones, see Lions [24], we usually establish a sequence of "*approximate solutions*" which is bounded in an appropriate space. Generally, the most convenient approximate solutions are obtained by the standard method, here we choose the method "Faedo – Galerkin approximation". The difficulty is to pass to the limit, this can be overcome in the following way. Using many different techniques in estimations with respect to every form of μ or f, we find sufficiently many "a priori estimates" on the approximate solutions to obtain-via compactness argument in which the compact imbedding theorems are used-strong convergence in appropriate spaces. The passage to the limit is now possible and the existence of solution follows.

On the other hand, the linearization method for nonlinear terms is usually used. Let us present this technique as follows. At first, we note that, for each $v = v(x,t)$ belonging to a suitable function space X, we can give some suitable assumptions to obtain a unique solution $u \in X$ of the problem with respect to

$$\mu(x, t, v, v_x, v_t, \|v\|^2, \|v_x\|^2, \|v_t\|^2) = \widetilde{\mu}(x, t)$$

and

$$f(x, t, v, v_x, v_t, \|v\|^2, \|v_x\|^2, \|v_t\|^2) = \widetilde{f}(x, t).$$

It is obvious that u depends on v, so we can suppose that $u = \widehat{A}(v)$. Therefore, the above prolem can be reduced to the fixed point problem for the operator $\widehat{A} : X \to X$. Based on these ideas, with the first term u_0 being chosen, the usual iteration $u_m = \widehat{A}(u_{m-1})$, $m = 1, 2, \ldots$, is applied to establish a sequence $\{u_m\}$, which converges to the solution of the probem, hence we get the existence results as was stated above.

After this, the conditions are given such that the stability, regularity in time variable, asymptotic behavior or asymptotic expansion of the solution are established.

The paper consists of five sections. In Sec. 2, we consider the nonlinear wave equation with the Kirchhoff-Carrier operator. Sec. 3 is devoted to the nonlinear wave equation associated with the nonhomogeneous conditions. In Sec. 4, we consider the linear wave equation associated with two-point boundary conditions. Finally, Sec. 5 is concerned with the nonlinear wave equations involving convolution.

The results given here were presented in [3, 4, 12, 13, 25-53, 62-68, 79-81].

2. The Nonlinear Kirchhoff-Carrier Wave Equations

In this section we consider the nonlinear wave equation with the Kirchhoff-Carrier operator

$$\begin{cases} u_{tt} - \mu\left(t, \|u\|^2, \|u_x\|^2, \|u_t\|^2\right) u_{xx} = f(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2, \|u_t\|^2), \\ \qquad\qquad x \in \Omega = (0,1),\ 0 < t < T, \\ B_0 u \equiv u_x(0,t) - h_0 u(0,t) = 0,\ B_1 u \equiv u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x), \end{cases} \tag{10}$$

where μ, f, $\widetilde{u}_0$, $\widetilde{u}_1$ are given functions satisfying conditions specified later and $h_0 \geq 0$, $h_1 \geq 0$ are given constants and $h_0 + h_1 > 0$. We shall first associate with the problem (10) a linear recurrent sequence which is bounded in a suitable function space. The existence of a local solution is proved by a standard compactness argument [24]. If $\mu \in C^{N+1}(\mathbb{R}^4_+)$, $\mu_1 \in C^N(\mathbb{R}^4_+)$, $\mu \geq \mu_0 > 0$, $\mu_1 \geq 0$, $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3_+)$ and $f_1 \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3_+)$, then an asymptotic expansion of order $N+1$ in ε of the solution is obtained for the right – hand side of the form $f(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2, \|u_t\|^2) + \varepsilon f_1(x, t, u, u_x, u_t, \|u\|^2, \|u_x\|^2, \|u_t\|^2)$ and μ stands for $\mu + \varepsilon\mu_1$, for ε sufficiently small. This result is a relative generalization of [4, 28, 31-35, 37, 38, 40, 41, 44, 48, 63, 64, 66-69, 82].

2.1. The Existence and Uniqueness Theorem

We will omit the definitions of the usual function spaces and denote them by the notations $L^p = L^p(0,1)$, $H^m = H^m(0,1)$. The notion of Sobolev space, H^m, in the paper can be found in [24, p. 5].

Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \le p \le \infty$ the Banach space of real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < +\infty \text{ for } 1 \le p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \text{ for } p = \infty.$$

Let $u(t)$, $u_t(t) = u'(t)$, $u_{tt}(t) = u''(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

With $f = f(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2,\|u_t\|^2) = f(x,t,u,v,w,U,V,W)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial f}{\partial u}$, $D_4 f = \frac{\partial f}{\partial v}$, $D_5 f = \frac{\partial f}{\partial w}$, $D_6 f = \frac{\partial f}{\partial U}$, $D_7 f = \frac{\partial f}{\partial V}$ and $D_8 f = \frac{\partial f}{\partial W}$.

We make the following assumptions:

(A_1) $h_0 > 0$, $h_1 \ge 0$,

(A_2) $(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$,

(A_3) $\mu \in C^1(\mathbb{R}_+^4)$, $\mu(t,U,V,W) \ge \mu_0 > 0$,

(A_4) $f \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+^3)$.

Remark 2.1. (i) In this section, the above derivatives of the unknown function u are defined in the sense of distributions, (see [24, p. 6-7]). So are the derivatives of the unknown function u in the whole paper.

(ii) The results as it will be said below (Theorems 2.4-2.8) still hold if (A_1) is replaced by the following assumption:

$$h_0 \ge 0, \ h_1 \ge 0, \ h_0 + h_1 > 0.$$

We put

$$a(u,v) = \int_0^1 u_x(x) v_x(x) dx + h_0 u(0) v(0) + h_1 u(1) v(1). \tag{11}$$

Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by (11) is continuous on $H^1 \times H^1$ and coercive on H^1. In H^1 we use the equivalent norm

$$\|v\|_{H^1} = \left(v^2(0) + \int_0^1 |v_x(x)|^2 \, dx \right)^{1/2}. \tag{12}$$

For each $M > 0$ and $T > 0$, we put

$$\begin{aligned} W(M,T) = \{ & v \in L^\infty(0,T;H^2) : v_t \in L^\infty(0,T;H^1),\ v_{tt} \in L^\infty(0,T;L^2), \\ & \|v\|_{L^\infty(0,T;H^2)}, \|v_t\|_{L^\infty(0,T;H^1)}, \|v_{tt}\|_{L^\infty(0,T;L^2)} \le M \}. \end{aligned} \tag{13}$$

We choose the first term $u_0 = \widetilde{u}_0$. Suppose that

$$u_{m-1} \in W(M,T). \tag{14}$$

We associate with the problem (10) the following variational problem and define the weak solution of (10) as follows.

Find $u_m \in W(M,T)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t), v \rangle + \mu_m(t) a(u_m(t), v) = \langle F_m(t), v \rangle \text{ for all } v \in H^1, \\ u_m(0) = \widetilde{u}_0,\ u_m'(0) = \widetilde{u}_1, \end{cases} \tag{15}$$

where

$$\begin{cases} \mu_m(t) & = \mu(t, \|u_{m-1}(t)\|^2, \|\bigtriangledown u_{m-1}(t)\|^2, \|u_{m-1}'(t)\|^2), \\ F_m(x,t) & = f(x,t,u_{m-1}(t), \bigtriangledown u_{m-1}(t), u_{m-1}'(t), \|u_{m-1}(t)\|^2, \\ & \qquad \|\bigtriangledown u_{m-1}(t)\|^2, \|u_{m-1}'(t)\|^2). \end{cases} \tag{16}$$

Definition 2.2. We say that u is a weak solution of (10) if

$$u \in \widetilde{W} = \left\{ v \in L^\infty(0,T;H^2) : v' \in L^\infty(0,T;H^1),\ v'' \in L^\infty(0,T;L^2) \right\}$$

and for each $v \in H^1$ one has

$$\begin{aligned} & \langle u''(t), v \rangle + \mu\left(t, \|u\|^2, \|u_x\|^2, \|u'\|^2\right) a(u(t), v) \\ & = \langle f(x,t,u,u_x,u_t, \|u\|^2, \|u_x\|^2, \|u'\|^2), v \rangle, \end{aligned}$$

and

$$u(0) = \widetilde{u}_0,\ u'(0) = \widetilde{u}_1.$$

Remark 2.3. The notion of weak solution of the problems given in next sections is also defined in the same manner.

We then have the following results, the proofs of which can be found in the paper [38].

Theorem 2.4. *Let* $(A_1) - (A_4)$ *hold. Then there exist positive constants* M, T *and the linear recurrent sequence* $\{u_m\} \subset W(M,T)$ *defined by* (14)-(16).

Theorem 2.5. *Let $(A_1) - (A_4)$ hold. Then there exist positive constants M, T such that the problem* (10) *has a unique weak solution $u \in W(M,T)$.*

The linear recurrent sequence $\{u_m\}$ defined by (14)-(16) *converges to the solution u strongly in the space $W_1(T) = \{v \in L^\infty(0,T;H^1) : v' \in L^\infty(0,T;L^2)\}$.*

Furthermore, we have also the estimation

$$\|u_m - u\|_{L^\infty(0,T;H^1)} + \|u_m' - u'\|_{L^\infty(0,T;L^2)} \le C k_T^m \text{ for all } m, \tag{17}$$

where $k_T < 1$ and C are positive constants depending only on T, u_0, and u_1.

Remark 2.6. If the condition (A_4) on f is weakened to

$(\widetilde{A}_4)$ $f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+^3)$ satisfies the conditions $D_j f \in C^0([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+^3),\ 1 \le j \le 8,\ j \ne 2,$

then Theorems 2.4, 2.5 still hold. It is not necessary that $f \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+^3)$, where the set $W(M,T)$ appearing in (13) is replaced by the following set

$$W_1(M,T) = \{v \in W_0(M,T) : v_{tt} \in L^\infty(0,T;L^2)\}, \tag{18}$$

with

$$\begin{aligned} W_0(M,T) = \{&v \in L^\infty(0,T;H^2) : v_t \in L^\infty(0,T;H^1), v_{tt} \in L^2(Q_T), \\ &\|v\|_{L^\infty(0,T;H^2)} \le M, \|v_t\|_{L^\infty(0,T;H^1)} \le M, \|v_{tt}\|_{L^2(Q_T)} \le M\}. \end{aligned} \tag{19}$$

2.2. Asymptotic Expansion of the Solutions in a Small Parameter ε

In this part, let $(A_1) - (A_4)$ hold. We make more the following assumptions:

(A_5) $\mu_1 \in C^1(\mathbb{R}_+^4),\ \mu_1(t,U,V,W) \ge 0,$

(A_6) $f_1 \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+^3).$

We consider the following perturbed problem, where ε is a small parameter, $|\varepsilon| \le 1$:

$$(P_\varepsilon) \begin{cases} u_{tt} - \mu_\varepsilon(t, \|u\|^2, \|u_x\|^2, \|u_t\|^2)\Delta u \\ \quad = F_\varepsilon(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2,\|u_t\|^2), 0<x<1,\ 0<t<T, \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x), \\ F_\varepsilon(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2,\|u_t\|^2) \\ \quad = f(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2,\|u_t\|^2) \\ \qquad + \varepsilon f_1(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2,\|u_t\|^2), \\ \mu_\varepsilon(t,\|u\|^2,\|u_x\|^2,\|u_t\|^2) \\ \quad = \mu(t,\|u\|^2,\|u_x\|^2,\|u_t\|^2) + \varepsilon\mu_1(t,\|u\|^2,\|u_x\|^2,\|u_t\|^2). \end{cases}$$

Let $u_0 \in W(M,T)$ be a weak solution of the problem (P_0) corresponding to $\varepsilon = 0$.

We then have the following results, the proofs of which can be found in the paper [38].

Theorem 2.7. *Let $(A_1)-(A_6)$ hold. Then there exist constants $M>0$ and $T>0$ such that, for every ε with $|\varepsilon|\leq 1$, the problem (P_ε) has a unique weak solution $u_\varepsilon\in W(M,T)$ satisfying the asymptotic estimation*

$$\|u_\varepsilon-u_0\|_{L^\infty(0,T;H^1)}+\|u_\varepsilon'-u_0'\|_{L^\infty(0,T;L^2)}\leq C\,|\varepsilon|\,, \tag{20}$$

where C is a constant depending only on μ_0, h_0, h_1, T, M, f, f_1, μ and μ_1.

The next result gives an asymptotic expansion of the weak solution u_ε of order $N+1$ in ε, for ε sufficiently small. We use the following notations

$$f[u]=f(x,t,u,u_x,u_t,\|u\|^2,\|u_x\|^2,\|u_t\|^2),\ \ \mu[u]=\mu(t,\|u\|^2,\|u_x\|^2,\|u_t\|^2).$$

Now, we assume that
(A_7) $\mu\in C^{N+1}(\mathbb{R}_+^4),\ \mu_1\in C^N(\mathbb{R}_+^4),\ \mu(t,U,V,W)\geq\mu_0>0,\ \mu_1(t,U,V,W)\geq 0,$
(A_8) $f\in C^{N+1}([0,1]\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+^3),\ f_1\in C^N([0,1]\times\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+^3).$

Let us consider the weak solutions $u_1,\ u_2,\dots,\ u_N\in W(M,T)$ with suitable constants $M>0$ and $T>0$ of the following corresponding problems:
a)

$$(Q_1)\begin{cases} u_1''-\mu[u_0]\Delta u_1=\widetilde{F}_1[u_1],\ 0<x<1,\ 0<t<T,\\ \nabla u_1(0,t)-h_0u_1(0,t)=\nabla u_1(1,t)+h_1u_1(1,t)=0,\\ u_1(x,0)=u_1'(x,0)=0,\end{cases}$$

where

$$\widetilde{F}_1[u_1]=\pi_1[f]+\pi_0[f_1]+(\rho_1[\mu]+\rho_0[\mu_1])\,\Delta u_0, \tag{21}$$

with $\pi_0[f]$, $\pi_1[f]$, $\rho_0[\mu]$, $\rho_1[\mu]$ defined as follows

$$\pi_0[f]=f[u_0]\equiv f(x,t,u_0,\nabla u_0,u_0',\|u_0\|^2,\|\nabla u_0\|^2,\|u_0'\|^2), \tag{22}$$

$$\begin{aligned}\pi_1[f]=\ &\pi_0[D_3f]u_1+\pi_0[D_4f]\bigtriangledown u_1+\pi_0[D_5f]u_1'+2\pi_0[D_6f]\langle u_0,u_1\rangle\\ &+2\pi_0[D_7f]\langle\bigtriangledown u_0,\bigtriangledown u_1\rangle+2\pi_0[D_8f]\langle u_0',\ u_1'\rangle,\end{aligned} \tag{23}$$

$$\rho_0[\mu]=\mu[u_0]\equiv\mu(t,\|u_0\|^2,\|\nabla u_0\|^2,\|u_0'\|^2), \tag{24}$$

$$\rho_1[\mu]=2\rho_0[D_6\mu]\langle u_0,u_1\rangle+2\rho_0[D_7\mu]\langle\bigtriangledown u_0,\bigtriangledown u_1\rangle+2\rho_0[D_8\mu]\langle u_0',\ u_1'\rangle, \tag{25}$$

b) with $2\leq i\leq N$,

$$(Q_i)\begin{cases} u_i''-\mu[u_0]\Delta u_i=\widetilde{F}_i[u_i],\ \ 0<x<1,\ 0<t<T,\\ \nabla u_i(0,t)-h_0u_i(0,t)=\nabla u_i(1,t)+h_1u_i(1,t)=0,\\ u_i(x,0)=u_i'(x,0)=0,\ \ i=1,2,\dots,N,\end{cases}$$

where

$$\widetilde{F}_i[u_i]=\pi_i[f]+\pi_{i-1}[f_1]+\sum_{k=1}^{i}(\rho_k[\mu]+\rho_{k-1}[\mu_1])\,\Delta u_{i-k}, \tag{26}$$

with $\pi_i[f] = \pi_i[f, u_0, u_1, \ldots, u_i]$, $\rho_i[\mu] = \rho_i[\mu, u_0, u_1, \ldots, u_i]$, $2 \le i \le N$ defined by the recurrent formulas

$$\begin{aligned}\pi_i[f] = &\sum_{k=0}^{i-1} \frac{i-k}{i} \{\pi_k[D_3 f]u_{i-k} + \pi_k[D_4 f]\nabla u_{i-k} + \pi_k[D_5 f]u'_{i-k}\} \\ &+ \frac{2}{i} \sum_{k=0}^{i-1} \sum_{j=0}^{i-k-1} (i-k-j)\{\pi_k[D_6 f]\langle u_j, u_{i-k-j}\rangle \\ &+ \pi_k[D_7 f]\langle \bigtriangledown u_j, \bigtriangledown u_{i-k-j}\rangle + \pi_k[D_8 f]\langle u'_j, u'_{i-k-j}\rangle\},\ 2 \le i \le N, \quad (27)\end{aligned}$$

$$\begin{aligned}\rho_i[\mu] = &\frac{2}{i} \sum_{k=0}^{i-1} \sum_{j=0}^{i-k-1} (i-k-j)\{\rho_k[D_6 \mu]\langle u_j, u_{i-k-j}\rangle \\ &+ \rho_k[D_7 \mu]\langle \bigtriangledown u_j, \bigtriangledown u_{i-k-j}\rangle + \rho_k[D_8 \mu]\langle u'_j,\ u'_{i-k-j}\rangle\},\ 2 \le i \le N. \quad (28)\end{aligned}$$

We then have the following results, the proofs of which can be found in the paper [38].

Theorem 2.8. *Let* (A_1), (A_2), (A_7) *and* (A_8) *hold. Then there exist constants* $M > 0$ *and* $T > 0$ *such that, for every* ε, *with* $|\varepsilon| \le 1$, *the problem* (P_ε) *has a unique weak solution* $u_\varepsilon \in W(M,T)$ *satisfying an asymptotic estimation up to order* $N+1$ *as follows*

$$\left\| u'_\varepsilon - \sum_{i=0}^{N} \varepsilon^i u'_i \right\|_{L^\infty(0,T;L^2)} + \left\| u_\varepsilon - \sum_{i=0}^{N} \varepsilon^i u_i \right\|_{L^\infty(0,T;H^1)} \le C_T \left|\varepsilon\right|^{N+1}, \quad (29)$$

where $u_0,\ u_1, \ldots,\ u_N$ *are the weak solutions of the problems* $(P_0),\ (Q_1), \ldots,\ (Q_N)$, *respectively.*

Remark 2.9. In the proof of Theorem 2.8, there is a great need of the following lemma on MacLaurin's expansion.

Lemma 2.10. *Let* $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3_+)$. *Put*

$$f[h] = f(x, t, h, h_x, h_t, \|h\|^2, \|h_x\|^2, \|h_t\|^2),$$

where $h = h(x,t) = \sum_{i=0}^{N} u_i(x,t)\varepsilon^i$. *We have*

$$f[h] = \sum_{i=0}^{N} \pi_i[f]\varepsilon^i + o(\varepsilon^N), \quad (30)$$

where the coefficients $\pi_i[f] = \frac{1}{i!}\frac{\partial^i}{\partial \varepsilon^i} f[h]\Big|_{\varepsilon=0}$ *of* ε^i *are defined by the recurrent formulas* (22), (23) *and* (27).

The proof of Lemma 2.10 can be found in the paper [38].

Remark 2.11. In case $f \in C^2([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$, $f_1 \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3)$ and $N = 1$, we have also obtained some similar results in the papers [28, 32, 33] for various μ_ε.

2.3. The Nonlinear Kirchhoff-Carrier Wave Equation Associated with Mixed Nonhomogeneous Conditions and a High Order Iterative Method

The above results (Theorems 2.4- 2.8) still hold for the following nonhomogeneous boundary value problem

$$\begin{cases} u_{tt} - \mu\left(t, \|u_x\|^2\right) u_{xx} = f(x,t,u,u_x,u_t,\|u_x\|^2),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = g_0(t),\ u(1,t) = g_1(t), \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x), \end{cases} \tag{31}$$

where $h_0 \geq 0$ is a given constant and μ, f, $\widetilde{u}_0$, $\widetilde{u}_1$ are given functions satisfying conditions specified later. In order to solve the problem (31), for all $x \in [0,1]$, $z \geq 0$ and $t \geq 0$, we put

$$\begin{cases} v(x,t) = u(x,t) - \varphi(x,t), \\ \varphi(x,t) = \frac{1}{1+h_0}\left[(x-1)g_0(t) + (h_0 x + 1)g_1(t)\right], \\ \widetilde{f}(x,t,v,v_x,v_t,z) = f(x,t,v+\varphi,v_x+\varphi_x,v_t+\varphi_t,z) - \varphi_{tt}, \\ \widetilde{v}_0(x) = \widetilde{u}_0(x) - \varphi(x,0),\ \widetilde{v}_1(x) = \widetilde{u}_1(x) - \varphi_t(x,0), \end{cases} \tag{32}$$

in which

$$g_0(0) = u_x(0,0) - h_0 u(0,0) = \widetilde{u}_0'(0) - h_0\widetilde{u}_0(0), \quad g_1(0) = u(1,0) = \widetilde{u}_0(1). \tag{33}$$

Then problem (31) reduces to the following problem with the homogeneous boundary conditions:

$$\begin{cases} v_{tt} - \mu(t, \|v_x(t) + \varphi_x(t)\|^2) v_{xx} = \widetilde{f}(x,t,v,v_x,v_t,\|v_x(t)+\varphi_x(t)\|^2), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < x < 1,\ 0 < t < T, \\ v_x(0,t) - h_0 v(0,t) = v(1,t) = 0, \\ v(x,0) = \widetilde{v}_0(x),\ v_t(x,0) = \widetilde{v}_1(x). \end{cases} \tag{34}$$

We shall use the function space $V = \{v \in H^1 : v(1) = 0\}$ instead of H^1 and the symmetric bilinear form $a(\cdot,\cdot)$ on $V \times V$ is as follows

$$a(v,w) = \int_0^1 v_x(x) w_x(x) dx + h_0 v(0) w(0),\ v, w \in V. \tag{35}$$

We make the following assumptions:

(G) $g_0,\, g_1 \in C^3(\mathbb{R}_+)$;
(A_1') $h_0 \geq 0$;
(A_2') $(\widetilde{v}_0,\, \widetilde{v}_1) \in (V \cap H^2) \times V$;
(A_3') $\mu \in C^1(\mathbb{R}_+^2),\; \mu(t,z) \geq \mu_0 > 0$;
(A_4') $f \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+)$ satisfies the conditions

$$f(1,t,u,v,w,z) = 0 \;\forall t,\, z \geq 0 \text{ and } (u,v,w) \in \mathbb{R}_+^2 \times \mathbb{R}^3.$$

For each $M > 0$ and $T > 0$, we put

$$W(M,T) = \{v \in L^\infty(0,T;V \cap H^2) : v_t \in L^\infty(0,T;V),\, v_{tt} \in L^\infty(0,T;L^2),\\ \|v\|_{L^\infty(0,T;V\cap H^2)} \leq M,\, \|v_t\|_{L^\infty(0,T;V)} \leq M,\, \|v_{tt}\|_{L^\infty(0,T;L^2)} \leq M\}. \tag{36}$$

We establish the linear recurrent sequence $\{v_m\}$ as follows.

We choose the first term $v_0 = \widetilde{v}_0$, suppose that

$$v_{m-1} \in W_1(M,T), \tag{37}$$

and associate with the problem (34) the following problem.

Find $v_m \in W_1(M,T)$ which satisfies the linear variational problem

$$\begin{cases} \langle v_m''(t), w\rangle + \mu_m(t) a(v_m(t), w) = \langle F_m(t), w\rangle, \text{ for all } w \in V, \\ v_m(0) = \widetilde{v}_0,\; v_m'(0) = \widetilde{v}_1, \end{cases} \tag{38}$$

where

$$\begin{cases} \mu_m(t) = \mu(t, \|\bigtriangledown v_{m-1}(t) + \nabla\varphi(t)\|^2), \\ F_m(x,t) = \widetilde{f}(x,t,v_{m-1}(t), \bigtriangledown v_{m-1}(t), v_{m-1}'(t), \|\bigtriangledown v_{m-1}(t) + \nabla\varphi(t)\|^2). \end{cases} \tag{39}$$

We then have the following results, the proofs of which can be found in the paper [37].

Theorem 2.12. *Let* (G), $(A_1') - (A_4')$ *hold. Then there exist positive constants* M, T *and the linear recurrent sequence* $\{v_m\} \subset W(M,T)$ *defined by* (37)-(39).

Theorem 2.13. *Let* (G), $(A_1') - (A_4')$ *hold. Then there exist positive constants* M, T, *such that the problem* (34) *has a unique weak solution* $v \in W(M,T)$.

The linear recurrent sequence $\{v_m\}$ *defined by* (37)-(39) *converges to the solution* v *strongly in the space* $W_1(T) = \{v \in L^\infty(0,T;V) : v' \in L^\infty(0,T;L^2)\}$.

Furthermore, we have also the estimation

$$\|v_m - v\|_{L^\infty(0,T;V)} + \|v_m' - v'\|_{L^\infty(0,T;L^2)} \leq C_T k_T^m, \textit{ for all } m, \tag{40}$$

where $k_T < 1$ *and* C *are positive constants depending only on* T, v_0, v_1.

We note that, the convergence of $\{v_m\}$ in Theorem 2.13 is only at the rate of order 1, i.e., in the estimation (40), the error obtained at the m^{th}-step is $(k_T)^m$, with $0 < k_T < 1$.

In order to get the convergence of $\{v_m\}$ at the rate of order N ($N \geq 2$), i.e., the above error in (40) has to be $(k_T)^{N^m}$, we need to consider special forms of the problem and then establish a special approximation as follows.

We consider the nonlinear wave equation with the Kirchhoff–Carrier operator

$$\begin{cases} u_{tt} - \mu\left(t, \|u\|^2, \|u_x\|^2\right) u_{xx} = f(u),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = g_0(t),\ u(1,t) = 0, \\ u(x,0) = \widetilde{u}_0(x),\ \ u_t(x,0) = \widetilde{u}_1(x), \end{cases} \tag{41}$$

where μ, f, $\widetilde{u}_0$, $\widetilde{u}_1$ are given functions and $h_0 \geq 0$ is a given constant. Put $\varphi(x,t) = \frac{x-1}{1+h_0} g_0(t)$. By the transformation $v(x,t) = u(x,t) - \varphi(x,t)$, we shall reformulate (41) as the problem with homogeneous boundary conditions as follows

$$\begin{cases} v_{tt} - \mu(t, \|v(t) + \varphi(t)\|^2, \|v_x(t) + \varphi_x(t)\|^2) v_{xx} = f(\varphi + v) - \varphi_{tt}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < x < 1,\ \ 0 < t < T, \\ v_x(0,t) - h_0 v(0,t) = v(1,t) = 0, \\ v(x,0) = \widetilde{v}_0(x),\ \ v_t(x,0) = \widetilde{v}_1(x), \end{cases} \tag{42}$$

where

$$\widetilde{v}_0(x) = \widetilde{u}_0(x) - \varphi(x,0), \widetilde{v}_1(x) = \widetilde{u}_1(x) - \varphi_t(x,0), \tag{43}$$

and h_0, g_0, $\widetilde{u}_0$ satisfying the consistency condition $g_0(0) = \widetilde{u}_{0x}(0) - h_0 \widetilde{u}_0(0)$.

We strengthen the following assumptions

(A_3'') $\mu \in C^1(\mathbb{R}_+^3)$, and there exist constants $p > 1$, $\mu_0 > 0$, $\widetilde{\mu}_i > 0$, $i = 0, 1, 2, 3$ such that

(i) $\mu_0 \leq \mu(t,y,z) \leq \mu_0(1 + y^p + z^p)$, $\forall (t,y,z) \in \mathbb{R}_+^3$,

(ii) $|D_1 \mu(t,y,z)| \leq \widetilde{\mu}_1(1 + y^p + z^p)$, $\forall (t,y,z) \in \mathbb{R}_+^3$,

(iii) $|D_2 \mu(t,y,z)| \leq \widetilde{\mu}_2(1 + y^{p-1} + z^p)$, $\forall (t,y,z) \in \mathbb{R}_+^3$,

(iv) $|D_3 \mu(t,y,z)| \leq \widetilde{\mu}_3(1 + y^p + z^{p-1})$, $\forall (t,y,z) \in \mathbb{R}_+^3$,

(A_4'') $f \in C^N(\mathbb{R})$. (A_4'') $f \in C^N(\mathbb{R})$.

In this paragraph, we still use the function space $V = \{v \in H^1 : v(1) = 0\}$, the set $W(M,T)$ and the symmetric bilinear form $a(\cdot,\cdot)$ on $V \times V$ are defined by (35), (36), respectively.

We choose the first term $v_0 \equiv 0$. Suppose that

$$v_{m-1} \in W(M,T). \tag{44}$$

We associate with the problem (42) the following problem.

Find $v_m \in W(M,T)$ satisfying the nonlinear variational problem

$$\begin{cases} \langle v_m''(t), w \rangle + \mu_m(t) a(v_m(t), w) = \langle F_m(t), w \rangle \text{ for all } w \in V, \\ v_m(0) = \widetilde{v}_0,\ \ v_m'(0) = \widetilde{v}_1, \end{cases} \tag{45}$$

where

$$\begin{cases} \mu_m(t) = \mu(t, \|v_m(t) + \varphi(t)\|^2, \|\triangledown v_m(t) + \nabla\varphi(t)\|^2), \\ F_m(x,t) = -\varphi_{tt} + \sum_{i=0}^{N-1} \frac{1}{i!} f^{(i)}(v_{m-1} + \varphi)(v_m - v_{m-1})^i . \end{cases} \tag{46}$$

We then have the following result, the proof of which can be found in the paper [79].

Theorem 2.14. *Let assumptions* $g_0 \in C^3(\mathbb{R}_+)$, (A_1'), (A_2'), (A_3''), (A_4'') *hold. Then there exist a constant* $M > 0$ *depending on* $\widetilde{u}_0$, $\widetilde{u}_1$, μ, g_0, h_0 *and a constant* $T > 0$ *depending on* $\widetilde{u}_0$, $\widetilde{u}_1$, μ, g_0, h_0, f, *such that*
(i) *The problem* (42) *has a unique weak solution* $v \in W(M,T)$.
(ii) *The recurrent sequence* $\{v_m\}$ *defined by* (45), (46) *converges at the rate of order* N *to the solution* v *strongly in the space* $W_1(T) = \{v \in L^\infty(0,T;V) : v' \in L^\infty(0,T;L^2)\}$ *in the sense*

$$\begin{aligned} &\|v_m - v\|_{L^\infty(0,T;V)} + \|v_m' - v'\|_{L^\infty(0,T;L^2)} \\ &\le C\left(\|v_{m-1} - v\|_{L^\infty(0,T;V)} + \left\|v_{m-1}' - v'\right\|_{L^\infty(0,T;L^2)}\right)^N , \end{aligned} \tag{47}$$

for all $m \ge 1$, *where* C *is a suitable constant.*

Furthermore, we have also the estimation

$$\|v_m - v\|_{L^\infty(0,T;V)} + \|v_m' - v'\|_{L^\infty(0,T;L^2)} \le C_T\,(k_T)^{N^m} , \tag{48}$$

for all $m \ge 1$, *where* C_T *and* $k_T < 1$ *are positive constants depending only on* T.

Remark 2.15. Theorem 2.14 still holds for the boundary value problems in [63, 64, 81].

2.4. The Nonlinear Kirchhoff-Carrier Wave Equation in the Unit Membrane
Here, we will consider the following initial and boundary value problem

$$\begin{cases} u_{tt} - \mu\left(t, \|u\|_0^2, \|u_r\|_0^2, \|u'\|_0^2\right)(u_{rr} + \frac{1}{r}u_r) = f(r,t,u,u_r), \\ \qquad\qquad\qquad\qquad 0 < r < 1,\ 0 < t < T, \\ |\lim\limits_{r\to 0_+} \sqrt{r}u_r(r,t)| < \infty,\ u_r(1,t) + hu(1,t) = 0, \\ u(r,0) = \widetilde{u}_0(r),\ u_t(r,0) = \widetilde{u}_1(r), \end{cases} \tag{49}$$

where μ, f, $\widetilde{u}_0$, $\widetilde{u}_1$ are given functions, $h > 0$ is a given constant,

$$\|u\|_0^2 = \int_0^1 r\,|u(r,t)|^2\,dr,\ \ \|u_r\|_0^2 = \int_0^1 r\,|u_r(r,t)|^2\,dr,\ \ \|u'\|_0^2 = \int_0^1 r\,|u_t(r,t)|^2\,dr.$$

Equation $(49)_1$ herein is the bi-dimensional nonlinear wave equation describing nonlinear vibrations of the unit membrane $\Omega_1 = \{(x,y) : x^2 + y^2 < 1\}$. In the vibration process, the area of the unit membrane and the tension at various points change in time. The condition on the boundary $\partial\Omega_1$ describes elastic constraints, where the constant h has a mechanical signification. Boundary condition $|\lim_{r\to 0_+}\sqrt{r}u_r(r,t)|<\infty$ is satisfied automatically if u is a classical solution of the problem (49), for example, with $u\in C^1([0,1]\times(0,T))\cap C^2((0,1)\times(0,T))$. This condition is also used in connection with Sobolev spaces with weight r (see [4, 29, 44, 48]).

In the first part, we study the problem (49) with right-hand side $f(r,t,u,u_r)$, where $f\in C^0([0,1]\times\mathbb{R}_+\times\mathbb{R}^2)$ satisfies the condition $\partial f/\partial r$, $\partial f/\partial u$, $\partial f/\partial u_r\in C^0([0,1]\times\mathbb{R}_+\times\mathbb{R}^2)$, and $\mu\in C^1(\mathbb{R}_+^4)$, $\mu\geq\mu_0>0$, with μ_0 a given constant. First, we shall associate with Equation $(49)_1$ a linear recurrent sequence which is bounded in a suitable function space. The existence of a local solution is proved by a standard compactness argument.

In the second part, we consider problem (49) corresponding to $f=f(r,u)$, $f\in C^2([0,1]\times\mathbb{R})$, and $\mu=\mu(\|u_r\|_0^2)$, $\mu\in C^1(\mathbb{R}_+)$, $\mu_0\leq\mu(z)\leq\overline{d}_0(1+z^p)$, $|\mu'(z)|\leq\overline{d}_1(1+z^{p-1})$, with given constants $\mu_0>0$, $p>1$ and $\overline{d}_0$, $\overline{d}_1\geq 0$. We associate with Equation $(49)_1$ a recurrent sequence $\{u_m\}$ defined by

$$\frac{\partial^2 u_m}{\partial t^2}-\mu\left(\|u_{mr}\|_0^2\right)\left(\frac{\partial^2 u_m}{\partial r^2}+\frac{1}{r}\frac{\partial u_m}{\partial r}\right)=f(r,u_{m-1})+(u_m-u_{m-1})\frac{\partial f}{\partial u}(r,u_{m-1}), \tag{50}$$

$0<r<1$, $0<t<T$, with u_m satisfying $(49)_{2-4}$. The first term u_0 is chosen as $u_0\equiv 0$. If $f\in C^2([0,1]\times\mathbb{R})$, we prove that the sequence $\{u_m\}$ converges quadratically to the solution u in the sense which will be given in Theorem 2.18.

The results obtained here are in part generalizations of those in [4, 44, 59].

Finally, if $\mu\in C^{N+1}(\mathbb{R}_+)$, $\mu_1\in C^N(\mathbb{R}_+)$, $\mu\geq\mu_0>0$, $\mu_1\geq 0$, $f\in C^{N+1}([0,1]\times\mathbb{R})$ and $f_1\in C^N([0,1]\times\mathbb{R})$, then an asymptotic expansion of order $N+1$ in ε is obtained with a right-hand side of the form $f(r,u)+\varepsilon f_1(r,u)$ and μ stands for $\mu+\varepsilon\mu_1$, for ε sufficiently small. This result is a relative generalization of [28, 32, 69].

Put $\Omega=(0,1)$. For any function $v\in C^0(\overline{\Omega})$ we define $\|v\|_0$ as

$$\|v\|_0=\left(\int_0^1 rv^2(r)dr\right)^{1/2}$$

and define the space V_0 as the completion of the space $C^0(\overline{\Omega})$ with respect to the norm $\|\cdot\|_0$. Similarly, for any function $v\in C^1(\overline{\Omega})$, we define $\|v\|_1$ as

$$\|v\|_1=\left(\|v\|_0^2+\|v_r\|_0^2\right)^{1/2}$$

and define the space V_1 as the completion of the space $C^1(\overline{\Omega})$ with respect to the norm $\|\cdot\|_1$. Note that the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ can be defined, respectively, from the inner products $\langle u,v\rangle=\int_0^1 ru(r)v(r)dr$, $\langle u,v\rangle+\langle u_r,v_r\rangle$. Identifying V_0 with its dual V_0', we obtain the dense and continuous embeddings $V_1\hookrightarrow V_0\equiv$

$V_0' \hookrightarrow V_1'$. The inner product notation will be used to denote the duality pairing between V_1 and V_1'.

Now, let the bilinear form $a(\cdot,\cdot)$ be defined by

$$a(u,v) = hu(1)v(1) + \int_0^1 r u_r(r) v_r(r) dr, \ u, v \in V_1, \tag{51}$$

where h is a positive constant. Then for some uniquely defined bounded linear operator $A : V_1 \to V_1'$ we have $a(u,v) = \langle Au, v\rangle$ for all u, $v \in V_1$.

For any function $v \in C^2(\overline{\Omega})$ we define $\|v\|_2$ as

$$\|v\|_2 = \left(\|v\|_0^2 + \|v_r\|_0^2 + \|Av\|_0^2\right)^{1/2}$$

and define the space V_2 as the completion of $C^2(\overline{\Omega})$ with respect to the norm $\|\cdot\|_2$.

Note that V_2 is also a Hilbert space with respect to the scalar product $\langle u,v\rangle + \langle u_r, v_r\rangle + \langle Au, Av\rangle$ and that V_2 can be defined also as $V_2 = \{v \in V_1 : Av \in V_0\}$.

2.4.1. The general case: $f = f(r,t,u,u_r)$

In this part, we consider the initial and boundary value problem (49) with the general right-hand side $f = f(r,t,u,u_r)$ under the following assumptions:

(B_1) $(\widetilde{u}_0, \widetilde{u}_1) \in V_2 \times V_1$,
(B_2) $\mu \in C^1(\mathbb{R}_+^4)$ with $\mu(t,\xi,\eta,\lambda) \geq \mu_0 > 0$ for all $t,\ \xi,\ \eta,\ \lambda \geq 0$,
(B_3) $f \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^2)$ and $\partial f/\partial r,\ \partial f/\partial u,\ \partial f/\partial u_r \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^2)$. (52)

For each $M > 0$ and $T > 0$ we set

$$\begin{aligned} W(M,T) = \{&v \in L^\infty(0,T;V_2) : v' \in L^\infty(0,T;V_1) \text{ and } v'' \in L^2(0,T;V_0), \\ &\text{with } \|v\|_{L^\infty(0,T;V_2)},\ \|v'\|_{L^\infty(0,T;V_1)},\ \|v''\|_{L^2(0,T;V_0)} \leq M\}, \\ W_1(M,T) = \{&v \in W(M,T) : v'' \in L^\infty(0,T;V_0)\}. \end{aligned}$$

We choose the first term $u_0 \equiv 0$, suppose that $u_{m-1} \in W_1(M,T)$, and associate with the problem (49) the following variational problem: Find $u_m \in W_1(M,T)$ so that

$$\begin{cases} \langle u_m''(t), v\rangle + \mu_m(t) a(u_m(t), v) = \langle F_m(t), v\rangle \ \forall v \in V_1, \\ u_m(0) = \widetilde{u}_0,\ u_m'(0) = \widetilde{u}_1, \end{cases} \tag{53}$$

where

$$\begin{aligned} \mu_m(t) &= \mu\left(t, \|u_{m-1}(t)\|_0^2, \|\nabla u_{m-1}(t)\|_0^2, \left\|u_{m-1}'(t)\right\|_0^2\right), \\ F_m(r,t) &= f(r,t,u_{m-1}(t), \nabla u_{m-1}(t)). \end{aligned} \tag{54}$$

We then have the following result, the proof of which can be found in the paper [48].

Theorem 2.16. *Let assumptions* $(B_1)-(B_3)$ *hold. Then there exist a constant* $M>0$ *depending on* $\widetilde{u}_0$, $\widetilde{u}_1$, μ, h, *and a constant* $T>0$ *depending on* $\widetilde{u}_0$, $\widetilde{u}_1$, μ, h, f *such that*
(i) *The problem* (49) *has a unique weak solution* $u\in W_1(M,T)$.
(ii) *The linear recurrent sequence* $\{u_m\}$ *defined by* (53), (54) *converges to the solution* u *of the problem* (49) *strongly in the space*

$$W_1(T)=\{v\in L^\infty(0,T;V_1): v'\in L^\infty(0,T;V_0)\}.$$

Furthermore, we have also the estimation

$$\|u_m-u\|_{L^\infty(0,T;V_1)}+\|u'_m-u'\|_{L^\infty(0,T;V_0)}\le C_T k_T^m,$$

for all $m\ge 1$, *where* C_T *and* $k_T<1$ *are positive constants depending only on* T.

Remark 2.17. In the case $\mu\equiv 1$, some results have been obtained in [28]. In the case Equation $(49)_1$ not involving the term $\frac{1}{r}u_r$, f being in $C^1(\overline{\Omega}\times\mathbb{R}_+\times\mathbb{R}^2)$ and $\mu\equiv 1$, we have obtained some results in [34]. Let us emphasize here that we do not need the condition $f\in C^1(\overline{\Omega}\times\mathbb{R}_+\times\mathbb{R}^2)$ as was stated above.

2.4.2. The special case: $f=f(r,u)$, $\mu=\mu(z)$

In this part, we consider the initial boundary value problem (49) with the autonomous right-hand side independent of u_r and $\mu=\mu(z)$ satisfies the following conditions

(B'_2) $\mu\in C^1(\mathbb{R}_+)$ and there exist constants $\mu_0>0,\ p>1,\ \overline{d}_0,\ \overline{d}_1\ge 0$ such that
(i) $\mu_0\le\mu(z)\le\overline{d}_0(1+z^p)$, for all $z\ge 0$,
(ii) $|\mu'(z)|\le\overline{d}_1(1+z^{p-1})$, for all $z\ge 0$,
(B'_3) $f\in C^2(\overline{\Omega}\times\mathbb{R})$.

We choose the first term $u_0\equiv 0$, suppose that $u_{m-1}\in W_1(M,T)$, and associate with the problem (49) the following variational problem: Find $u_m\in W_1(M,T)$ so that

$$\begin{cases}\langle u''_m(t),v\rangle+\mu_m(t)a(u_m(t),v)=\langle F_m(t),v\rangle\ \forall v\in V_1,\\ u_m(0)=\widetilde{u}_0,\ u'_m(0)=\widetilde{u}_1,\end{cases}\tag{55}$$

where

$$\mu_m(t)=B\left(\|\nabla u_m(t)\|_0^2\right),\ F_m(r,t)=f(r,u_{m-1})+(u_m-u_{m-1})\frac{\partial f}{\partial u}(r,u_{m-1}).\tag{56}$$

We then have the following result, the proof of which can be found in the paper [48].

Theorem 2.18. *Let assumptions* (B_1), (B_2') *and* (B_3') *hold. Then:*
(i) *There exist constants* $M > 0$ *and* $T > 0$ *such that the problem* (49) *corresponding to* $f = f(r,u)$ *and* $\mu = \mu(z)$ *has a unique weak solution* $u \in W_1(M,T)$.
(ii) *The recurrent sequence* $\{u_m\}$ *defined by* (55), (56) *converges quadratically to the solution* u *strongly in the space* $W_1(T)$ *in the sense*

$$\begin{aligned} &\|u_m - u\|_{L^\infty(0,T;V_1)} + \|u_m' - u'\|_{L^\infty(0,T;V_0)} \\ &\le C\left(\|u_{m-1} - u\|_{L^\infty(0,T;V_1)} + \left\|u_{m-1}' - u'\right\|_{L^\infty(0,T;V_0)}\right)^2, \end{aligned} \tag{57}$$

where C *is a suitable constant. Furthermore, we have also the estimation*

$$\|u_m - u\|_{L^\infty(0,T;V_1)} + \|u_m' - u'\|_{L^\infty(0,T;V_0)} \le C_T\,(k_T)^{2^m} \tag{58}$$

for all $m \ge 1$, *where* C_T *and* $k_T < 1$ *are positive constants depending only on* T.

Remark 2.19. In the case $\mu \equiv 1$, the result in [4] is a corollary of Theorem 2.18.

2.4.3. Asymptotic Expansion of the Solutions
In this part, we assume $(\widetilde{u}_0, \widetilde{u}_1)$ satisfies (B_1). We also assume

(B_4) $\mu \in C^{N+1}(\mathbb{R}_+),\ \mu_1 \in C^N(\mathbb{R}_+)$ with $\mu(z) \ge \mu_0 > 0,\ \mu_1(z) \ge 0\ \forall z \ge 0$,
(B_5) $f \in C^{N+1}(\overline{\Omega} \times \mathbb{R}),\ f_1 \in C^N(\overline{\Omega} \times \mathbb{R})$.

We consider the following perturbed problem, where ε is a small parameter, $|\varepsilon| \le 1$:

$$(P_\varepsilon)\begin{cases} u_{tt} - \mu_\varepsilon\left(\|u_r\|_0^2\right)\left(u_{rr} + \frac{1}{r}u_r\right) = F_\varepsilon(r,u),\ 0 < r < 1,\ 0 < t < T, \\ |\lim\limits_{r\to 0_+} \sqrt{r}u_r(r,t)| < \infty,\ u_r(1,t) + hu(1,t) = 0, \\ u(r,0) = \widetilde{u}_0(r),\ u_t(r,0) = \widetilde{u}_1(r), \\ F_\varepsilon(r,u) = f(r,u) + \varepsilon f_1(r,u),\ \mu_\varepsilon\left(\|u_r\|_0^2\right) = \mu\left(\|u_r\|_0^2\right) + \varepsilon\mu_1\left(\|u_r\|_0^2\right). \end{cases}$$

We shall study the asymptotic expansion of the weak solution u_ε of the problem (P_ε) with respect to ε. We use the following notations

$$\begin{aligned} f[u] &= f(r,u), & \mu[u] &= \mu\left(\|u_r(t)\|_0^2\right), \\ D_1 f &= \partial f/\partial r, & D_2 f &= \partial f/\partial u, \\ \mu' &= \tfrac{d\mu}{dz}, & \mu^{(i)} &= \tfrac{d^i\mu}{dz^i}. \end{aligned}$$

Let $u_0 \in W_1(M,T)$ be the weak solution of the problem (P_0) corresponding to $\varepsilon = 0$.

For $M > 0$ and $T > 0$ being suitable constants, let us consider the weak solutions $u_1,\ u_2, \ldots,\ u_N \in W_1(M,T)$ of the following corresponding problems:

a)

$$(Q_1)\begin{cases} u_1'' + \mu[u_0]Au_1 = \widetilde{F}_1[u_1],\ 0 < r < 1,\ 0 < t < T, \\ |\lim\limits_{r\to 0_+} \sqrt{r}u_{1r}(r,t)| < \infty,\ u_{1r}(1,t) + hu_1(1,t) = 0, \\ u_1(r,0) = u_1'(r,0) = 0, \end{cases}$$

where

$$\widetilde{F}_1[u_1] = \pi_1[f] + \pi_0[f_1] - (\rho_1[\mu] + \rho_0[\mu_1])\, Au_0, \tag{59}$$

with $\pi_0[f]$, $\pi_1[f]$, $\rho_0[\mu]$, $\rho_1[\mu]$ defined as follows

$$\begin{cases} \pi_0[f] = f[u_0] \equiv f(r,u_0),\ \pi_1[f] = \pi_0[D_2 f]u_1, \\ \rho_0[\mu] = \mu[u_0] \equiv \mu\left(\|u_{0r}(t)\|_0^2\right),\ \rho_1[\mu] = 2\rho_0[\mu']\langle u_{0r}, u_{1r}\rangle, \end{cases} \tag{60}$$

b) with $2 \le i \le N$,

$$(Q_i)\begin{cases} u_i'' + \mu[u_0]Au_i = \widetilde{F}_i[u_i],\ 0 < r < 1,\ 0 < t < T, \\ |\lim\limits_{r\to 0_+} \sqrt{r}u_{ir}(r,t)| < \infty,\ u_{ir}(1,t) + hu_i(1,t) = 0, \\ u_i(r,0) = u_i'(r,0) = 0,\ i = 1,2,\ldots,N, \end{cases}$$

where

$$\widetilde{F}_i[u_i] = \pi_i[f] + \pi_{i-1}[f_1] - \sum_{k=1}^{i} (\rho_k[\mu] + \rho_{k-1}[\mu_1])\, Au_{i-k}, \tag{61}$$

with $\pi_i[f] = \pi_i[f,u_0,u_1,\ldots,u_i]$, $\rho_i[\mu] = \rho_i[\mu,u_0,u_1,\ldots,u_i]$, $2 \le i \le N$ defined by the recurrent formulas

$$\pi_i[f] = \sum_{k=0}^{i-1} \frac{i-k}{i}\pi_k[D_2 f]u_{i-k},\ 2 \le i \le N, \tag{62}$$

$$\rho_i[\mu] = \frac{2}{i}\sum_{k=0}^{i-1}\sum_{j=0}^{i-k-1} (i-k-j)\rho_k[\mu']\left\langle u_{jr},\ (u_{i-k-j})_r\right\rangle,\ 2 \le i \le N. \tag{63}$$

We then have the following result, the proof of which can be found in the paper [48].

Theorem 2.20. *Let (B_1), (B_4) and (B_5) hold. Then there exist constants $M > 0$ and $T > 0$ such that, for every ε, with $|\varepsilon| \le 1$, the problem (P_ε) has a unique weak solution $u_\varepsilon \in W_1(M,T)$ satisfying an asymptotic estimation up to order $N+1$ as follows*

$$\left\|u_\varepsilon' - \sum_{i=0}^{N}\varepsilon^i u_i'\right\|_{L^\infty(0,T;V_0)} + \left\|u_\varepsilon - \sum_{i=0}^{N}\varepsilon^i u_i\right\|_{L^\infty(0,T;V_1)} \le C_T\,|\varepsilon|^{N+1}, \tag{64}$$

where u_0, u_1, ..., u_N are the weak solutions of problems (P_0), (Q_1), ..., (Q_N), respectively.

Remark 2.21. In the case of equation $(49)_1$ not involving the term $\frac{1}{r}u_r$, we have also obtained some results in the papers [28, 31, 32, 34].

3. A Nonlinear Wave Equation Associated with Nonhomogeneous Conditions

In this section, we consider the following problem

$$\begin{cases} u_{tt} - \frac{\partial}{\partial x}\left(\mu(x,t)u_x\right) + f(u,u_t) = F(x,t),\ 0 < x < 1,\ 0 < t < T, \\ \mu(0,t)u_x(0,t) = P(t), \\ -\mu(1,t)u_x(1,t) = K_1 \left|u(1,t)\right|^{p_1-2} u(1,t) + \lambda_1 \left|u_t(1,t)\right|^{q_1-2} u_t(1,t), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \end{cases} \tag{65}$$

and

$$P(t) = g(t) + K_0 \left|u(0,t)\right|^{p_0-2} u(0,t) + \lambda_0 \left|u_t(0,t)\right|^{q_0-2} u_t(0,t) - \int_0^t k(t-s)u(0,s)ds, \tag{66}$$

where p_0, $q_0 \geq 2$, p_1, $q_1 \geq 2$, $K_0 \geq 0$, $K_1 \geq 0$, $\lambda_0 > 0$, $\lambda_1 > 0$ are given constants and μ, u_0, u_1, f, F, g, k are given functions satisfying conditions specified later. In Equation $(65)_1$ the nonlinear term $f(u,u_t)$ is supposed to be continuous with respect to two variables (u,u_t) and nondecreasing with respect to the second variable.

This section consists of three main parts. In Part 1, under some conditions, first we consider the existence and next, we consider the uniqueness of solution. The proof is based on the Faedo-Galerkin method and the weak compact method associated with a monotone operator. For the case of $q_0 = q_1 = 2$; p_0, $p_1 \geq 2$, in Part 2, we prove that the unique solution u belongs to $(L^\infty(0,T;H^2) \cap C^0(0,T;H^1) \cap C^1(0,T;L^2)) \times H^1(0,T)$, with $u_t \in L^\infty(0,T;H^1)$, $u_{tt} \in L^\infty(0,T;L^2)$, $u(0,\cdot)$, $u(1,\cdot) \in H^2(0,T)$, if we assume $(u_0,\ u_1) \in H^2 \times H^1$, $f \in C^1\left(\mathbb{R}^2\right)$ and some other conditions. Finally, in Part 3, with $q_0 = q_1 = 2$; p_0, $p_1 \geq N+1$, $f \in C^{N+1}\left(\mathbb{R}^2\right)$, $N \geq 2$, we obtain an asymptotic expansion of the solution of the problem (65), (66) up to order $N+1$ in two small parameters K_0, K_1. The results obtained here may be considered as the generalizations of those in [1-3, 12, 13, 22, 25, 27, 39, 42, 43, 45-47, 50, 51, 61, 62, 73, 78].

3.1. The Existence and Uniqueness Theorem

Without loss of generality we can suppose that $\lambda_0 = \lambda_1 = 1$. We make the following assumptions:
(H_1) $(u_0,\ u_1) \in H^1 \times L^2$,
(H_2) $F \in L^1(0,T;L^2)$,
(H_3) $g \in L^{q_0'}(0,T)$, $q_0' = q_0(q_0-1)^{-1}$,
(H_4) $k \in L^1(0,T)$,

(H_5) $\mu \in C^0\left(\overline{Q_T}\right)$, $\mu(x,t) \geq \mu_0 > 0$, $\mu_t \in L^1(0,T;L^\infty)$, $\mu_t(x,t) \leq 0$, a.e. $(x,t) \in Q_T$,
(H_6) $p_0,\ p_1,\ q_0,\ q_1 \geq 2$,
(H_7) $K_0,\ K_1 \geq 0$,
(F) the function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies the following conditions:
(F_1) $(f(u,v) - f(u,\widetilde{v}))(v - \widetilde{v}) \geq 0$ for all $u,\ v, \widetilde{v} \in \mathbb{R}$,
(F_2) there exist constants $C_1,\ C_1',\ C_2' \geq 0$, $C_2 > 0$ and $q > p > 1$, such that

(i) $\int_0^u f(s,0)ds \geq -C_1|u|^p - C_1'$ for all $u \in \mathbb{R}$,
(ii) $(f(u,v) - f(u,0))\,v \geq C_2\,|v|^q - C_2'$ for all $u, v \in \mathbb{R}$,

(F_3) there exist a constant $C_3 > 0$ and a function $f_1 \in C^0(\mathbb{R}_+)$, such that $|f(u,v)| \leq f_1(|u|) + C_3\,|v|^{q-1}$ for all $u, v \in \mathbb{R}$,
(F_4) for each $M > 0$, there exists a function $K_M : \mathbb{R}_+ \to \mathbb{R}_+$ continuous and satisfying

(i) $|f(u,v) - f(\widetilde{u},v)| \leq K_M(|v|)\,|u - \widetilde{u}|$ for all $u,\ \widetilde{u} \in [-M,M]$ and $v \in \mathbb{R}$,
(ii) $K_M(|v|) \in L^1(0,T;L^2)$ for all $v \in L^\infty(0,T;L^2)$.

Remark 3.1. (i) In assumption (F_2, i), we require that $q > p$; if $p = q$ then the positive constant C_1 needs to be sufficiently small.

(ii) As an example of the function f, we can take $f(u,u_t) = |u|^{\alpha-2}u + |u_t|^{q-2}u_t + |u|^\gamma\,|u_t|^{\beta-2}u_t$, where $\alpha,\ \beta,\ \gamma,\ q$ are constants, with $\alpha > 1$, $\gamma > 0$, $1 < \beta < q$. Then, f satisfies assumptions $(F_1) - (F_3)$. Furthermore, if $\alpha \geq 2$, $\gamma \geq 1$, $1 < \beta < \min\{2, q\}$, then f satisfies assumptions $(F_1) - (F_4)$.

We have the following result, the proof of which can be found in the paper [50].

Theorem 3.2. *Let $(H_1) - (H_7)$ and $(F_1) - (F_3)$ hold. For every $T > 0$, there exists a weak solution of the problem (65), (66) such that*

$$\begin{cases} u \in L^\infty(0,T;H^1),\ u_t \in L^\infty(0,T;L^2) \cap L^q(Q_T), \\ u(0,\cdot) \in W^{1,q_0}(0,T),\ u(1,\cdot) \in W^{1,q_1}(0,T). \end{cases} \tag{67}$$

Furthermore, if $k \in W^{1,1}(0,T)$ in (H_4); $p_0,\ p_1 \in \{2\} \cup [3,+\infty)$, and (F_4) holds, then the solution is unique.

Remark 3.3. (i) Theorem 3.2 gives no conclusion about the uniqueness of solution when $2 < p_0 < 3$ or $2 < p_1 < 3$.

(ii) The corresponding results in [3, 43] are two special cases of Theorem 3.2 with $\mu(x,t) \equiv 1$; $f(u,u_t) = Ku + \lambda u_t$, where $K,\ \lambda \geq 0$, $\lambda_0 = 0$, $p_0 = p_1 = q_1 = 2$, $(u_0,\ u_1) \in H^2 \times H^1$ and $\mu(x,t) \equiv 1$, $f(u,u_t) = K\,|u|^{p-2}u + \lambda\,|u_t|^{q-2}u_t$, where $K,\ \lambda \geq 0$, $p,\ q \geq 2$; $\lambda_0 = 0$, $p_0 = p_1 = q_1 = 2$, $(u_0,\ u_1) \in H^2 \times H^1$, respectively.

(iii) Theorem 3.2 still holds if (F_2) is replaced by the following assumption:
$(\widetilde{F_2})$ There exist constants $C_1,\ C_1',\ C_2' \geq 0$, $C_2 > 0$, $q > p > 1$ and a function $f_2 : \mathbb{R}_+ \to \mathbb{R}$ continuous, such that

(j) $\int_0^u f_2(s)ds \geq -C_1|u|^p - C_1'$ for all $u \in \mathbb{R}$,

(jj) $(f(u,v) - f_2(u))\,v \geq C_2\,|v|^q - C_2'$ for all $u,\ v \in \mathbb{R}$.

(iv) In the proof of Theorem 3.2, we require the following lemma:

Lemma 3.4. *Let u be the weak solution of the following problem*

$$\begin{cases} u'' - \frac{\partial}{\partial x}\left(\mu(x,t)u_x\right) = \Phi,\ 0 < x < 1,\ 0 < t < T, \\ \mu(0,t)u_x(0,t) = G_0(t), \quad -\mu(1,t)u_x(1,t) = G_1(t), \\ u(x,0) = u_0(x),\ u'(x,0) = u_1(x), \\ u \in L^\infty(0,T;H^1),\ u' \in L^\infty(0,T;L^2), \\ u'(0,\cdot) \in L^{q_0}(0,T),\ u'(1,\cdot) \in L^{q_1}(0,T), \\ G_0 \in L^{q_0'}(0,T),\ G_1 \in L^{q_1'}(0,T),\ \Phi \in L^1(0,T;L^2). \end{cases} \tag{68}$$

Then we have

$$\begin{aligned} &\frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\int_0^1 \mu(x,t)u_x^2(x,t)dx - \frac{1}{2}\int_0^t ds \int_0^1 \mu'(x,s)u_x^2(x,s)dx \\ &\quad + \int_0^t G_0(s)u'(0,s)ds + \int_0^t G_1(s)u'(1,s)ds \\ &\geq \frac{1}{2}\left\|u_1\right\|^2 + \frac{1}{2}\int_0^1 \mu(x,0)u_0^2(x)dx + \int_0^t \langle \Phi(s), u'(s)\rangle\, ds,\ \ a.e.\ t \in [0,T]. \end{aligned} \tag{69}$$

Furthermore, if $u_0 = u_1 = 0$ there is equality in (69).

Lemma 3.4 was proved in [50] with the same method as in [22, Lemma 2.1, p. 79]. Let us note more that, since the conditions $u'(0,\cdot) \in L^{q_0}(0,T)$, $u'(1,\cdot) \in L^{q_1}(0,T)$ are necessary, this lemma cannot be applied to the problem in our recent paper [62]. So, the other lemma established in [62] allows us to weaken the regularity conditions as above, which is similar to Lemma 3.4 but there is no need of the conditions $u'(0,\cdot) \in L^{q_0}(0,T)$, $u'(1,\cdot) \in L^{q_1}(0,T)$.

3.2. The Regularity of the Solution

In this part, we study the regularity of the solution of the problem (65), (66) corresponding to $q_0 = q_1 = 2$, $(u_0,\ u_1) \in H^2 \times H^1$. Henceforth, we strengthen the hypotheses and assume that:

(H_1') $(u_0,\ u_1) \in H^2 \times H^1$,

(H_2') $F,\ F_t \in L^1(0,T;L^2)$,

(H_3') $g \in H^1(0,T)$,

(H_4') $k \in W^{1,1}(0,T)$,

(H_5') $\mu \in C^1\left(\overline{Q_T}\right)$, $\mu_{tt} \in L^1(0,T;L^\infty)$, $\mu(x,t) \geq \mu_0 > 0$, $\forall (x,t) \in \overline{Q_T}$,

(H_6') $p_0,\ p_1 \geq 2;\ q_0 = q_1 = 2$,

(F') $f \in C^1\left(\mathbb{R}^2\right)$ satisfies (F_2) and the following conditions: There exist constants $\widetilde{C}_1 > 0$, $q > 2$ and two continuous functions $\widetilde{b}_1,\ \widetilde{b}_2 : \mathbb{R}_+ \to \mathbb{R}_+$, such that

(F_1') $|vD_1f(u,v)| \leq \widetilde{b}_1(|u|) + \widetilde{b}_2(|u|)\,|v|^{q/2}$ for all $u,\ v \in \mathbb{R}$,

(F_2') $D_2f(u,v) \geq -\widetilde{C}_1$ for all $u,\ v \in \mathbb{R}$.

Let K_0, $K_1 \geq 0$. We have the following theorem, the proof of which can be found in the paper [50].

Theorem 3.5. *Let* $(H_1') - (H_6')$, (F_1') *and* (F_2') *hold. Then, for every* $T > 0$, *there exists a unique weak solution* u *of the problem* (65), (66) *such that*

$$\begin{cases} u \in L^\infty\left(0,T;H^2\right),\ u_t \in L^\infty\left(0,T;H^1\right),\ u_{tt} \in L^\infty\left(0,T;L^2\right), \\ u(0,\cdot),\ u(1,\cdot) \in H^2\left(0,T\right). \end{cases} \tag{70}$$

Remark 3.6. (i) With the regularity obtained by (70), it follows that the problem (65), (66) has a unique strong solution u that satisfies

$$\begin{cases} u \in C^0\left(0,T;H^1\right) \cap C^1\left(0,T;L^2\right) \cap L^\infty\left(0,T;H^2\right),\ u_t \in L^\infty\left(0,T;H^1\right), \\ u_{tt} \in L^\infty\left(0,T;L^2\right),\ u(0,\cdot),\ u(1,\cdot) \in H^2\left(0,T\right). \end{cases} \tag{71}$$

(ii) From (70) we can see that $u \in H^2\left(Q_T\right)$. If $(u_0,\ u_1) \in H^2 \times H^1$, the weak solution u of the problem (65), (66) belongs to $H^2\left(Q_T\right)$. So the solution is almost classical which is rather natural since the initial data $(u_0,\ u_1)$ does not belong necessarily to $C^2\left(\overline{\Omega}\right) \times C^1\left(\overline{\Omega}\right)$.

(iii) In [5], Browder has studied the operator differential equation

$$u_{tt} + Au + M(u) = 0,\ t > 0, \tag{72}$$

with the initial conditions

$$u(0_+) = u_0,\ u_t(0_+) = u_1, \tag{73}$$

where A is a positive densely defined self-adjoint linear operator in a Hilbert space H with $A^{1/2}$ being its positive square root. $M(u)$ is a (possibly) nonlinear function from $D\left(A^{1/2}\right)$ to H and some other conditions. In genegral, the results in Theorem 3.2 and in the papers [5, 23] overlap and do not include each other as particular cases.

(iv) In the special case with $f(u,u_t) = K\left|u\right|^{p-2}u + \lambda\left|u_t\right|^{q-2}u_t$, where K, $\lambda \geq 0$; p, $q \geq 2$, $q_0 = q_1 = q_1 = 2$, $\mu(x,t) \equiv 1$ and $(u_0,u_1) \in H^2 \times H^1$, we have obtained the results in the paper [47].

3.3. Asymptotic Expansion of the Solution with Respect to Two Small Parameters K_0, K_1.

In this part, we consider two given functions u_0, u_1 as $\widetilde{u}_0$, $\widetilde{u}_1$, respectively. Then we assume that $\lambda_0 = \lambda_1 = 1$, $q_0 = q_1 = 2$; p_0, $p_1 \geq N+1$; $N \in \mathbb{N}$, $N \geq 2$, and $(\widetilde{u}_0, \widetilde{u}_1, F, g, k, \mu, f)$ satisfy the assumptions $(H_1') - (H_5')$ and (F_1') and (F_2').

Let $(K_0,\ K_1) \in \mathbb{R}_+^2$. By Theorem 3.2, the problem (65), (66) has a unique weak solution depending on $(K_0,\ K_1) : u = u_{K_0,\,K_1}$.

We consider the following perturbed problem, where K_0, K_1 are small parameters such that $0 \leq K_0 \leq K_{0*}$, $0 \leq K_1 \leq K_{1*}$:

$$(\widetilde{P}_{K_0,\,K_1})\begin{cases} Lu \equiv u'' - \frac{\partial}{\partial x}\left(\mu(x,t)u_x\right) = -f(u,u') + F(x,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < x < 1,\ 0 < t < T, \\ L_0 u \equiv \mu(0,t)u_x(0,t) = P(t),\ L_1 u \equiv -\mu(1,t)u_x(1,t) = Q(t), \\ u(x,0) = \widetilde{u}_0(x),\ u'(x,0) = \widetilde{u}_1(x), \\ P(t) = g(t) + K_0 H_{p_0}\left(u(0,t)\right) + u'(0,t) - \int_0^t k(t-s)u(0,s)ds, \\ Q(t) = K_1 \left|u(1,t)\right|^{p_1-2} u(1,t) + u'(1,t). \end{cases}$$

We shall study the asymptotic expansion of the solution of the problem $(\widetilde{P}_{K_0,K_1})$ with respect to $(K_0,\,K_1)$. We use the following notation. For a multi-index $\gamma = (\gamma_0,\,\gamma_1) \in \mathbb{Z}_+^2$, and $\overrightarrow{K} = (K_0,\,K_1) \in \mathbb{R}_+^2$, we put

$$|\gamma| = \gamma_0 + \gamma_1,\ \gamma! = \gamma_0!\gamma_1!,\ \overrightarrow{K}^{\gamma} = K_0^{\gamma_0}K_1^{\gamma_1},\ ||\overrightarrow{K}|| \ = \sqrt{K_0^2 + K_1^2}.$$

Now, we assume that

$$(F_3')\ \ f \in C^{N+1}\left(\mathbb{R}^2\right),\ N \geq 2.$$

First, we need the following lemma, the proof of which can be found in the paper [49].

Lemma 3.7. *Let $m,\ N \in \mathbb{N}$ and $v_\alpha \in \mathbb{R}$, $\alpha \in \mathbb{Z}_+^2$, $1 \leq |\alpha| \leq N$. Then*

$$\left(\sum\nolimits_{1\leq|\alpha|\leq N} v_\alpha \overrightarrow{K}^{\alpha}\right)^m = \sum\nolimits_{m\leq|\alpha|\leq mN} T^{(m)}[v]_\alpha \overrightarrow{K}^{\alpha}, \tag{74}$$

where the coefficents $T^{(m)}[v]_\alpha$, $m \leq |\alpha| \leq mN$, depending on $v = (v_\alpha)$, $\alpha \in \mathbb{Z}_+^2$, $1 \leq |\alpha| \leq N$, are defined by the recurrent formulas

$$\begin{cases} T^{(1)}[v]_\alpha = v_\alpha,\ 1 \leq |\alpha| \leq N, \\ T^{(m)}[v]_\alpha = \sum\limits_{\beta \in A_\alpha^{(m)}} v_{\alpha-\beta} T^{(m-1)}[v]_\beta,\ m \leq |\alpha| \leq mN,\ m \geq 2, \\ A_\alpha^{(m)} = \{\beta \in \mathbb{Z}_+^2 : \beta \leq \alpha,\ 1 \leq |\alpha-\beta| \leq N,\ m-1 \leq |\beta| \leq (m-1)N\}. \end{cases} \tag{75}$$

Let $u_0 \equiv u_{0,0}$ be a unique weak solution of the problem $(\widetilde{P}_{0,0})$ (as in Theorem 3.2) corresponding to $(K_0,\,K_1) = (0,\,0)$, i.e.,

$$\left(\widetilde{P}_{0,0}\right)\begin{cases} Lu_0 = -f(u_0,\ u_0') + F(x,t),\ 0 < x < 1,\ 0 < t < T, \\ L_0 u_0 = P_0(t),\ L_1 u_0 = Q_0(t), \\ u_0(x,0) = \widetilde{u}_0(x),\ u_0'(x,0) = \widetilde{u}_1(x), \\ P_0(t) = g(t) + u_0'(0,t) - \int_0^t k(t-s)u_0(0,s)ds,\ Q_0(t) = u_0'(1,t), \\ u_0 \in C^0\left(0,T;H^1\right) \cap C^1\left(0,T;L^2\right) \cap L^\infty\left(0,T;H^2\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad u_0' \in L^\infty\left(0,T;H^1\right), \\ u_0'' \in L^\infty\left(0,T;L^2\right),\ u_0(0,\cdot),\ u_0(1,\cdot) \in H^2\left(0,T\right). \end{cases}$$

Let us consider the sequence of weak solutions u_γ, $\gamma \in \mathbb{Z}_+^2$, $1 \le |\gamma| \le N$, defined by the following problems:

$$
\left(\widetilde{P}_\gamma\right)\begin{cases}
Lu_\gamma = F_\gamma,\ 0<x<1,\ 0<t<T,\\
L_0u_\gamma = P_\gamma(t),\ L_1u_\gamma = Q_\gamma(t),\\
u_\gamma(x,0)=u_\gamma'(x,0)=0,\\
P_\gamma(t)=\widehat{P}_\gamma(t)+u_\gamma'(0,t)-\int_0^t k(t-s)u_\gamma(0,s)ds,\\
Q_\gamma(t)=\widehat{Q}_\gamma(t)+u_\gamma'(1,t),\\
u_\gamma\in C^0\left(0,T;H^1\right)\cap C^1\left(0,T;L^2\right)\cap L^\infty\left(0,T;H^2\right),\\
u_\gamma'\in L^\infty\left(0,T;H^1\right),\\
u_\gamma''\in L^\infty\left(0,T;L^2\right),\ u_\gamma(0,\cdot),\ u_\gamma(1,\cdot)\in H^2\left(0,T\right),
\end{cases}
$$

where F_γ, $\widehat{P}_\gamma$, $\widehat{Q}_\gamma$, $|\gamma|\le N$, are defined by the recurrent formulas

$$
F_\gamma=\begin{cases}
-f(u_0,u_0')+F, \qquad |\gamma|=0,\\
-D_1f(u_0,u_0')u_\gamma-D_2f(u_0,u_0')u_\gamma', \qquad |\gamma|=1,\\
-D_1f(u_0,u_0')u_\gamma-D_2f(u_0,u_0')u_\gamma'\\
\quad -\sum_{m=2}^{|\gamma|}\sum_{i=0}^{m}\frac{1}{i!(m-i)!}D_1^{m-i}D_2^if(u_0,u_0')\times\\
\quad \times\sum\limits_{\beta\le\gamma,\ m-i\le|\gamma-\beta|\le(m-i)N,\ i\le|\beta|\le iN} T^{(m-i)}[u]_{\gamma-\beta}T^{(i)}[u']_\beta,\\
\qquad\qquad 2\le|\gamma|\le N,
\end{cases}
\tag{76}
$$

$$
\widehat{P}_\gamma(t)=\begin{cases}
g(t), & |\gamma|=0,\\
0, & \gamma_0=0,\ 1\le\gamma_1\le N,\\
H_{p_0}\left(u_0(0,t)\right), & \gamma_0=1,\ \gamma_1=0,\\
\sum_{m=1}^{|\gamma|-1}\frac{1}{m!}H_{p_0}^{(m)}\left(u_0(0,t)\right)T^{(m)}[u(0,t)]_{\gamma_0-1,\gamma_1}, & \gamma_0\ge1,\ 2\le|\gamma|\le N,
\end{cases}
\tag{77}
$$

$$
\widehat{Q}_\gamma(t)=\begin{cases}
0, & \gamma_1=0,\ 1\le\gamma_0\le N,\\
H_{p_1}\left(u_0(1,t)\right), & \gamma_1=1,\ \gamma_0=0,\\
\sum_{m=1}^{|\gamma|-1}\frac{1}{m!}H_{p_1}^{(m)}\left(u_0(1,t)\right)T^{(m)}[u(1,t)]_{\gamma_0,\gamma_1-1}, & \gamma_1\ge1,\ 2\le|\gamma|\le N,
\end{cases}
\tag{78}
$$

where we have used the notations $u=(u_\gamma)$, $u'=(u_\gamma')$, $|\gamma|\le N$.

We have the following theorem, the proof of which can be found in the paper [50].

Theorem 3.8. *Let p_0, $p_1 \ge N+1$, $N\ge2$ and $(H_1')-(H_5')$, $(F_1')-(F_3')$ hold. Then, for every $\overrightarrow{K}=(K_0,\ K_1)\in\mathbb{R}_+^2$, with $0\le K_0\le K_{0*}$, $0\le K_1\le K_{1*}$, the problem $(\widetilde{P}_{K_0,K_1})$ has a unique weak solution $u=u_{K_0,K_1}$ satisfying the asymptotic estimations up to order $N+1$ as follows*

$$
\begin{aligned}
&\left\|u' - \sum_{|\gamma|\le N} u'_\gamma \overrightarrow{K}^\gamma\right\|_{L^\infty(0,T;L^2)} + \left\|u - \sum_{|\gamma|\le N} u_\gamma \overrightarrow{K}^\gamma\right\|_{L^\infty(0,T;H^1)} \\
&\quad + \left\|u'(0,\cdot) - \sum_{|\gamma|\le N} u'_\gamma(0,\cdot) \overrightarrow{K}^\gamma\right\|_{L^2(0,T)} \\
&\quad + \left\|u'(1,\cdot) - \sum_{|\gamma|\le N} u'_\gamma(1,\cdot) \overrightarrow{K}^\gamma\right\|_{L^2(0,T)} \\
&\le \widetilde{C}_N^* ||\overrightarrow{K}||^{N+1},
\end{aligned} \tag{79}
$$

where $\widetilde{C}_N^*$ is a positive constant independent of $\overrightarrow{K}$, the function u_γ is the weak solution of the problem $(\widetilde{P}_\gamma)$, $\gamma \in \mathbb{Z}_+^2$, $|\gamma| \le N$.

Remark 3.9. In case $f(u, u_t) = Ku + \lambda u_t$, where K, $\lambda \ge 0$, $p_0 = p_1 = q_1 = 2$, $\lambda_0 = 0$ and $(u_0, u_1) \in H^2 \times H^1$, this special problem was considered by Long, Dinh and Diem in [43] and the assymptotic expansion of the solutions with respect to two parameters (K, λ) up to order $N + 1$ was obtained.

4. The Linear Wave Equations Associated with Two-point Boundary Conditions

In this section we consider the following initial-boundary value problem for the linear wave equation

$$
\begin{cases}
u_{tt} - u_{xx} + Ku + \lambda u_t = f(x,t), \ 0 < x < 1, \ t > 0, \\
u_x(0,t) = h_0 u(0,t) + \lambda_0 u_t(0,t) + \widetilde{h}_1 u(1,t) + \widetilde{\lambda}_1 u_t(1,t) + g_0(t), \\
-u_x(1,t) = h_1 u(1,t) + \lambda_1 u_t(1,t) + \widetilde{h}_0 u(0,t) + \widetilde{\lambda}_0 u_t(0,t) + g_1(t), \\
u(x,0) = \widetilde{u}_0(x), \ u_t(x,0) = \widetilde{u}_1(x),
\end{cases} \tag{80}
$$

where h_0, h_1, λ_0, λ_1, $\widetilde{h}_0$, $\widetilde{h}_1$, $\widetilde{\lambda}_0$, $\widetilde{\lambda}_1$, K, λ are constants and $\widetilde{u}_0$, $\widetilde{u}_1$, f, g_0, g_1 are given functions.

This section consists of three parts. In Part 1, we present the existence and uniqueness of a weak solution and the ones of a strong solution of the problem (80) with the suitable conditions. Part 2 is devoted to the study of the regularity of the solution. Finally, in Part 3, we prove that the exponential decay properties of the global solution are similar to the ones of the functions f, g_0, g_1. The theorems given below have been presented in [80].

4.1. Existence and Uniqueness of the Solution

In this part, we assume that h_0, λ_0, λ_1 are positive constants, h_1 is a nonnegative constant and K, λ, $\widetilde{h}_0$, $\widetilde{h}_1$, $\widetilde{\lambda}_0$, $\widetilde{\lambda}_1$ are constants. Let

$$
|\widetilde{\lambda}_0 + \widetilde{\lambda}_1| < 2\sqrt{\lambda_0 \lambda_1}. \tag{81}
$$

We have the following theorems [80].

Theorem 4.1. *Let $T > 0$ and assume that g_0, $g_1 \in L^2(0,T)$, $f \in L^1\left(0,T;L^2\right)$. Then, for each $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$, the problem* (80) *has a unique weak solution* *u satisfying*

$$u \in L^{\infty}(0,T;H^1),\ u_t \in L^{\infty}(0,T;L^2),\ u(0,\cdot),\ u(1,\cdot) \in H^1(0,T). \tag{82}$$

Theorem 4.2. *Let $T>0$ and assume that $g_0,\ g_1 \in H^1(0,T)$; $f,\ f_t \in L^2(Q_T)$. Then, for each $(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$, the problem* (80) *has a unique strong solution u satisfying*

$$\begin{cases} u \in H^2(Q_T) \cap C^0\left(0,T;H^1\right) \cap C^1\left(0,T;L^2\right) \cap L^{\infty}\left(0,T;H^2\right), \\ u_t \in L^{\infty}\left(0,T;H^1\right),\ u_{tt} \in L^{\infty}\left(0,T;L^2\right);\ u(0,\cdot),\ u(1,\cdot) \in H^2(0,T). \end{cases} \tag{83}$$

4.2. The Regularity of the Solution

In this part, we study the regularity of solution of the problem (80). For this purpose, we also assume that the constants $h_0,\ h_1,\ \lambda_0,\ \lambda_1,\ K,\ \lambda,\ \widetilde{h}_0,\ \widetilde{h}_1,\ \widetilde{\lambda}_0,\ \widetilde{\lambda}_1$ satisfy the conditions as in Part 1.

Furthermore, we will impose the following stronger assumptions, with $r \in \mathbb{N}$,

(A_1) $\widetilde{u}_0 \in H^{r+2}$ and $\widetilde{u}_1 \in H^{r+1}$,

(A_2) The function $f(x,t)$ satisfies

(i) $\dfrac{\partial^r f}{\partial x^j \partial t^{r-j}} \in L^{\infty}(0,T;L^2)$, $0 \le j \le r$,

(ii) $\dfrac{\partial^{\nu} f}{\partial t^{\nu}} \in L^2(0,T;L^2)$, $0 \le \nu \le r+1$,

(iii) $\dfrac{\partial^{\mu} f}{\partial t^{\mu}}(\cdot,0) \in H^1$, $0 \le \mu \le r-1$,

(A_3) $g_0,\ g_1 \in H^{r+2}(0,T),\ r \ge 1.$

We have the following theorem, [80].

Theorem 4.3. *Let $(A_1)-(A_3)$ hold. Then the unique solution u of the problem* (80) *satisfies*

$$\begin{cases} u \in C^{r-1}\left(0,T;H^2\right) \cap C^r\left(0,T;H^1\right) \cap C^{r+1}\left(0,T;L^2\right), \\ \dfrac{\partial^r u}{\partial t^r} \in L^{\infty}\left(0,T;H^2\right) \cap C^0\left(0,T;H^1\right) \cap C^1\left(0,T;L^2\right), \\ \dfrac{\partial^{r+1} u}{\partial t^{r+1}} \in L^{\infty}\left(0,T;H^1\right), \\ \dfrac{\partial^{r+2} u}{\partial t^{r+2}} \in L^{\infty}\left(0,T;L^2\right), \\ u(0,\cdot),\ u(1,\cdot) \in H^{r+2}(0,T), \end{cases} \tag{84}$$

$$\frac{\partial^{r+2-j} u}{\partial t^{r+2-j}} \in L^{\infty}(0,T;H^j),\ 0 \le j \le r+2. \tag{85}$$

Furthermore

$$u \in H^{r+2}(Q_T) \cap \left(\bigcap_{j=0}^{r+1} C^{r+1-j}(0,T;H^j) \right). \tag{86}$$

4.3. Exponential Decay of the Solution

In this part, we assume that $K > 0$ and $\lambda > 0$. Let u be a strong solution of the problem (80).

We then have the following theorem, [80].

Theorem 4.4. *Assume that* $\|f(t)\|^2 + g_0^2(t) + g_1^2(t) \le \sigma_1 \exp(-\sigma_2 t)$ *for all* $t \ge 0$, *where* σ_1, σ_2 *are two positive constants. Then there exist positive constants* γ_1, γ_2 *such that*

$$\|u'(t)\|^2 + \|u(t)\|_{H^1}^2 \le \gamma_1 \exp(-\gamma_2 t) \text{ for all } t \ge 0, \tag{87}$$

for any strong solution of the problem (80)*, where* $\widetilde{h}_0$ *and* $\widetilde{h}_1$ *are chosen small enough.*

Remark 4.5. We can extend the above theorem to weak solutions by using density arguments.

5. Nonlinear Wave Equations Involving Convolution

In this section, we consider the following initial and boundary value problem:

$$\begin{cases} u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + F(u, u_t) = f(x,t), \ 0 < x < 1, \ 0 < t < T, \\ u_x(0,t) = K_0 u(0,t), \ u_x(1,t) + K_1 u(1,t) = g(t), \\ u(x,0) = \widetilde{u}_0(x), \ u_t(x,0) = \widetilde{u}_1(x), \end{cases} \tag{88}$$

where $F(u,u_t) = K|u|^{p-2}u + \lambda |u_t|^{q-2} u_t$, with $K_0 > 0$; $K_1 \ge 0$, $p \ge 2$, $q \ge 2$; K, λ are given constants and $\widetilde{u}_0$, $\widetilde{u}_1$, f, g, k are given functions satisfying conditions specified later.

Various problems of the type (88) have been considered by many authors and several results concerning existence, nonexistence, and asymptotic behavior have been established.

In a recent paper [6], Berrimia and Messaoudi considered the problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t k(t-s)\Delta u(s)ds = |u|^{p-2} u, \ x \in \Omega, \ t > 0, \\ u = 0, \text{ on } \partial\Omega, \\ u(x,0) = \widetilde{u}_0(x), \ u_t(x,0) = \widetilde{u}_1(x), \ x \in \Omega, \end{cases} \tag{89}$$

where $p > 2$ is a constant, k is a given positive function, and Ω is a bounded domain of $\mathbb{R}^n$ $(n \ge 1)$, with a smooth boundary $\partial\Omega$. In [9], Cavalcanti et al. studied the following equation

$$u_{tt} - \Delta u + \int_0^t k(t-s)\Delta u(s)ds + |u|^{p-2} u + a(x)u_t = 0, \text{ in } \Omega \times (0, \infty), \tag{90}$$

for $a : \Omega \to \mathbb{R}_+$ being a function, which may be null on a part of the domain Ω. Under the conditions that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometrical restrictions and

$$-\zeta_1 k(t) \leq k'(t) \leq -\zeta_2 k(t),\ t \geq 0, \tag{91}$$

the authors established an exponential rate of decay.

When $k \equiv 0$, and $f(x,t) = 0$, the problem $(88)_{1,3}$ with the mixed nonhomogeneous condition

$$u_x(0,t) = K_0 u(0,t) + g(t),\ u(1,t) = 0, \tag{92}$$

where $K_0 > 0$ is a given constant; or with the more generalized boundary condition

$$u_x(0,t) = g(t) + K_0 u(0,t) - \int_0^t H(t-s)u(0,s)ds,\ u(1,t) = 0. \tag{93}$$

has been studied in [25, 27], respectively, by Long and Dinh.

In [28], Long and Diem have studied the problem $(88)_{1,3}$ with $k \equiv 0$, and the mixed homogeneous condition

$$u_x(0,t) - K_0 u(0,t) = u_x(1,t) + K_1 u(1,t) = 0, \tag{94}$$

where K_0, K_1 are given non-negative constants with $K_0 + K_1 > 0$ and a right-hand side of the form

$$F = F(x, t, u, u_x, u_t). \tag{95}$$

In [3] Bergounioux et al. studied the problem $(88)_{1,3}$ with $k \equiv 0$, $F(u, u_t) = Ku + \lambda u_t$, and the mixed boundary conditions $(88)_2$ standing for

$$\begin{cases} u_x(0,t) = g(t) + K_0 u(0,t) - \int_0^t H(t-s)u(0,s)ds, \\ u_x(1,t) + K_1 u(1,t) + \lambda_1 u_t(1,t) = 0, \end{cases} \tag{96}$$

where K_0, K, λ, $K_1 \geq 0$, $\lambda_1 > 0$ are given constants and g, H are given functions.

In [43], Long et al. obtained the unique existence, regularity and asymptotic expansion of the problem $(88)_{1,3}$ and (96) in the case of $k \equiv 0$, $F(u, u_t) = K\left|u\right|^{p-2} u + \lambda \left|u_t\right|^{q-2} u_t$, with $p \geq 2$, $q \geq 2$; K, λ being given constants.

In [42], Long et al. gave the unique existence, stability, regularity in time variable and asymptotic expansion for the solution of the problem (88) when $F(u, u_t) = Ku + \lambda u_t$, $\widetilde{u}_0 \in H^2$ and $\widetilde{u}_1 \in H^1$. In this case, the problem (88) is the mathematical model describing a shock problem involving a linear viscoelastic bar.

This section consists of three main parts. In Part 1, under the conditions $(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$; f, $f_t \in L^2(Q_T)$, $k \in W^{2,1}(0,T)$, $g \in H^2(0,T)$; K, λ, $K_1 \geq 0$, $K_0 > 0$; p, $q \geq 2$, we investigate the unique existence of a weak solution u of the problem (88). The proof is based on the Faedo – Galerkin method associated to a priori estimates, weak convergence and compactness techniques. Part 2 is devoted to the asymptotic behavior of the solution u as $K_1 \to 0_+$. Finally, in Part 3 we obtain an asymptotic expansion of the solution u of the problem (88) up to order $N+1$ in three small parameters K, λ, K_1. The results obtained here are in part generalizations of those in [3, 6, 9, 25, 27, 28, 39, 42, 43, 45, 47, 57, 58, 75-77].

5.1. The Existence and Uniqueness Theorem

Without loss of generality, we can suppose that $K_0 = 1$. We make the following assumptions:
(H_1) $K,\ \lambda,\ K_1 \geq 0,\ p,\ q \geq 2,$
(H_2) $(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1,$
(H_3) $k \in W^{2,1}(0,T),\ g \in H^2(0,T),$
(H_4) $f,\ f_t \in L^2(Q_T),$ for all $T > 0.$

We then have the following result, the proof of which can be found in the paper [49].

Theorem 5.1. *Let* $(H_1) - (H_4)$ *hold. Then, for every* $T > 0$, *there exists a unique weak solution* u *of the problem* (88) *such that*

$$\begin{cases} u \in H^2(Q_T) \cap C^0\left(0,T;H^1\right) \cap C^1\left(0,T;L^2\right) \cap L^\infty\left(0,T;H^2\right), \\ u_t \in L^\infty\left(0,T;H^1\right),\ u_{tt} \in L^\infty\left(0,T;L^2\right). \end{cases} \tag{97}$$

Remark 5.2. In case $p,\ q > 2$ and $K < 0$, $\lambda < 0$, the existence of solutions for the problem (88) is still open.

5.2. Asymptotic Expansion of the Solution with Respect to the Small Parameter K_1

In this part, we assume that $(\widetilde{u}_0, \widetilde{u}_1, f, g, k, K, \lambda)$ satisfy $(H_1) - (H_4)$. Let $K_1 > 0$. By Theorem 5.1, the problem (88) has a unique weak solution $u = u_{K_1}$ depending on K_1. We consider the following perturbed problem, where K_1 is a small parameter:

$$(P_{K_1}) \begin{cases} Au \equiv u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds = -K\Psi_p(u) - \lambda\Psi_q(u_t) + f(x,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) = u(0,t),\ u_x(1,t) + K_1 u(1,t) = g(t), \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x), \\ \Psi_r(z) = |z|^{r-2} z,\ r \in \{p,\ q\}. \end{cases}$$

We shall study the asymptotic expansion of the solution of the problem (P_{K_1}) with respect to K_1. We then have the following result, the proof of which can be found in the paper [49].

Theorem 5.3. *Let* $T > 0$. *Let* $(H_1) - (H_4)$ *hold. Then*
(i) *The problem* (P_0) *corresponding to* $K_1 = 0$ *has only the solution* $\widehat{u}_0$ *satisfying*

$$\widehat{u}_0 \in L^\infty(0,T;H^2),\ \widehat{u}_0' \in L^\infty(0,T;H^1),\ \widehat{u}_0'' \in L^\infty(0,T;L^2). \tag{98}$$

(ii) *The solution u_{K_1} converges strongly in $W_1(T)$ to $\widehat{u}_0$, as $K_1 \to 0_+$, where*

$$W_1(T) = \{v \in L^\infty(0,T;H^1) : v_t \in L^\infty(0,T;L^2)\}.$$

Furthermore, we have the estimation

$$\left\| u'_{K_1} - \widehat{u}'_0 \right\|_{L^\infty(0,T;L^2)} + \| u_{K_1} - \widehat{u}_0 \|_{L^\infty(0,T;H^1)} \le \widetilde{C}_T K_1, \tag{99}$$

where $\widetilde{C}_T$ is a posistive constant depending only on T.

The next result gives an asymptotic expansion of the weak solution u_{K_1} of order $N+1$ in K_1, for K_1 sufficiently small.

We use the following notations. For a multi-index $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{Z}_+^N$, and $x = (x_1, ..., x_N) \in \mathbb{R}^N$, we put

$$|\alpha| = \alpha_1 + ... + \alpha_N,\ \alpha! = \alpha_1!...!\alpha_N!,\ x^\alpha = x_1^{\alpha_1}...x_N^{\alpha_N}.$$

First, we shall need the following lemma.

Lemma 5.4. *Let m, $N \in \mathbb{N}$, $x = (x_1, ..., x_N) \in \mathbb{R}^N$, and $t \in \mathbb{R}$. Then*

$$\left(\sum_{i=1}^{N} x_i t^i \right)^m = \sum_{k=m}^{mN} P_k^{[m]}[x] t^k, \tag{100}$$

where the coefficents $P_k^{(m)}[x]$, $m \le k \le mN$ depending on $x = (x_1, ..., x_N)$ are defined by the formula

$$\begin{cases} P_k^{[m]}[x] = \sum\limits_{\alpha \in A_k^{(m)}} \frac{m!}{\alpha!} x^\alpha,\ m \le k \le mN, \\ A_k^{(m)} = \{\alpha \in \mathbb{Z}_+^N : |\alpha| = m,\ \sum_{i=1}^N i\alpha_i = k\}. \end{cases} \tag{101}$$

The proof of this lemma is easy, hence we omit the details.

Let $\widehat{u}_0$ be a weak solution of the problem (P_{K_1}) corresponding to $K_1 = 0$ as in Theorem 5.3.

$$(P_0) \begin{cases} A\widehat{u}_0 = -K\Psi_p(\widehat{u}_0) - \lambda\Psi_q(\widehat{u}'_0) + f(x,t),\ 0 < x < 1,\ 0 < t < T, \\ \widehat{u}_{0x}(0,t) = \widehat{u}_0(0,t),\ \widehat{u}_{0x}(1,t) = g(t), \\ \widehat{u}_0(x,0) = \widetilde{u}_0(x),\ \widehat{u}'_0(x,0) = \widetilde{u}_1(x), \\ \widehat{u}_0 \in L^\infty(0,T;H^2),\ \widehat{u}'_0 \in L^\infty(0,T;H^1),\ \widehat{u}''_0 \in L^\infty(0,T;L^2). \end{cases}$$

Let us consider the sequence of the weak solutions $\widehat{u}_i$, $i = 1, 2, ..., N$, defined by the following problems:

$$(\widehat{P}_i)\begin{cases} A\widehat{u}_i \equiv F_i,\ 0<x<1,\ 0<t<T, \\ \widehat{u}_{ix}(0,t)=\widehat{u}_i(0,t),\ \widehat{u}_{ix}(1,t)=-\ \widehat{u}_{i-1}(1,t), \\ \widehat{u}_i(x,0)=\widehat{u}_i'(x,0)=0, \\ \widehat{u}_i\in L^\infty(0,T;H^2),\ \widehat{u}_i'\in L^\infty(0,T;H^1),\ \widehat{u}_i''\in L^\infty(0,T;L^2), \end{cases}$$

where

$$\begin{cases} F_i=-\ \sum_{m=1}^{i}\frac{1}{m!}\left[K\Psi_p^{(m)}\left(\widehat{u}_0\right)P_i^{[m]}[\widehat{u}]+\lambda\Psi_q^{(m)}\left(\widehat{u}_0'\right)P_i^{[m]}[\widehat{u}']\right], \\ \widehat{u}=\left(\widehat{u}_1,...,\widehat{u}_N\right),\ \widehat{u}'=\left(\widehat{u}_1',...,\widehat{u}_N'\right). \end{cases} \tag{102}$$

We then have the following theorem, the proof of which can be found in the paper [49].

Theorem 5.5. *Let $p,\ q\geq N+2,\ N\geq 1,\ K_{1*}>0$ and $(H_2)-(H_4)$ hold. Then, for every $K_1\in(0,K_{1*})$, the problem (P_{K_1}) has a unique weak solution u_{K_1} satisfying the asymptotic estimation up to order $N+1$ as follows*

$$\left\|u_{K_1}'-\sum_{i=0}^{N}\widehat{u}_i'K_1^i\right\|_{L^\infty(0,T;L^2)}+\left\|u_{K_1}-\sum_{i=0}^{N}\widehat{u}_iK_1^i\right\|_{L^\infty(0,T;H^1)}\leq D_TK_1^{N+1}, \tag{103}$$

where D_T is a constant independent of K_1, the functions $\widehat{u}_i,\ i=0,1,\ldots,N$ are the weak solutions of the problems (P_0), $(\widehat{P}_i)$, $i=1,...,N$, respectively.

5.3. Asymptotic Expansion of the Solution with Respect to Three Small Parameters $(K,\ \lambda,\ K_1)$

In this part, we assume that $p,\ q\geq N+1,\ N\geq 2$ and $(\widetilde{u}_0,\widetilde{u}_1,f,g,k)$ satisfy the assumptions $(H_2)-(H_4)$. Let $(K,\ \lambda,\ K_1)\in\mathbb{R}_+^3$. By Theorem 5.1, the problem (88) has a unique weak solution u depending on $(K,\ \lambda,\ K_1):u=u(K,\ \lambda,\ K_1)$.

We consider the following perturbed problem, where $K,\ \lambda,\ K_1$ are small parameters such that $0\leq K\leq K_*,\ 0\leq\lambda\leq\lambda_*,\ 0\leq K_1\leq K_{1*}:$

$$(P_{K,\lambda,K_1})\begin{cases} Au & \equiv u_{tt}-u_{xx}+\int_0^t k(t-s)u_{xx}(s)ds \\ & =-K\Psi_p(u)-\lambda\Psi_q(u_t)+f(x,t),\ 0<x<1,\ 0<t<T, \\ u_x(0,t) & =u(0,t),\ u_x(1,t)=-K_1u(1,t)+g(t), \\ u(x,0) & =\widetilde{u}_0(x),\ u_t(x,0)=\widetilde{u}_1(x). \end{cases}$$

We shall study the asymptotic expansion of the solution of the problem (P_{K,λ,K_1}) with respect to $K,\ \lambda,\ K_1$.

We use the following notations. For a multi-index $\gamma=(\gamma_1,\ \gamma_2,\ \gamma_3)\in\mathbb{Z}_+^3$, and $\overrightarrow{K}=(K,\lambda,K_1)\in\mathbb{R}_+^3$, we put

$$\begin{aligned} |\gamma| &= \gamma_1+\gamma_2+\gamma_3,\ \gamma!=\gamma_1!\gamma_2!\gamma_3!, \\ ||\overrightarrow{K}|| &= \sqrt{K^2+\lambda^2+K_1^2}, \\ \overrightarrow{K}^\gamma &= K^{\gamma_1}\lambda^{\gamma_2}K_1^{\gamma_3}. \end{aligned}$$

Let $u_0 \equiv u_{0,0,0}$ be a unique weak solution of the problem $(P_{0,0,0})$ (as in Theorem 5.1) corresponding to $\overrightarrow{K} = (K,\ \lambda,\ K_1) = (0,\ 0,\ 0)$, i.e.,

$$(P_{0,0,0}) \begin{cases} Au_0 = P_{0,0,0} \equiv f(x,t),\ 0 < x < 1,\ 0 < t < T, \\ u_{0x}(0,t) = u_0(0,t),\ u_{0x}(1,t) = g(t), \\ u_0(x,0) = \widetilde{u}_0(x),\ u_0'(x,0) = \widetilde{u}_1(x), \\ u_0 \in L^\infty\left(0,T;H^2\right),\ u_0' \in L^\infty\left(0,T;H^1\right),\ u_0'', \in L^\infty\left(0,T;L^2\right). \end{cases}$$

Let us consider the sequence of weak solutions u_γ, $\gamma \in \mathbb{Z}_+^3$, $1 \le |\gamma| \le N$, defined by the following problems:

$$(P_\gamma) \begin{cases} Au_\gamma = P_\gamma,\ 0 < x < 1,\ 0 < t < T, \\ u_{\gamma x}(0,t) = u_\gamma(0,t),\ u_{\gamma x}(1,t) = Q_\gamma(t), \\ u_\gamma(x,0) = u_\gamma'(x,0) = 0, \\ u_\gamma \in L^\infty\left(0,T;H^2\right),\ u_\gamma' \in L^\infty\left(0,T;H^1\right),\ u_\gamma'' \in L^\infty\left(0,T;L^2\right), \end{cases}$$

where P_γ, Q_γ, $|\gamma| \le N$ are defined by the recurrent formulas

$$Q_\gamma(t) = \begin{cases} g(t), & |\gamma| = 0, \\ 0, & 1 \le |\gamma| \le N,\ \gamma_3 = 0, \\ -u_{\gamma_1,\gamma_2,\gamma_3-1}(1,t), & 1 \le |\gamma| \le N,\ \gamma_3 \ge 1, \end{cases} \tag{104}$$

and

$$P_\gamma = \begin{cases} f, & |\gamma| = 0, \\ 0, & 1 \le |\gamma| \le N,\ \gamma_1 = \gamma_2 = 0, \\ -\Psi_p\left(u_0\right), & |\gamma| = 1,\ \gamma_1 = 1, \\ -\Psi_q\left(u_0'\right), & |\gamma| = 1,\ \gamma_2 = 1, \\ -\sum_{m=1}^{|\gamma|-1} \frac{1}{m!}\Psi_q^{(m)}\left(u_0'\right) T^{(m)}[u']_{0,\gamma_2-1,\gamma_3}, & 2 \le |\gamma| \le N,\ \gamma_2 \ge 1,\ \gamma_1 = 0, \\ -\sum_{m=1}^{|\gamma|-1} \frac{1}{m!}\Psi_p^{(m)}\left(u_0\right) T^{(m)}[u]_{\gamma_1-1,0,\gamma_3}, & 2 \le |\gamma| \le N,\ \gamma_1 \ge 1,\ \gamma_2 = 0, \\ -\sum_{m=1}^{|\gamma|-1} \frac{1}{m!}\Big[\Psi_p^{(m)}\left(u_0\right) T^{(m)}[u]_{\gamma_1-1,\gamma_2,\gamma_3} & \\ \qquad + \Psi_q^{(m)}\left(u_0'\right) T^{(m)}[u']_{\gamma_1,\gamma_2-1,\gamma_3}\Big], & 2 \le |\gamma| \le N,\ \gamma_1 \ge 1,\ \gamma_2 \ge 1. \end{cases} \tag{105}$$

Here we have used the notation $u = (u_\gamma)$, $|\gamma| \le N$.

We then have the following theorem, the proof of which can be found in the paper [49].

Theorem 5.6. *Let $p,\ q \ge N+1$, $N \ge 2$ and $(H_2)-(H_4)$ hold. Then, for every $\overrightarrow{K} = (K,\ \lambda,\ K_1) \in \mathbb{R}_+^3$, with $0 \le K \le K_*$, $0 \le \lambda \le \lambda_*$, $0 \le K_1 \le K_{1*}$, the problem (P_{K,λ,K_1}) has a unique weak solution $u = u_{K,\lambda,K_1}$ satisfying the asymptotic estimation up to order $N+1$ as follows*

$$\left\|u' - \sum_{|\gamma| \le N} u'_\gamma \overrightarrow{K}^\gamma\right\|_{L^\infty(0,T;L^2)} + \left\|u - \sum_{|\gamma| \le N} u_\gamma \overrightarrow{K}^\gamma\right\|_{L^\infty(0,T;H^1)}$$
$$\le \widetilde{D}_N^* ||\overrightarrow{K}||^{N+1}, \tag{106}$$

for all $\overrightarrow{K} \in \mathbb{R}_+^3$, $||\overrightarrow{K}|| \le ||\overrightarrow{K_*}||$, *where* $\widetilde{D}_N^*$ *is a positive constant independent of* $\overrightarrow{K}$, *the functions* u_γ, $|\gamma| \le N$ *are the weak solutions of the problems* (P_γ), $\gamma \in \mathbb{Z}_+^3$, $|\gamma| \le N$.

5.4. Asymptotic Behavior of the Solution as $t \to +\infty$.

In this part, we will consider the following initial and boundary value problem:

$$\begin{cases} u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + |u_t|^{q-2}u_t = f(x,t,u),\ 0<x<1,\ 0<t<T, \\ u_x(0,t) = K_0 u(0,t),\ u_x(1,t) + K_1 u(1,t) = g(t), \\ u(x,0) = \widetilde{u}_0(x),\ \ u_t(x,0) = \widetilde{u}_1(x), \end{cases} \tag{107}$$

where $K_0 > 0$, $K_1 \ge 0$, $q \ge 2$ are given constants and $\widetilde{u}_0$, $\widetilde{u}_1$, g, k, f are given functions.

At first, under a certain local Lipschitzian condition on f with $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$; k, $g \in H^1(0,T)$, $\lambda > 0$, $K_0 > 0$; $K_1 \ge 0$; $q \ge 2$, a global existence and uniqueness theorem is proved. The proof is based on the results in [49], the contraction mapping theorem and standard arguments of density. Next, the asymptotic behavior of the solution u as $t \to \infty$ is studied. It is proved that under more restrictive conditions, the unique solution $u(t)$ exists on $\mathbb{R}_+$ such that $\|u'(t)\| + \|u(t)\|_{H^1}$ decays exponentially to 0 as $t \to +\infty$. The results obtained here are in part generalizations of those in [3, 6, 9, 42, 43, 45, 49].

5.4.1. Global Existence

In Part 1, we study the global existence of solutions for the problem (107). For this purpose, first, we consider a related nonlinear problem. Then, we use the well-known Banach's fixed point theorem to prove the existence of solutions to the nonlinear problem (107).

We make the following assumptions:

(H_1') $K_0 > 0,\ K_1 \ge 0,\ q \ge 2$,
(H_2') $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$,
(H_3') $k,\ g \in H^1(0,T)$,
(H_4') $f \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R})$ satisfies the conditions $D_2 f$, $D_3 f \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R})$.

For each $T > 0$, we put

$$W(T) = \{v \in L^\infty(0,T;H^1) : v_t \in L^\infty(0,T;L^2) \cap L^q(Q_T)\}.$$

We then have the following theorem, the proof of which can be found in the paper [52].

Theorem 5.7. *Let* $T > 0$ *and* $(H_1') - (H_4')$ *hold. Then there exists* $T_1 \in (0,T)$ *such that problem* (107) *has a unique weak solution* $u \in W(T_1)$ *and such that* u'', $u_{xx} \in L^{q'}(0,T_1;(H^1)')$.

Remark 5.8. In case $\lambda = 0$, $f(x,t,u) = |u|^{p-2}u$, $p > 2$, $k \in W^{2,1}(\mathbb{R}_+)$, $k \geq 0$, $k(0) > 0$, $0 < \int_0^{+\infty} k(t)dt < 1$, $k'(t) + \zeta k(t) \leq 0$ for all $t \geq 0$, with $\zeta > 0$, and the boundary condition $u(0,t) = u(1,t) = 0$ standing for $(107)_2$, S. Berrimia, S. A. Messaoudi [6] have obtained a global existence and uniqueness theorem.

5.4.2. Decay of the Solution as $t \to +\infty$

In Part 2, we will consider asymptotic behavior of the solution u as $t \to +\infty$. We assume that $g(t) = 0$, $f(x,t,u) = F(x,t) - |u|^{p-2}u$, $p \geq 2$ and consider the following problem

$$\begin{cases} u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(s)ds + |u|^{p-2}u + |u_t|^{q-2}u_t = F(x,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < x < 1,\ t > 0, \\ u_x(0,t) - K_0 u(0,t) = u_x(1,t) + K_1 u(1,t) = 0, \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x). \end{cases} \tag{108}$$

We make the following assumptions:

(H_1'') $K_0 > 0$, $K_1 \geq 0$, p, $q \geq 2$,

(H_3'') $k \in W^{2,1}(\mathbb{R}_+)$, $k \geq 0$, satisfying

(i) $k(0) > 0$, $0 < 1 - \int_0^{+\infty} k(t)dt = k_\infty < 1$,

(ii) there exists a positive constant ζ such that $k'(t) + \zeta k(t) \leq 0$ for all $t \geq 0$,

(H_4'') $F \in L^1(0,\infty;L^2) \cap L^2(0,\infty;L^2)$, $F_t \in L^1(0,\infty;L^2)$,

(H_5'') There exists a constant $\sigma > 0$ such that $\int_0^{+\infty} e^{st}\|F(t)\|^2 dt < +\infty$.

We then have the following theorem, the proof of which can be found in the paper [52].

Theorem 5.9. *Suppose that* (H_1''), (H_2) *and* $(H_3'') - (H_5'')$ *hold. Then the solution* $u(t)$ *of the problem* (108) *decays exponentially to zero as* $t \to +\infty$ *in the following sense: There exist positive constants* C *and* γ *such that*

$$\|u'(t)\| + \|u(t)\|_{H^1} \leq Ce^{-\gamma t} \text{ for all } t \geq 0. \tag{109}$$

Remark 5.10. The estimation (109) holds for any regular solution corresponding to $(\widetilde{u}_0, \widetilde{u}_1) \in H^2 \times H^1$. This is still true for solutions corresponding to $(\widetilde{u}_0, \widetilde{u}_1) \in H^1 \times L^2$ by simple density arguments.

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