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Geometry of Domains in $\mathbb{C}^n$ with Noncompact Automorphism Groups

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Abstract. We survey results arising from the study of domains in $\mathbb{C}^n$ with non-compact automorphism group in the past ten years. Beginning with well-known characterizations of smoothly bounded domains with non-compact automorphism group, we develop ideas toward characterizations of arbitrary (not necessary bounded) domains in $\mathbb{C}^n$.

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1. Notation

Let D be a domain in $\mathbb{C}^n$. In a neighborhood U of a fixed point $p \in \partial D$ we can write

$$D \cap U = \{z \in U : \rho(z) < 0\}.$$

Such a function ρ is called a defining function for D near p. We say that, for $1 \leq k \leq \infty$, D has C^k-smooth or real analytic boundary near p if there is a defining function ρ for D near p which is, respectively, either C^k-smooth or real analytic and $\nabla\rho \neq 0$ on ∂D. The boundary is said to be globally C^k-smooth or real analytic if it is such at every point. When the boundary is globally C^k-

smooth then it is easy to patch together local defining functions to obtain a single global defining function for the entire boundary. From now on, when speaking about defining functions of domains with smooth boundary, we will be assuming that these functions satisfy the conditions just discussed.

If D has at least C^2-smooth boundary near $p \in \partial D$, then ∂D is said to be pseudoconvex at p if there is a defining function ρ for D near p such that

$$\mathcal{L}_\rho(p)(w,w) := \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq 0 \quad (*),$$

for all $w = (w_1, \cdots, w_n) \in T_p^c(\partial D)$; here $T_p^c(\partial D)$ is the complex tangent space to ∂D at p.

We call $p \in \partial D$ a point of strong or strict pseudoconvexity if (*) is strict for non-zero $w \in T_p^c(\partial D)$.

The Hermite form $\mathcal{L}_\rho$ defined in (*) is called the Levi form of ∂D at p.

We now recall the definition of finite type in the sense of J. P. D'Angelo (see [9]).

Let $D \subset \mathbb{C}^n$ be a domain with $C^\infty-$smooth boundary and let $p \in \partial D$. Then the type $\tau(p)$ of ∂D at p is defined as

$$\tau(p) = \sup_F \frac{\nu(\tau \circ F)}{\nu(F)},$$

where ρ is a defining function of D near p, the supremum is taken over all holomorphic mappings F defined in a neighborhood of $0 \in \mathbb{C}$ into $\mathbb{C}^n$ such that $F(0) = p$, and $\nu(\phi)$ is the order of vanishing of a function ϕ at the origin.

The boundary ∂D is said to be of finite type at p if $\tau(p) < \infty$.

The domain D is a domain of finite type if ∂D is of finite type at every point.

2. Characterization of Domains in $\mathbb{C}^n$ by Their Noncompact Automorphism Groups

Let Ω be a domain, i.e. connected open subset, in a complex manifold M. Let the *automorphism group* of Ω (denoted $\mathrm{Aut}(\Omega)$) be the collection of biholomorphic self-maps of Ω with composition of mappings as its binary operation. The topology on $\mathrm{Aut}(\Omega)$ is that of uniform convergence on compact sets (i.e., the compact-open topology).

One of the important problems in several complex variables is to study the interplay between the geometry of a domain and the structure of its automorphism group. More precisely, we wish to see to what extent a domain is determined by its automorphism group.

It is a standard and classical result of H. Cartan that if Ω is a bounded domain in $\mathbb{C}^n$ and the automorphism group of Ω is noncompact then there exist a point $x \in \Omega$, a point $p \in \partial\Omega$, and automorphisms $\varphi_j \in \mathrm{Aut}(\Omega)$ such that $\varphi_j(x) \to p$. In this circumstance we call p a *boundary orbit accumulation point.*

Work in the past twenty years has suggested that the local geometry of the so-called "boundary orbit accumulation point" p in turn gives global information about the characterization of model of the domain. For instance, Wong [28] and Rosay [22] proved the following theorem.

Wong-Rosay Theorem. (see [28, 22]) *Any bounded domain $\Omega \Subset \mathbb{C}^n$ with a C^2 strongly pseudoconvex boundary orbit accumulation point is biholomorphic to the unit ball in $\mathbb{C}^n$.*

By using the scaling technique, introduced by S. Pinchuk, in 1991 E. Bedford and S. Pinchuk proved the theorem about the characterization of the complex ellipsoids.

Bedford-Pinchuk Theorem. (see [4]) *Let $\Omega \subset \mathbb{C}^{n+1}$ be a bounded pseudoconvex domain of finite type whose boundary is smooth of class C^∞, and suppose that the Levi form has rank at least $n-1$ at each point of the boundary. If $\mathrm{Aut}(\Omega)$ is noncompact, then Ω is biholomorphically equivalent to the domain*

$$E_m = \{(w, z_1, \cdots, z_n) \in \mathbb{C}^{n+1} : |w|^2 + |z_1|^{2m} + |z_2|^2 + \cdots + |z_n|^2 < 1\},$$

for some integer $m \geq 1$.

The approach of Bedford-Pinchuk involves two steps. In the first step they use the method of scaling to show that the domain Ω in consideration is holomorphically equivalent to a domain D of the form

$$D = \{(z_1, \tilde{z}) \in \mathbb{C}^{n+1} : \mathrm{Re}\, z_1 + Q(\tilde{z}, \bar{\tilde{z}}) < 0\},$$

where Q is a polynomial. The domain D has a non-trivial holomorphic vector field. In the second step this vector field is transported back to Ω, the result is analyzed at the parabolic fixed point, and this information is used to determine the original domain.

There has been also certain progress, by other authors, on the first step of the above procedure of Bedford-Pinchuk. The following completely local result for domains (not necessary bounded) in $\mathbb{C}^2$ was obtained by F. Berteloot in 1994.

Berteloot Theorem. (see [7]) *Let Ω be a domain in $\mathbb{C}^2$ and let $\xi_0 \in \partial\Omega$. Assume that there exist a sequence (φ_p) in $\mathrm{Aut}(\Omega)$ and a point $a \in \Omega$ such that $\lim \varphi_p(a) = \xi_0$. If $\partial\Omega$ is pseudoconvex and of finite type near ξ_0 then Ω is biholomorphically equivalent to $\{(w, z) \in \mathbb{C}^2 : \mathrm{Re}\, w + H(z, \bar{z}) < 0\}$, where H is a homogeneous subharmonic polynomial on $\mathbb{C}$ with degree $2m$.*

The first aim in this article is to show a completely local result on the first step of the above procedure of Bedford-Pinchuk for domains (not necessary bounded) in $\mathbb{C}^n$. Namely, we prove the following.

Theorem 2.1. (see [25]) *Let Ω be a domain in $\mathbb{C}^n$ and let $\xi_0 \in \partial\Omega$. Assume that*

(a) *$\partial\Omega$ is pseudoconvex, of finite type and smooth of class C^∞ in some neighbourhood of $\xi_0 \in \partial\Omega$.*

(b) *The Levi form has rank at least $n-2$ at ξ_0.*

(c) *There exists a sequence (φ_p) in $\mathrm{Aut}(\Omega)$ such that $\lim \varphi_p(a) = \xi_0$ for some $a \in \Omega$.*

Then Ω is biholomorphically equivalent to a domain of the form

$$M_H = \{(w_1, \cdots, w_n) \in \mathbb{C}^n : \mathrm{Re}\ \ w_n + H(w_1, \bar{w}_1) + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 < 0\},$$

where H is a homogeneous subharmonic polynomial with $\Delta H \not\equiv 0$.

Without the assumption (b) we show that Theorem 2.1 also is true for linearly convex domains in $\mathbb{C}^n$. On the other words, Berteloot Theorem holds for linearly convex domains in $\mathbb{C}^n$.

Theorem 2.2. (see [26]) *Let Ω be a domain in $\mathbb{C}^n$, and let ξ_0 be a point of $\partial\Omega$. Assume that ξ_0 is an accumulating point for a sequence of automorphisms of Ω. If $\partial\Omega$ is smooth, linearly convex, and of finite type $2m$ near ξ_0, then Ω is biholomorphically equivalent to a rigid polynomial domain*

$$D = \{z \in \mathbb{C}^n : \mathrm{Re}\, z_1 + P(z') < 0\},$$

where P is a real nondegenerate plurisubharmonic polynomial of degree less than or equal to $2m$.

The nondegeneracy of P is given by condition "$\{P = 0\}$ without nontrivial analytic set".

Open questions.

1. We would like to emphasize here that the assumption on boundedness of domains in the Bedford-Pinchuk theorem is essential in their proofs. It seems to us that some key techniques in their proofs could not be used for unbounded domains in $\mathbb{C}^n$. Thus, the first natural question that whether the Bedford-Pinchuk theorem is true for any domain in $\mathbb{C}^n$.

2. Is it true that the theorems on characterization of smoothly bounded domains in $\mathbb{C}^n$ with noncompact automorphism groups holds without an extra assumption such as the finiteness of type or pseudoconvexity?

Many experts believe that the answers are positive.

Main idea of the proof of Theorem 2.1. (See the detailed proof in [25])

Without loss of generality we may assume that $\xi_0 = 0$ and the rank of the Levi form at ξ_0 is exactly $n-2$.

Put $\eta_j = \varphi_j(a)$. Then $\{\eta_j\} \to (0, \cdots, 0)$. We construct the sequence $\{\psi_j\}$ of automorphisms of $\mathbb{C}^n$ which associates with the sequence $\{\eta_j\}$ by the method of the dilation of coordinates, i.e. $\psi_j = \Delta^{\epsilon_j}_{\eta'_j} \circ \Phi_{\eta'_j}$.

We show that the sequence $\{\psi_j \circ \varphi_j\}$ is normal and its limits are holomorphic mappings from Ω to the domain of the form

$$R_P = \{(w_1, \cdots, w_n) \in \mathbb{C}^n : \operatorname{Re} w_n + P(w_1, \overline{w}_1) + \sum_{\alpha=2}^{n-1} |w_\alpha|^2 < 0\},$$

where P is a homogeneous subharmonic polynomial with $\Delta P \not\equiv 0$.

Put $D_j = \psi_j(\Omega)$ and $f_j = \psi_j \circ \varphi_j$. By the previous steps, there exists a subsequence of $\{f_j\}$ which converges uniformly on compact subsets of Ω to a holomorphic mapping from Ω into the domain R_P.

After taking a subsequence, the following properties are satisfied

(i) D_j converges to R_P.

(ii) f_j is convergent uniformly on compact subsets of Ω.

(iii) f_j^{-1} is convergent uniformly on compact subsets of R_P.

(iv) If $f = \lim f_j$, then $f(\Omega) \subset R_P$.

Now Theorem 2.1 is deduced from the Green-Krantz theorem (see [10, p.161] and [24]).

3. Green-Krantz Conjecture

In 1993, R. E. Green and S. G. Krantz introduced the following.

Green-Krantz Conjecture. (see [11]) *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain with noncompact automorphism group. Then $\partial\Omega$ is of finite type at any boundary orbit accumulation point.*

The conjecture in its full generality is open.

We now try to analyze the causes of the successes of E. Bedford, S. Pinchuk and F. Berteloot we presented in the previous section. They showed that if p is of finite type, then the domain is biholomorphic to the domain of the following form

$$M_P = \{(w, z) \in \mathbb{C}^2 : \operatorname{Re} w + P(z, \bar{z}) < 0\},$$

where P is an homogeneous polynomial in z and $\bar{z}$. Each domain M_P is called a model of Ω at p.

To prove this, they first applied the Scaling method to point out that $\operatorname{Aut}(\Omega)$ contains a parabolic subgroup, i.e., there is a point $p_\infty \in \partial\Omega$ and a one-parameter subgroup $\{h^t\}_{t\in\mathbb{R}} \subset \operatorname{Aut}(\Omega)$ such that for all $z \in \Omega$

$$\lim_{t\to\pm\infty} h^t(z) = p_\infty. \qquad (1)$$

Each boundary point satisfying (1) is called a parabolic boundary point of Ω. After that, the local analysis of a holomorphic vector field H which generates the above subgroup h^t was carried out to show that Ω is biholomorphic to the desired homogeneous model.

This allows us to give the following definition.

Let Ω be a domain in $\mathbb{C}^n$. A boundary point $p \in \partial\Omega$ is called a *parabolic orbit accumulation point* if there is a one-parameter subgroup

$$\{\psi_t \in \mathrm{Aut}(\Omega), -\infty < t < \infty\}$$

of automorphisms such that

$$\lim_{t\to\pm\infty} \psi_t(x_0) = p$$

for some $x_0 \in \Omega$.

We now recall the condition (R) of Bell (see [6]).

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with a C^∞ smooth boundary. Consider the Bergman projection $P : C^\infty(\Omega) \to C^\infty(\Omega)$ given by

$$P(f)(w) = \int_\Omega f(z)\overline{K(z,w)}dz,$$

where $K(z,w)$ is the Bergman kernel.

We say that Ω satisfies Bell's condition (R) if the Bergman projection $P : C^\infty(\Omega) \to C^\infty(\Omega)$ extends to a map $C^\infty(\bar{\Omega}) \to C^\infty(\bar{\Omega})$.

Theorem of Kim-Krantz. (see [16]) *Let $\Omega \subset \mathbb{C}^2$ be a pseudoconvex domain with a C^∞ smooth boundary satisfying Bell's condition (R). Assume also that $\partial\Omega$ does not contain any non-trivial analytic variety. Then every parabolic orbit accumulation boundary point is of finite D'Angelo type.*

This theorem provides a proof of an important special case of the Greene-Krantz Conjecture. Unfortunately, their proof was incorrect.

In fact, we show gaps in their proof.

Let $p \in \partial\Omega$ be a parabolic orbit accumulation point of infinite D'Angelo type. Choose a holomorphic local coordinate system at p so that p now becomes the origin and the local defining function of Ω takes the form

$$\rho(z) = \mathrm{Re}\, z_1 + \Psi(z_2, \mathrm{Im}\ z_1).$$

Then they pointed out that Ψ vanishes to infinite order at the origin. But, in general, it is not true, e.g., $\psi(z_2, \mathrm{Im}\, z_1) = e^{-1/|z_2|^2} + |z_2|^4.|\mathrm{Im}\, z_1|^2$.

By an another approach, we proved the following.

Theorem 3.1. (see [8]) *Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain in $\mathbb{C}^2$ and $0 \in \partial\Omega$. Assume that*

(1) $\partial\Omega$ is $C^\infty-$ smooth satisfying the Bell's condition (R),

(2) $\partial\Omega$ does not contain any non-trivial analytic variety,

(3) There exists a neighborhood U of $0 \in \partial\Omega$ such that

$$\Omega \cap U = \{(z_1, z_2) \in \mathbb{C}^2 : \rho = \operatorname{Re} z_1 + P(z_2) + Q(z_2, \operatorname{Im} z_1) < 0\},$$

where P and Q satisfy the following

(i) $\lim_{z_2 \to 0} \dfrac{P(z_2)}{|z_2|^N} = 0,\ N = 0, 1, 2, \cdots,$

(ii) $Q(0, \operatorname{Im} z_1) = Q(z_2, 0) = 0,$

$\dfrac{\partial}{\partial z_1} Q(z_2, 0) = \dfrac{\partial}{\partial z_2} Q(z_2, 0) = 0,$

$\dfrac{\partial^2}{\partial z_2 \partial \bar{z}_2} Q(0, \operatorname{Im} z_1) = 0$ *and*

$\dfrac{\partial^N}{\partial z_2^N} Q(0, \operatorname{Im} z_1) = 0,\ N = 0, 1, 2, \cdots.$

Then, $(0, 0)$ *is not a parabolic orbit accumulation point.*

The detailed proof of the above theorem is presented in [8].

Remark 3.2. By a simple computation, we see that

• The funtions $P(z_2) = e^{-1/|z_2|^2}$ and $Q(z_2, \operatorname{Im} z_1) = |z_2|^4.|\operatorname{Im} z_1|^2$ satisfy the above conditions.

• $(0, 0)$ is of infinite type. Hence every parabolic orbit accumulation boundary point of the above-mentioned domain is of finite D'Angelo type.

We now consider a bounded domain $\Omega \subset \mathbb{C}^2$. Suppose that Ω is biholomorphic to the domain D defined by $D = \{(w, z) \in \mathbb{C}^2 : \operatorname{Re} w + \sigma(z) < 0\}$ with some smooth real-valued function σ on the complex plane. The one-parameter group of translations $\{L^t\}_{t \in \mathbb{R}}$ given by $L^t(w, z) = (w + it, z)$ acts on the domain D. The transformation $\psi : D \to \Omega$ allows us to define the one-parameter group of biholomorphic mappings $\{h^t := \psi^{-1} \circ L^t \circ \psi\}_{t \in \mathbb{R}}$ acting on Ω. We now show that this one-parameter group is parabolic. Namely, we prove the following theorem.

Theorem 3.3. (see [8]) *Let Ω be a C^1-smooth, bounded, strictly geometrically convex domain in $\mathbb{C}^2$. Let $\psi : \Omega \to D$ be a biholomorphism, where $D := \{(w, z) \in \mathbb{C}^2 : \operatorname{Re} w + \sigma(z) < 0\}$ and σ is a C^1-smooth nonnegative function on the complex plane such that $\sigma(0) = 0$. Then, there exists some point $a_\infty \in \partial\Omega$ such that $\lim_{t \to +\infty} \psi^{-1}(w \pm it, z) = a_\infty$ for any $(w, z) \in D$.*

The detailed proof of the above theorem is also presented in [8].

4. Miscellaneous

From the viewpoint of F. Klein, geometry of domains in $\mathbb{C}^n$ is geometry of their automorphism groups. Thus, this subject has been intensively studied by several authors since the end of the 19th century and has grown into a huge theory. Many contributed. We refer the reader to the excellent survey article of Isaev and Krantz [15] and references therein for the development of related subjects. We now would like to present some results arising after their article.

4.1. Compactness of Automorphism Groups of Domains in $\mathbb{C}^n$

First of all, we consider some classical examples.

$$B^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\},$$
$$E_{1,2} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^4 < 1\},$$
$$E_{2,2} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^4 + |z_2|^4 < 1\}.$$

The unit ball B^2 is a homogeneous domain. Hence $\operatorname{Aut}(B^2)$ is non-compact.
The domain $E_{1,2}$ has non-compact automorphism group.
The domain $E_{2,2}$ has compact automorphism group.

To show explicit examples (outside well-known examples) on domains in $\mathbb{C}^n$ with compact automorphism groups is not easy. Recently, M. Landucci showed domains of this kind.

Theorem 4.1. (see [21]) *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$, with C^1- boundary, and satisfying the following two conditions:*

(a) *There exists a point $P \in \partial\Omega$ and a system of complex coordinates (z, w) defined on a connected neighborhood U of P so that $(z(P), w(P)) = (0, 0)$ and*

$$\Omega \cup U = \{Q \in U : \operatorname{Re} w(Q) + \Phi(z(Q), \overline{z(Q)}) < 0\},$$

where Φ is smooth, subharmonic and strictly positive at all points different from the origin, where it vanishes at any order, i.e.,

$$\lim_{z\to 0} \frac{\Phi(z)}{|z|^N} = 0 \ \forall N \geq 0.$$

(b) *Any point $P \in \partial\Omega \setminus \bar{U}$ is a boundary point of finite type.*

Then $\operatorname{Aut}(\Omega)$ *is compact.*

4.2. Realizing Connected Lie Groups as Automorphism Groups of Complex Manifolds

As we know, if D is a bounded domain in $\mathbb{C}^n$, then $\operatorname{Aut}(D)$ turns out to be a real Lie group. Moreover, if this group is positive-dimensional, then it is never a complex Lie group. Generally, S. Kobayashi showed that

Theorem 4.2. (see [17])

(i) *Let M be a hyperbolic complex manifold of dimension n. Then* $\operatorname{Aut}(M)$ *is a real Lie group of dimension $\leq n^2 + n$.*

(ii) *Let M be a compact complex manifold. Then* $\operatorname{Aut}(M)$ *is a complex Lie group.*

(iii) *Let M be a compact hyperbolic complex manifold. Then* $\mathrm{Aut}(M)$ *is finite.*

There is a natural question: What real Lie groups can be realized as the automorphism groups of hyperbolic complex manifolds?

In 1987, Saerens-Zame and independently Bedford-Dadok have answered affirmatively to connected compact real Lie groups.

Theorem 4.3. (see [23, 2]) *Given a connected compact real Lie group K. There always exists a strictly pseudoconvex bounded domain $\Omega \Subset \mathbb{C}^n$ such that* $\mathrm{Aut}(\Omega)$ *is isomorphic to K.*

In 2004, J. Winkelmann showed that the above assertion holds without the compactness of K.

Theorem 4.4. (see [27]) *Given a connected real Lie group K. There always exists a strictly pseudoconvex bounded domain $\Omega \Subset \mathbb{C}^n$ such that* $\mathrm{Aut}(\Omega)$ *is isomorphic to K.*

We now present the main idea of Saerens-Zame, Bedford-Dadok and Winkelmann. First of all, they found a domain D on which K acts by automorphisms, and then perturbed it to a K-invariant strictly pseudoconvex subdomain in such a way that the additional automorphisms were ruled out by assigning CR-invariants to each K-orbit on the boundary. To find such a domain D to start with, Saerens-Zame first embedded the compact Lie group K into the unitary group $U(N_1)$ and then constructed a domain D in $GL(N_1, \mathbb{C}) \times \mathbb{C}^{N_2}$ on which K acts by automorphisms, where N_1 and N_2 are large in general. Having observed every Lie algebra is linear and hence the universal covering of a Lie group could be viewed as linear, Winkelmann has been able to embed $\tilde{K}$ into $Sp(N_3, \mathbb{R})$ and then find a suitable domain $D \subset \mathbb{C}^{N_4}$ to start with. The complexification $K_{\mathbb{C}}$ of a compact Lie group K is Stein. Starting from domains in $K_{\mathbb{C}}$, Bedford-Dadok were able to give a more concrete construction. They found bounded strictly pseudoconvex domains $\Omega \subset K_{\mathbb{C}}$ or $\Omega \subset K_{\mathbb{C}} \times \mathbb{C}$ such that $\mathrm{Aut}(\Omega) = K$.

4.3. Cancellation Problem

The cancellation problem is posed as follows:

Let M, N be connected complex manifolds of dimension n such that $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are isomorphic as topological groups equipped with the compact-open topology. Then M is holomorphically equivalent to N.

In general, the cacellation problem does not hold. A simple example of this kind with non-trivial automorphism groups is given by spherical shells

$$S_r := \{z \in \mathbb{C}^n : r < ||z|| < 1\}, 0 \leq r < 1.$$

It is straightforward to see that for $n \geq 2$ the group $\mathrm{Aut}(S_r)$ coincides with the unitary group $U(n)$ for all r. However, S_{r_1} and S_{r_2} are not equivalent for $r_1 \neq r_2$.

Much attention has been given to this problem in the past ten years, and some interesting results have been obtained by several authors. For instance, we have the following.

Theorem 4.5. *Let M be a connected complex manifold of dimension n.*

(i) (see [14]) *If* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(\mathbb{C}^n)$ *are isomorphic as topological groups equipped with the compact-open topology, then M is holomorphically equivalent to $\mathbb{C}^n$.*

(ii) (see [12]) *If* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(B^n)$, *where B^n is the unit ball in $\mathbb{C}^n$, are isomorphic as topological groups equipped with the compact-open topology, then M is holomorphically equivalent to B^n.*

Recently, A. Kodama and S. Shimizu obtained the following characterization of another classical domains, the space obtained by omitting the coordinate hyperplanes from $\mathbb{C}^n$ and the unit polydisc $\Delta^n \subset \mathbb{C}^n$.

Theorem 4.6. *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy.*

(i) (see [19]) *If* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(\mathbb{C}^k \times (\mathbb{C}^*)^{n-k})$ *are isomorphic as topological groups, then M is holomorphically equivalent to $\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$.*

(ii) (see [20]) *If* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(\Delta^n)$ *are isomorphic as topological groups, then M is holomorphically equivalent to Δ^n.*

In particular, the above theorem holds for Stein manifolds and for all domains in $\mathbb{C}^n$.

Here is the main idea of A. Kodama and S. Shimizu in the proof of Theorem 4.6(ii). The connected component of the identity $\mathrm{Aut}(\Delta^n)^0$ of the group $\mathrm{Aut}(\Delta^n)$ is isomorphic to the direct product of n copies of the group $\mathrm{Aut}(\Delta) \simeq SU_{1,1}/\mathbb{Z}_2$, and therefore contains a subgroup (which is a maximal compact subgroup) isomorphic to the n-torus $\mathbb{T}^n$. A topological group isomorphism between $\mathrm{Aut}(M)$ and $\mathrm{Aut}(\Delta^n)$ yields a smooth action by holomorphic transformations of $\mathbb{T}^n$ on M. The assumptions of holomorphic separability and smoothness of the envelope of holomorphy are used by the authors to linearize this action thus representing the manifold M as a Reinhardt domain in $\mathbb{C}^n$. This is possible due to a theorem by Barrett, Bedford and Dadok (see [1]).

Recently, A. V. Isaev showed that the assertion of Theorem 4.6 (ii) remains true if the assumptions of holomorphic separability and smoothness of the envelope of holomorphy are dropped.

Theorem 4.7. (see [13]) *Let M be a connected complex manifold of dimension n such that for every $p \in M$ the isotropy subgroup* $\mathrm{Aut}_p(M) := \{g \in \mathrm{Aut}(M) : g(p) = p\}$ *is compact in* $\mathrm{Aut}(M)$. *Then M is holomorphically equivalent to Δ^n.*

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