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Parametric Optimization Problems and Parametric Variational Inequalities

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Abstract. This survey is a re-edited form of a plenary lecture on Optimization Theory and Scientific Computing at the Seventh Congress of Vietnamese Mathematicians, August 4–8, 2008, Quy Nhon University, Quy Nhon, Binh Dinh. It describes some aspects of the author's and his coauthors' research on optimization problems and variational inequalities during the last 18 years (1992–2009). The focus point is made on parametric problems and qualitative results (stability, sensitivity of the solution set and the alike objects when the problem undergoes small perturbations).

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1. Introduction

This survey paper aims at describing some aspects of the author's and his coauthors' research on optimization problems and variational inequalities during the last 18 years (1992–2009). We will focus on parametric problems and qualitative results (stability, sensitivity of the solution set and the alike objects when the problem undergoes small perturbations). The choice of the time period is not casual. Although the author had had some publications on stability of optimization problems before, it was not until February 1992 − January 1993, working at University of Pisa (Pisa, Italy) under the guidance of Professor F. Giannessi in The World Laboratory's project "Transportation Network Equilibrium Problems and Variational Inequalities", that he became aware of the mathematical models called *variational inequality* and *complementarity problem*.

Through the paper, we wish to show how it is useful to combine the studies of optimization problems and variational inequalities in one task. The idea is, indeed, an old one. It has been known from the very beginning of the theory of variational inequalities (about 45 years ago). However, since we have benefited a lot from the idea, we would like to share with the interested reader the experiences accumulated in our research group upon its realizations.

This survey was initially presented as a plenary lecture on Optimization Theory and Scientific Computing at the Seventh Congress of Vietnamese Mathematicians, August 4–8, 2008, Quy Nhon University, Quy Nhon, Binh Dinh. Having in mind that not all the listeners of the talk, as well as not all the readers of this article, are researchers in optimization and equilibrium theories, we tried to keep our exposition as informal as possible.

The rest of the paper has four sections. Sec. 2 describes some relationships between optimization problems and variational inequalities. Sec. 3 states some theorems on solution stability/sensitivity of parametric variational inequalities together with an application to the traffic equilibrium problem. Sec. 4 is devoted to solution stability/sensitivity of parametric optimization problems. The last section briefly mentions several recent results of the author and his coauthors on optimization problems, variational inequalities, set-valued and variational analysis.

2. The Fermat Rule: Relationships between Optimization Problems and Variational Inequalities

About Fermat's life and mathematical research, Dieudonné [15, p. 263] wrote: "FERMAT, Pierre de (1601–1665) Fermat was born at Beaumont, the son of a leather-merchant who was rich enough to allow Pierre to study law at the University of Toulouse. After receiving his bachelor's degree in 1631, at Orléans, Fermat bought the office of *conseiller* in the Toulouse *parlement*. From 1648 he was a member of the Chambre de l'Edit at Castres. Undoubteadly the profoundest mathematician of the seventeenth century, Fermat originated, with Pascal, the theory of probability, and discovered, before Descartes, the method of coordinates. He was the first to provide a general method for the determination of the tangents to a plane curve; but it was above all in number theory that his genius was manifested."

Hiriart-Urruty [23, p. 3–4] wrote: "In 1629, (that is thirteen years before the birth of Newton), Fermat conceived his method "De maximis et minimis" that could be applied to the determination of values yielding the maximum or the minimum of a function, as well as tangents of curves, which boiled down to the foundations of differential calculus...

Consider the rule proposed by Fermat with a little modern symbolism, one obtains: 'in order to find the value where a function is maximum or minimum: write down $f(a)$, substitute $a + h$ for a, that is to say, write down $f(x + h)$; 'adequalize' $f(a)$ and $f(a + h)$ (adequalize is a term coined by Diophantus to recall that $f(x + h)$ is "almost equal" to $f(a)$); in this adequality only consider the terms of the first order in h; divide by h; then setting the result equal to zero, one obtains an equation in a form which has to be solved'. In other words, one way of using the derivative f' for determining the extrema is by seeking the solutions of $f'(a) = 0$."

From the above Fermat Rule we derive the next statement which is very helpful for finding local minimizers and local maximizers of a polynomial function.

Proposition 2.1. (Fermat Rule for one-variable polynomial functions) *If $\bar{x}$ is a local minimizer or a local maximizer of the polynomial function*

$$\varphi(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n,$$

where $x \in \mathbb{R}$ and $a_i \in \mathbb{R}$, then

$$\varphi'(\bar{x}) := n a_0 \bar{x}^{n-1} + (n-1) a_1 \bar{x}^{n-2} + \cdots + a_{n-1} = 0. \tag{1}$$

Concerning (1) we have the following remarks:

1. Finding $\bar{x}$ such that $\varphi(x) \geq \varphi(\bar{x})$ for all x from a neighborhood of $\bar{x}$ is more difficult than solving the *algebraic equation* (1).

2. Solutions of (1) are just candidates for the (local) solutions of the *optimization problem*

$$\min\{\varphi(x) \,:\, x \in \mathbb{R}\}.$$

3. The above Fermat Rule is a typical example of *necessary optimality conditions.*

4. First-order necessary optimality conditions in the modern optimization theory (including the Lagrange multiplier rules in constrained mathematical programming and the maximum principle in optimal control) are extensions of the Fermat Rule.

We now see how the first-order necessary optimality condition for a constrained optimization problem can be rewritten as a variational inequality. Suppose that X is a normed space with the dual X^* (if $X = \mathbb{R}^n$ or X is a Hilbert space then we can identify X^* with X), $\varphi : X \to \mathbb{R}$ a Fréchet differentiable function, $K \subset X$ a convex set (that is, $(1-t)u + tx \in K$ for all $u, x \in K$ and $t \in [0, 1]$).

Theorem 2.2. (Fermat Rule for differentiable constrained optimization problem) *If $\bar{x}$ is a local minimizer of the optimization problem*

$$\min\{\varphi(x) \,:\, x \in K\}, \tag{2}$$

then

$$\langle \varphi'(\bar{x}), x - \bar{x} \rangle \geq 0, \quad \forall x \in K. \tag{3}$$

This theorem can be proved easily by using the definition of Fréchet derivative (see [33]).

Definition 2.3. Let $K \subset X$ be a closed convex set, $f : K \to X^*$ a single-valued function. The problem of finding an $\bar{x} \in K$ satisfying

$$\langle f(\bar{x}), x - \bar{x} \rangle \geq 0, \quad \forall x \in K \tag{4}$$

is said to be a *variational inequality problem* or, simpler, a *variational inequality* (a VI, for short).

Problem (4) is denoted by $VI(f, K)$, where K is the constraint set, f the basic operator (if $X = \mathbb{R}^n$ then $f : K \to \mathbb{R}^n$ defines a *vector field* on K). The solution set of (4) is abbreviated to $\mathrm{Sol}(VI(f, K))$. Setting $f(x) = \varphi'(x)$ we see that (3) is a variational inequality.

In (4), since $f : K \to X^*$ can be given arbitrarily, it might not be the gradient of any Fréchet differentiable function. This means that (4) may not be the "Fermat equation" (first-order necessary optimality condition) of any optimization problem of the form (2). Thus, the concept of variational inequality is independent from the notion of optimization problem. It is a suitable mathematical model for different practical processes. For instance, many economic equilibrium problems can be modeled as VIs (see [47]).

Defining the normal cone to K at $\bar{x}$ by setting

$$N_K(\bar{x}) = \{x^* \in X^* : \langle x^*, x - \bar{x} \rangle \leq 0, \ \forall x \in K\}$$

if $\bar{x} \in K$, and $N_K(\bar{x}) = \emptyset$ if $\bar{x} \notin K$, we rewrite (4) equivalently as the inclusion $0 \in f(\bar{x}) + N_K(\bar{x})$. Hence variational inequality is a special case of the *generalized equation* of the form $0 \in T(x)$, where $T : K \rightrightarrows X^*$ is a multifunction (a set-valued map).

Unlike the solutions of optimization problems, solutions of VIs have a local character. From this point of view, VIs should be regarded as generalized equations, but not as something similar to optimization problems. The following proposition can be easily proven (see [35, Proposition 5.3]).

Proposition 2.4. *Let $\bar{x} \in K$. If there exists $\varepsilon > 0$ such that*

$$\langle f(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \forall y \in K \cap \bar{B}(\bar{x}, \varepsilon),$$

then $\bar{x} \in \mathrm{Sol}(VI(f, K))$.

The pioneering works of Robinson [56, 57, 58] showed clearly the benefits of treating variational inequalities as generalized equations.

Structure of the solution set and solution stability of $VI(f, K)$ depend greatly on the properties of the basic operator f.

Definition 2.5. One says that $f : K \to X^*$ is
(i) *strongly monotone* on K if $\langle f(y) - f(x), y - x\rangle \geq \alpha\|y - x\|^2$ for all $x, y \in K$, where $\alpha > 0$ is a constant;
(ii) *monotone* on K if $\langle f(y) - f(x), y - x\rangle \geq 0$ for all $x, y \in K$;
(iii) *pseudomonotone* [in the sense of Karmardian] on K if, for all $x, y \in K$, $\langle f(x), y - x\rangle \geq 0$ implies $\langle f(y), y - x\rangle \geq 0$;
(iv) *quasimonotone* on K if, for all $x, y \in K$, $\langle f(x), y - x\rangle > 0$ implies $\langle f(y), y - x\rangle \geq 0$.

Strongly monotone VIs correspond to strongly convex minimization problems (resp., boundary values problems for elliptic partial differential equations). Monotone VIs correspond to convex minimization problems (resp., initial and boundary values problems for parabolic PDEs). In some sense, nonmonotone VIs can be put in a correspondence with nonconvex minimization problems (resp., initial and boundary values problems for hyperbolic PDEs). Observe in addition that pseudomonotone VIs (resp., quasimonotone VIs) have the origin in pseudoconvex minimization problems (resp., quasiconvex minimization problems). Namely, if φ is Fréchet differentiable and pseudoconvex on K (i.e., from $x, y \in K$ and $\langle \varphi'(x), y - x\rangle \geq 0$ it follows that $\varphi(y) \geq \varphi(x)$), then the Fermat condition (3) is a pseudomonotone VI. Similarly, (3) is a quasimonotone VI if φ is Fréchet differentiable and quasiconvex on K (i.e., $\varphi((1 - t)x + ty) \leq \max\{\varphi(x), \varphi(y)\}$ for any $x, y \in K$ and $t \in (0, 1)$); see [21] for more details.

It is easy to verify that: a) a strongly monotone VI can have at most one solution, b) the solution set of a pseudomonotone VI is convex, c) strong monotonicity implies monotonicity, which implies pseudomonotonicity, d) pseudomonotonicity implies quasimonotonicity.

To have some concrete examples of strongly monotone VIs and monotone VIs, we may choose $X = \mathbb{R}^n$, $K = \mathbb{R}^n_+$ (the nonnegative orthant), $f(x) = Ax + b$, where A is a symmetric square matrix and $b \in \mathbb{R}$. Then f is monotone on K if and only if A is positive semidefinite, i.e., $v^T Av \geq 0$ for all $v \in \mathbb{R}^n$ (v^T stands for the transpose of the column vector v). Similarly, f is strongly monotone on K if and only if A is positive definite, i.e., $v^T Av > 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$.

Definition 2.6. Let $M \in \mathbb{R}^{n\times n}$, $q \in \mathbb{R}^n$. Suppose that $K \subset \mathbb{R}^n$ is a polyhedral convex set (i.e., K is the intersection of finitely many closed half-spaces of $\mathbb{R}^n$). The problem

$$\text{Find } \bar{x} \in K \text{ such that } \langle M\bar{x} + q, y - \bar{x}\rangle \geq 0, \quad \forall y \in K \tag{5}$$

is called the *affine variational inequality* (AVI, for brevity) defined by M, q and K.

We denote problem (5) and its solution set respectively by AVI(M, q, K) and Sol(AVI(M, q, K)). In the case $K = \mathbb{R}^n_+$, (5) becomes the *linear complementarity* *problem* (LCP, for short):

$$\bar{x} \geq 0, \quad M\bar{x} + q \geq 0, \quad \bar{x}^T(M\bar{x} + q) = 0. \tag{6}$$

The latter problem and its solution set are denoted by $\mathrm{LCP}(M, q)$ and $\mathrm{Sol}(M, q)$, respectively. In the last five decades, problem (6) has played a significant role in the development of nonlinear optimization and theory of variational inequalities; see e.g. [12, 35] and the references therein.

Necessary optimality conditions for a *quadratic programming problem* under linear constraints

$$\min \Big\{ f(x) := \frac{1}{2} x^T D x + c^T x \,:\, x \in \mathbb{R}^n,\ Ax \geq b \Big\},$$

where $D \in \mathbb{R}^{n\times n}_S$, $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $\mathbb{R}^{n\times n}_S$ denotes the set of $n \times n$ symmetric real matrices, is an AVI (see for instance [12, 35]).

We conclude this section by a remark that the Fermat Rule is a wonderful bridge connecting optimization problems with variational inequalities. Luckily, this is a two-way bridge. In fact, optimization theory has gained a lot from applying various results obtained first for variational inequalities.

3. Parametric Variational Inequalities

In this long section, we discuss two important sample models leading to parametric variational inequalities: one from the field of economic equilibria (the traffic equilibrium problem, Subsection 3.1), another from vector optimization (the linear fractional vector optimization problem, Subsection 3.4). In connection with each of the two models, we formulate several typical sensitivity/stability results. Among the latter, there is a fact about the Lipschitz continuity of the equilibrium flow in a traffic network, where the travel costs and the demands are subject to change.

3.1. The Traffic Equilibrium Problem

Here we will follow [35, Chap. 9]. We refer to the book [35] for the proofs of the cited-below propositions. Consider a traffic system with several cities and many roads connecting them. Suppose that the technical conditions (capacity and quality of roads, etc.) are established. Assume that we know the demands for transportation of some kind of materials or goods between each pair of two cities. The system is well functioning if all these demands are satisfied. The aim of the owner of the network is to keep the system well functioning. The users (drivers, passengers, etc.) do not behave blindly. To go from A to B they will choose one of the roads leading them from A to B with the minimum cost. This natural law is known as the user-optimizing principle or the Wardrop principle. The traffic flow satisfying demands and this law is said to be an *equilibrium flow* of the network. By using this principle, in most of the cases, the owner can compute or estimate the traffic flow on every road. The owner can affect on the network, for example, by requiring high fees from the users of the good roads to force them to use also some roads of lower quality. In this way, a new equilibrium flow, which is more suitable in the opinion of the owner, can be reached.

Traffic network is an example of networks acting in accordance with the Wardrop equilibrium principle. Other examples can be telephone networks or computer networks.

As it was proved by Smith [62] and Dafermos [13], a traffic network can be modeled as a variational inequality.

Consider a graph $\mathcal{G}$ consisting of a set $\mathcal{N}$ of nodes and a set $\mathcal{A}$ of arcs. Every *arc* is a pair of two nodes. The inclusion $a \in \mathcal{A}$ means that a is an arc. A *path* is an ordered family of arcs $a_1, \ldots, a_m$, where the second node of a_s coincides with the first node of a_{s+1} for $s = 1, \ldots, m-1$. We say that the path $\{a_1, \ldots, a_m\}$ connects the first node of a_1 with the second node of a_m.

Let I be a given set of the *origin-destination pairs* ($OD-$pairs, for brevity). Each $OD-$pair consists of two nodes: the origin (the first node of the pair) and the destination (the second node of the pair). Denote by P_i the family of all paths connecting the origin with the destination of an $OD-$pair $i \in I$. Let $P = \bigcup_{i \in I} P_i$ and let $|P|$ denote the number of elements of P.

A vector $v = (v_a : a \in \mathcal{A})$, where $v_a \geq 0$ for all $a \in \mathcal{A}$, is said to be a *flow* (or *flow on arcs*) on the graph. Each v_a indicates the amount of material flow on arc a.

Let there be given a vector function

$$c(v) = (c_a(v) : a \in \mathcal{A}),$$

where $c_a(v) \geq 0$ for all $a \in \mathcal{A}$. This function $c(\cdot)$, which maps $\mathbb{R}^{|\mathcal{A}|}$ to $\mathbb{R}^{|\mathcal{A}|}$, is called the *travel cost function*. Each number $c_a(v)$ is interpreted as the travel cost for *one unit* of material flow to go through an arc a provided that the flow v exists on the network. There are many examples explaining why the travel cost on one arc should depend on the flows on other arcs.

The travel cost on a path $p \in P_i$ $(i \in I)$ is given by the formula

$$C_p(v) = \sum_{a \in p} c_a(v).$$

Let $C(v) = (C_p(v) : p \in P)$. For each $i \in I$, define the *minimum travel cost* $u_i(v)$ for the $OD-$pair i by setting

$$u_i(v) = \min\{C_p(v) : p \in P_i\}.$$

Obviously, $C_p(v) - u_i(v) \geq 0$ for each $i \in I$ and for each $p \in P_i$. Let $D = (\delta_{ap})$ be *the incidence matrix* of the relations "arcs-paths"; that is

$$\delta_{ap} = \begin{cases} 0 & \text{if} \quad a \notin p, \\ 1 & \text{if} \quad a \in p \end{cases}$$

for all $a \in \mathcal{A}$ and $p \in P$.

It is natural to assume the fulfilment of the following *flow-invariant law*:

$$v_a = \sum_{i \in I} \sum_{p \in P_i} \delta_{ap} v_p. \tag{7}$$

Let there be given also a *vector of demands* $g = (g_i : i \in I)$. Every component g_i indicates the *demand* for an $OD-$pair i, that is the amount of the material flow going from the origin to the destination of the pair i. We say that a flow v on the network satisfies demands if

$$\sum_{p \in P_i} v_p = g_i \quad \forall i \in I. \tag{8}$$

Note that

$$\Delta := \left\{ v = (v_p : p \in P) \in \mathbb{R}_+^{|P|} : \sum_{p \in P_i} v_p = g_i \quad \forall i \in I \right\} \tag{9}$$

(the set of flows satisfying demands) is a polyhedral convex set.

If there are given upper bounds $(\gamma_p : p \in P)$, $\gamma_p > 0$ for all $p \in P$, for the capacities of the arcs, then the set of flows satisfying demands is given by the formula

$$\Delta = \left\{ v \in \mathbb{R}_+^{|P|} : \sum_{p \in P_i} v_p = g_i \;\; \forall i \in I, \;\; 0 \le v_p \le \gamma_p \;\; \forall\, p \in P \right\}. \tag{10}$$

In this case, Δ is a compact polyhedral convex set.

Definition 3.1. *A traffic network* $\{\mathcal{G}, I, c(v), g\}$ consists of a graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, a set I of $OD-$pairs, a travel cost function $c(v) = (c_a(v) : a \in \mathcal{A})$, and a vector of demands $g = (g_i : i \in I)$.

We now recall the *user-optimizing principle* introduced by Wardrop [70], which describes clearly the dependence of the equilibrium on the travel cost function.

Definition 3.2. (The Wardrop principle) A flow $\bar{v}$ on the network $\{\mathcal{G}, I, c(v), g\}$ is said to be *an equilibrium flow* if it satisfies demands and, for each $i \in I$ and for each $p \in P_i$, it holds

$$C_p(\bar{v}) - u_i(\bar{v}) = 0 \quad \text{if} \quad \bar{v}_p > 0.$$

The above principle can be stated equivalently as follows: If $C_p(\bar{v})$ (the travel cost on path $p \in P_i$) is greater than $u_i(\bar{v})$ (the minimum travel cost for the $OD-$pair i) then $\bar{v}_p = 0$ (the flow on p is zero). It is important to stress the following: *The fact that the flow on p is zero does not imply that the flows on all the arcs of p are zeros!*

The problem of finding an equilibrium flow $\bar{v}$ on the given network $\{\mathcal{G}, I, c(v), g\}$ is called *the network equilibrium problem.* Let

$$S(v) = (C_p(v) - u_i(v) : p \in P_i,\ i \in I)\,.$$

We see that the number of components of vector $S(v)$ is equal to $|P|$. Since $C_p(v) - u_i(v) \geq 0$ for all $p \in P_i$ and $i \in I$, we have $S(v) \geq 0$. Note that $v = (v_p : p \in P)$ is also a nonnegative vector.

Proposition 3.3. *A flow $\bar{v} \in \Delta$ on a network $\{\mathcal{G},I,c(v),g\}$ is an equilibrium flow if and only if*

$$\begin{cases} \bar{v} \geq 0, \quad S(\bar{v}) \geq 0, \\ \sum\limits_{p \in P_i} \bar{v}_p = g_i \quad \forall i \in I, \\ \langle S(\bar{v}), \bar{v} \rangle = 0. \end{cases} \tag{11}$$

Note that (11) is a (generalized) nonlinear complementarity problem under a polyhedral convex set constraint of the form

$$v \in \Delta, \quad f(v) \geq 0, \quad v^T f(v) = 0,$$

where $f(v) := S(v)$ and $\Delta \subset \mathbb{R}_+^{|P|}$.

Consider the incidence matrix $B = (\beta_{ip})$ of the relations "path-$OD-$pair", where

$$\beta_{ip} = \begin{cases} 1 & \text{if} \quad p \in P_i \\ 0 & \text{if} \quad p \notin P_i. \end{cases}$$

Note that a flow v satisfies demands if and only if

$$Bv = g. \tag{12}$$

Indeed, (12) means that $(Bv)_i = g_i$ for all $i \in I$. The latter is equivalent to (8). The set Δ of feasible flows can be defined either by (9) or by (10). For our convenience, in what follows we assume that Δ is given by (9).

The next proposition, which is due to Smith [62] and Dafermos [13], reduces the network equilibrium problem to a variational inequality.

Proposition 3.4. *A flow $\bar{v} \in \Delta$ is an equilibrium flow of the network $\{\mathcal{G},I,c(v),g\}$ if and only if*

$$\langle C(\bar{v}), v - \bar{v} \rangle \geq 0 \quad \forall v \in \Delta. \tag{13}$$

Note that (13) can be expressed as a variational inequality on the set of flows on arcs. Indeed, according to (7), $v_{\mathcal{A}} = Dv$ for every $v \in \Delta$. Therefore, the set Z of flows on arcs can be defined as follows

$$\begin{aligned} Z &= \{z : z = Dv, \;\; v \in \Delta\} \\ &= \{z : z = Dv, \;\; Bv = g, \;\; v \geq 0\}. \end{aligned} \tag{14}$$

Proposition 3.5. *A flow $\bar{v}_{\mathcal{A}} = (\bar{v}_a : a \in \mathcal{A}) \in Z$ is corresponding to an equilibrium flow of the network $\{\mathcal{G}, I, c(v), g\}$ if and only if*

$$\langle c(\bar{v}), v_{\mathcal{A}} - \bar{v}_{\mathcal{A}} \rangle \geq 0, \tag{15}$$

for all $v_{\mathcal{A}} = (v_a : a \in \mathcal{A}) \in Z$.

The variational inequality in (15), in some sense, is simpler than that one in (13). Both of them are variational inequalities on polyhedral convex sets, but the constraint set of (15) usually has a smaller dimension. Besides, in most of the cases we can assume that $c(v)$ is a *locally strongly monotone function*, while we cannot do so for $C(v)$.

If the costs on arcs and the demand g change, then the basic operator $c(v)$ of (15) and the constraint set Z (see (14)) are perturbed. Thus, it is natural to treat (15) as a parametric VI. Solution sensitivity of this problem will be addressed in Subsection 3.3 below.

3.2. Hölder Continuity of the Solution Map

Let $M \subset \mathbb{R}^m$ and $\Lambda \subset \mathbb{R}^r$ be two sets of parameters, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ a vector-valued function, $K : \Lambda \rightrightarrows \mathbb{R}^n$ a set-valued map with nonempty closed convex values. Consider the *parametric variational inequality* depending on a pair of parameters $(\mu, \lambda) \in M \times \Lambda$:

$$\text{Find } x \in K(\lambda) \text{ such that } \langle f(x,\mu), y - x\rangle \geq 0 \text{ for all } y \in K(\lambda). \tag{16}$$

Fix a pair $(\bar{\mu}, \bar{\lambda}) \in M \times \Lambda$ and suppose that $\bar{x}$ is a solution of the problem

$$\text{Find } x \in K(\bar{\lambda}) \text{ such that } \langle f(x,\bar{\mu}), y - x\rangle \geq 0 \text{ for all } y \in K(\bar{\lambda}). \tag{17}$$

Definition 3.6. (Pseudo-Lipschitz property, Aubin property, Lipschitz-like property; Aubin [5]) Let $\Lambda \subset \mathbb{R}^r$ be a nonempty subset, $K : \Lambda \rightrightarrows \mathbb{R}^n$ a set-valued map with nonempty closed convex values, $\bar{\lambda} \in \Lambda$ and $\bar{x} \in K(\bar{\lambda})$. K is said to be pseudo-Lipschitz at $(\bar{\lambda}, \bar{x})$ if a neighborhood V of $\bar{x}$, a neighborhood W of $\bar{x}$ and a constant $k > 0$ exist such that

$$K(\lambda) \cap W \subset K(\lambda') + k\|\lambda - \lambda'\|\bar{B}(0,1) \quad \forall \lambda, \lambda' \in \Lambda \cap V,$$

where $\bar{B}(a, \delta)$ denotes the closed ball with center a and radius δ.

Let $X \subset \mathbb{R}^n$ be a closed convex neighborhood of $\bar{x}$. Denote the metric projection of y onto the closed convex set $K(\lambda) \cap X$ by $P_{K(\lambda)\cap X}y$, where $y \in \mathbb{R}^n$ and

$\lambda \in \Lambda$, i.e., $P_{K(\lambda)\cap X}y$ is the unique point in $K(\lambda)\cap X$ with the minimal distance to y.

Theorem 3.7. (Yen [71, Lemma 1.1]) *Assume that K is pseudo-Lipschitz at $(\bar{\lambda}, \bar{x})$. Then a neighborhood V_1 of $\bar{\lambda}$, a neighborhood X_1 of $\bar{x}$, and a constant $k_1 > 0$ exist such that*

$$\begin{cases} \|P_{K(\lambda)\cap X}y - P_{K(\lambda')\cap X}y\| \le k_1\|\lambda - \lambda'\|^{1/2} \\ \quad \forall \lambda, \lambda' \in \Lambda \cap V_1, \ \ \forall y \in X_1. \end{cases}$$

The result in Theorem 3.7 and the direct proof of two pages given in [71] remain valid if $\mathbb{R}^n$ is replaced by a Hilbert space, and $\mathbb{R}^m$, $\mathbb{R}^r$ are replaced by two metric spaces. The only change is that $\|\mu' - \mu\|$ and $\|\lambda' - \lambda\|$ should now be $d_M(\mu', \mu)$ and $d_\Lambda(\lambda', \lambda)$, where d_M and d_Λ are the metrics in M and Λ.

Valuable extensions of the property of the metric projection in Theorem 3.7 from the Hilbert space setting to a uniformly convex Banach space setting were obtained by Kien [28, 29].

Consider the PVI (16). Let $\bar{x}$ be a solution to (17). We assume that a closed convex neighborhood X of $\bar{x}$, a neighborhood U of $\bar{\mu}$, and two constants $\alpha > 0$, $\ell > 0$ exist such that

$$\begin{cases} \|f(x', \mu') - f(x, \mu)\| \le \ell(\|x' - x\| + \|\mu' - \mu\|) \\ \forall \mu, \mu' \in M \cap U, \ \ \forall x, x' \in X, \end{cases} \tag{18}$$

and

$$\langle f(x', \mu) - f(x, \mu)\rangle \ge \alpha\|x' - x\|^2 \quad \forall \mu \in M \cap U, \ \forall x, x' \in X. \tag{19}$$

Theorem 3.8. (Yen [71, Theorem 2.1]) *Assume that $\bar{x}$ is a solution to* (17)*, conditions* (18) *and* (19) *hold, and the set-valued map $\lambda \mapsto K(\lambda)$ is pseudo-Lipschitz at $(\bar{\lambda}, \bar{x})$. Then constants $k_{\bar{\mu}}$ and $k_{\bar{\lambda}}$ and neighborhoods $\tilde{U}$ of $\bar{\mu}$ and $\tilde{V}$ of $\bar{\lambda}$ exist such that:*

(i) *For every $(\mu, \lambda) \in (M \cap \tilde{U}) \cap (\Lambda \cap \tilde{V})$, a unique solution to* (16)*, denoted by $x(\mu, \lambda)$, exists in X;*

(ii) *For all $(\mu', \lambda'), (\mu, \lambda) \in (M \cap \tilde{U}) \cap (\Lambda \cap \tilde{V})$,*

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \le k_{\bar{\mu}}\|\mu' - \mu\| + k_{\bar{\lambda}}\|\lambda' - \lambda\|^{1/2}.$$

Theorem 3.8 is proved by using Theorem 3.7, the Dafermos scheme [14], and the Banach contractive mapping principle. The result and the proof are valid if $\mathbb{R}^n$ is replaced by a Hilbert space, and $\mathbb{R}^m$, $\mathbb{R}^r$ are replaced by two metric spaces (then instead of $\|\mu' - \mu\|$ and $\|\lambda' - \lambda\|$ one should write $d_M(\mu', \mu)$ and $d_\Lambda(\lambda', \lambda)$).

Hölder sensitivity estimates for locally unique solutions in nonlinear parametric programming problems have been studied by Alt [3], Shapiro [61], Bonnans and Shapiro [9], and other authors.

Theorem 3.8 can be applied to

- the VI model introduced by M. A. Noor (see [75]),

- vector variational inequalities and vector optimization problems (see [34, 76]),

- the calculus of variations (see [30]).

Domokos [16] extended Theorem 3.8 to parametric VIs in uniformly convex Banach spaces. Kien [27] showed that an analogue of Theorem 3.8 is valid for generalized VIs (i.e., VIs with set-valued basic operators) in uniformly convex Banach spaces. Recently, interesting results on solution stability and sensitivity of finite-dimensional VIs, VIs in reflexive Banach spaces, pseudomonotone VIs, quasimonotone VIs, abstract equilibrium problems, as well as vector equilibrium problems, have been obtained by Mansour and Aussel [1], Mansour and Riahi [2], Anh and Khanh [4], Barbagallo and Cojocaru [6], Huy and Lee [24], Kien [31], Kien and Yao [32], and other authors.

3.3. Lipschitz Continuity of the Solution Map

We now discuss the sensitivity of solutions to a parametric VI with a parametric polyhedral constraint. Let

$$K(\lambda) = \{x \in \mathbb{R}^n : \, Ax \geq \lambda, \; x \geq 0\}, \tag{20}$$

$$\Lambda = \{\lambda \in \mathbb{R}^r : \, K(\lambda) \neq \emptyset\}, \tag{21}$$

where $A \in \mathbb{R}^{r \times n}$ is a given matrix. Let $M \subset \mathbb{R}^m$ be any subset and $f : \mathbb{R}^n \times M \to \mathbb{R}^n$ be a given function. Consider the following PVI depending on a pair of parameters $(\mu, \lambda) \in M \times \Lambda$:

$$\begin{cases} \text{Find} \quad x \in K(\lambda) \quad \text{such that} \\ \langle f(x,\mu), y - x \rangle \geq 0 \quad \text{for all} \quad y \in K(\lambda). \end{cases} \tag{22}$$

Assume that $\bar{x}$ is a solution of the problem

$$\begin{cases} \text{Find} \quad x \in K(\bar{\lambda}) \quad \text{such that} \\ \langle f(x,\bar{\mu}), y - x \rangle \geq 0 \quad \text{for all} \quad y \in K(\bar{\lambda}), \end{cases} \tag{23}$$

where $(\bar{\mu}, \bar{\lambda}) \in M \times \Lambda$ are given parameters.

Under some appropriate conditions on f in a neighborhood of $(\bar{x}, \bar{\mu})$ and *no* conditions on the matrix A, there exist $k > 0$ and neighborhoods X, U, V of $\bar{x}, \bar{\mu}$, and $\bar{\lambda}$, respectively, such that

(i) *For every* $(\mu, \lambda) \in (M \cap U) \times (\Lambda \cap V)$ *there is a unique solution* $x = x(\mu, \lambda)$ *of* (22) *in* X;

(ii) *For every* $(\mu, \lambda), (\mu', \lambda') \in (M \cap U) \times (\Lambda \cap V)$,

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \leq k(\|\mu' - \mu\| + \|\lambda' - \lambda\|).$$

To obtain this result, we need a property of the metric projection onto a moving polyhedral convex set which can be stated as follows.

Theorem 3.9. (Yen [72]) *Given a matrix $A \in \mathbb{R}^{r\times n}$, define the sets $K(\lambda)$ and Λ by* (20) *and* (21). *Then there exists a constant $k_1 > 0$ such that*

$$\|P_{K(\lambda')}y - P_{K(\lambda)}y\| \le k_1\|\lambda' - \lambda\|, \tag{24}$$

for all $y \in \mathbb{R}^n$ and $\lambda, \lambda' \in \Lambda$, where $P_{K(\lambda)}y$ is the metric projection of y onto $K(\lambda)$.

Theorem 3.9 is proved by using Mangasarian-Shau's lemma on complementarity cones [45], which is a nice tool from theory of the linear complementarity problems. As observed in [72], this result is a consequence of a fact stated as an exercise with no proof hints in [12, p. 696] about Lipschitz continuity of the solution map in strongly convex quadratic programming problems. Nevertheless, Theorem 3.9 and its proof may have an independent interest [59, p. 1049].

Consider problem (22) and suppose that for a pair $(\bar{\mu}, \bar{\lambda}) \in M \times \Lambda$ vector $\bar{x}$ is a solution of (23). Following [14] we assume that there exist neighborhoods X of $\bar{x}$, U of $\bar{\mu}$, and two constants $\alpha > 0, l > 0$, such that

$$\|f(x', \mu') - f(x, \mu)\| \le l(\|x' - x\| + \|\mu' - \mu\|) \tag{25}$$

for all μ, μ' in $M \cap U$, $x,\ x'$ in X, and

$$\langle f(x', \mu) - f(x, \mu), x' - x\rangle \ge \alpha\|x' - x\|^2 \tag{26}$$

for all $\mu \in M \cap U$, x and x' in X. Without loss of generality we can assume that X is a polyhedral convex set and $\alpha < l$. Condition (25) means that f is locally Lipschitz at $(\bar{x}, \bar{\mu})$. Condition (26) means that $f(\cdot, \mu)$ is locally strongly monotone around $\bar{x}$ with a common coefficient for all $\mu \in M \cap U$. Using the notation of [14] we put

$$G(x, \mu, \lambda) = P_{K(\lambda)\cap X}[x - \rho f(x, \mu)] \quad \text{for all } (x, \mu, \lambda) \in \mathbb{R}^n \times M \times \Lambda,$$

where $\rho > 0$ is a fixed number and $P_{K(\lambda)\cap X}y$ is the metric projection of y onto $K(\lambda) \cap X$. Let us consider a number ρ satisfying

$$0 < \rho \le \frac{\alpha}{l^2}. \tag{27}$$

For every $\lambda \in \Lambda$ such that $K(\lambda) \cap X \ne \emptyset$, Lemma 2.2 from [14] shows that

$$\|G(x', \mu, \lambda) - G(x, \mu, \lambda)\| \le \beta\|x' - x\|$$

for all x and x' in X, $\mu \in M \cap U$, where

$$\beta := (1 - \rho\alpha)^{1/2} < 1. \tag{28}$$

According to the Banach contractive mapping principle, there is a unique vector $x = x(\mu, \lambda) \in X$ satisfying

$$x(\mu, \lambda) = G(x(\mu, \lambda), \mu, \lambda). \tag{29}$$

For the map $K(\lambda)$ defined by (20) we apply the Walkup-Wets theorem [69] to find an $\theta > 0$ such that if $\lambda, \lambda' \in \Lambda$ and $x \in K(\lambda)$, then there exists $x' \in K(\lambda')$ satisfying

$$\|x' - x\| \leq \theta \|\lambda' - \lambda\|. \tag{30}$$

Since $\bar{x} \in K(\bar{\lambda})$, from (30) it follows that there is a neighborhood V_1 of $\bar{\lambda}$ such that

$$K(\lambda) \cap X \neq \emptyset \quad \text{for every} \quad \lambda \in \Lambda \cap V_1. \tag{31}$$

Since X is a polyhedral convex set we can find a matrix C of order $r_1 \times n$ and a vector $b \in \mathbb{R}^{r_1}$ such that $X = \{x \in \mathbb{R}^n : Cx \geq b\}$. Therefore

$$K(\lambda) \cap X = \{x \in \mathbb{R}^n : Ax \geq \lambda,\ Cx \geq b,\ x \geq 0\}. \tag{32}$$

So, taking (31) into account we can apply Theorem 3.9 for system (32) to choose a constant $k_1 > 0$ such that

$$\|P_{K(\lambda') \cap X} y - P_{K(\lambda) \cap X} y\| \leq k_1 \|\lambda' - \lambda\| \tag{33}$$

for all $y \in \mathbb{R}^n$, λ and λ' in $\Lambda \cap V_1$. (Note that k_1 depends not only on A but also on C, that is, on the neighborhood X.)

Lemma 3.10. (Yen [72]) *Let* (25) *and* (26) *be fulfilled. Assume that* $k_1 > 0$ *is a constant satisfying* (33). *Then for any* $\rho > 0$ *satisfying* (27) *there exist neighborhoods* $\bar{U}$ *and* $\bar{V}$ *of* $\bar{\mu}$ *and* $\bar{\lambda}$, *respectively, such that:*

(i) *For every* $(\mu, \lambda) \in (M \cap \bar{U}) \times (\Lambda \cap \bar{V})$ *vector* $x(\mu, \lambda) \in X$ *defined by* (29) *is the unique solution of* (22) *in* X*;*

(ii) *For all* $\mu,\ \mu' \in M \cap \bar{U}$ *and* $\lambda,\ \lambda' \in \Lambda \cap \bar{V}$,

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \leq \frac{1}{1 - \beta}(\rho l \|\mu' - \mu\| + k_1 \|\lambda' - \lambda\|),$$

where β *is defined in* (28).

This lemma can be proved similarly as Lemma 2.1 in [71]. Note that the scheme given by Dafermos [14, p. 424] is the key tool of our proof.

Theorem 3.11. (Yen [72]) *Let* $\bar{x}$ *be a solution of* (23). *If conditions* (25) *and* (26) *are satisfied, then there exist constants* $k_{\bar{\mu}} > 0$ *and* $k_{\bar{\lambda}} > 0$, *neighborhoods* $\bar{U}$ *of* $\bar{\mu}$ *and* $\bar{V}$ *of* $\bar{\lambda}$ *such that:*

(i) *For every* $(\mu, \lambda) \in (M \cap \bar{U}) \times (\Lambda \cap \bar{V})$ *there exists a unique solution of* (22) *in* X, *denoted by* $x(\mu, \lambda)$;

(ii) *For all* $(\mu', \lambda'), (\mu, \lambda) \in (M \cap \bar{U}) \times (\Lambda \cap \bar{V})$,

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \leq k_{\bar{\mu}} \|\mu' - \mu\| + k_{\bar{\lambda}} \|\lambda' - \lambda\|.$$

Let us consider problem (22) with $K(\lambda)$ defined in the following way:

$$K(\lambda) = \{x \in \mathbb{R}^n \, : \, x = Zh, \ \Gamma h = \lambda, \ h \geq 0\}, \tag{34}$$

where $h \in \mathbb{R}^p, \ \lambda \in \mathbb{R}^r, \ \Gamma$ is an $r \times p$ matrix, Z is an $n \times p$ matrix. This is the variational inequality model for the traffic equilibrium problem (see the above Subsection 3.1). The matrices, the vectors, and the function $f(x, \mu)$ in the model have the following interpretations (see Qiu and Magnanti [55], and Subsection 3.1):

$x =$ vector of flows on arcs, $h =$ vector of flows on paths,

$\Gamma =$ the incidence matrix of the relation "paths - *OD* (origin-destination) pairs",

$Z =$ the incidence matrix of the relation "arcs - paths",

$\lambda =$ vector of demands for the *OD* pairs,

$f(x, \mu) =$ vector of the costs on arcs when the network is loaded with flow x,

$\mu =$ parameter of the perturbation of the costs on arcs.

For a given pair (μ, λ), solutions of (22) are interpreted as the equilibrium flows on the traffic network, corresponding to vector λ of demands and function $f(., \mu)$ of the costs on arcs.

Since $K(\lambda)$ is defined by (34) rather by (20), Theorem 3.9 cannot be applied directly. However, a property like the one in (24) is valid.

Lemma 3.12. (Yen [72]) *Assume that* $K(\lambda)$ *is given by* (34)*,* $H(\lambda) := \{h \in \mathbb{R}^p \, : \, \Gamma h = \lambda, \ h \geq 0\}, \ \Lambda := \{\lambda \in \mathbb{R}^r \, : \, H(\lambda) \neq \emptyset\}$*. Then there exists a constant* $k > 0$ *such that*

$$\|P_{K(\lambda')}y - P_{K(\lambda)}y\| \leq k \, \|\lambda' - \lambda\|,$$

for every $y \in \mathbb{R}^n$ *and* $\lambda, \lambda' \in \Lambda$.

Now, let $K(\lambda)$ be defined by (34) and $\bar{x}$ be a solution of (23). Let the function $f(x, \mu)$ satisfy conditions (25) and (26). Again, we assume that X is a polyhedral convex set and $\alpha < l$. Let $\bar{h} \in H(\bar{\lambda})$ be a vector such that $\bar{x} = Z\bar{h}$. Then $(\bar{x}, \bar{h})$ is a solution at parameter $\lambda = \bar{\lambda}$ of the following system of linear inequalities and equalities

$$x - Zh = 0, \quad \Gamma h = \lambda, \quad h \geq 0. \tag{35}$$

Applying the cited Walkup-Wets theorem we find $\theta > 0$ such that for every $\lambda \in \Lambda$ there exists a solution (x, h) of (35) satisfying

$$\|(x, h) - (\bar{x}, \bar{h})\| \leq \theta \|\lambda - \bar{\lambda}\|.$$

This implies that $x \in K(\lambda)$ and $\|x - \bar{x}\| \le \theta\|\lambda - \bar{\lambda}\|$. Consequently, there is a neighborhood V_1 of $\bar{\lambda}$ such that

$$K(\lambda) \cap X \neq \emptyset \quad \text{for every} \quad \lambda \in \Lambda \cap V_1. \tag{36}$$

Now, let C be a matrix of order $r_1 \times n$ and b be a vector from $\mathbb{R}^{r_1}$, such that $X = \{x \in \mathbb{R}^n : Cx \ge b\}$. We have

$$\begin{aligned} K(\lambda) \cap X &= \{x \in \mathbb{R}^n : x = Zh,\ Cx \ge b,\ \Gamma h = \lambda,\ h \ge 0\} \\ &= \{x \in \mathbb{R}^n : x = Zh,\ CZh \ge b,\ \Gamma h = \lambda,\ h \ge 0\}. \end{aligned}$$

Since this set has the same structure as the one in (34), taking account of (36) we can apply Lemma 3.12 (see also the arguments for proving it in [72]) to find a constant $k > 0$ such that

$$\|P_{K(\lambda') \cap X} y - P_{K(\lambda) \cap X} y\| \le k\|\lambda' - \lambda\| \tag{37}$$

for all $y \in \mathbb{R}^n$ and $\lambda, \lambda' \in \Lambda \cap V_1$.

Using property (37) instead of (33) one can see that Lemma 3.10 (with k_1 being replaced by k) holds for the case where $K(\lambda)$ is given by (34). This gives us the following result.

Theorem 3.13. (Yen [72]) *Let $K(\lambda)$ be defined by* (34) *and $\bar{x}$ be a solution of* (23)*, where $\bar{\mu} \in M \times \Lambda$ is a given pair of parameters. If conditions* (25) *and* (26) *are satisfied, then there exist constants $k_{\bar{\mu}} > 0$ and $k_{\bar{\lambda}} > 0$, neighborhoods $\bar{U}$ of $\bar{\mu}$ and $\bar{V}$ of $\bar{\lambda}$ such that:*

(i) *For every $(\mu, \lambda) \in (M \cap \bar{U}) \times (\Lambda \cap \bar{V})$ there exists a unique solution of* (22) *in X, denoted by $x(\mu, \lambda)$;*

(ii) *For all $(\mu', \lambda'), (\mu, \lambda) \in (M \cap \bar{U}) \times (\Lambda \cap \bar{V})$,*

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \le k_{\bar{\mu}}\|\mu' - \mu\| + k_{\bar{\lambda}}\|\lambda' - \lambda\|.$$

Theorem 3.13 can be interpreted by saying that: *"In a traffic network with locally Lipschitz, locally strongly monotone function of costs on arcs, the equilibrium arcs flow is locally unique and is a locally Lipschitz function of the perturbations of costs on arcs and of the vector of demands."*

We refer to the papers by Robinson and Lu [42, 59, 60] for new and significant results on solution continuity in monotone affine variational inequalities whose convex polyhedral constraint sets are subject to perturbations.

Fresh information about the traffic equilibrium problem together with a new development of sensitivity analysis can be found in the papers by Patriksson and Rockafellar [51, 52, 53].

3.4. The Linear Fractional Vector Optimization Problem

In this subsection, we will follow [35, Chapter 8]. For detailed proofs and further analysis of the subject, we refer to the book [35].

Linear fractional vector optimization (LFVO) is an interesting area in the wider theory of vector optimization; see e.g. [10, 11, 43, 44, 63]. LFVO problems have applications in finance and production management (see Steuer [63]). In a LFVO problem, any point satisfying the first-oder necessary optimality condition is a solution. Therefore, solving a LFVO problem is to solve a monotone affine *vector variational inequality* (VVI, for short). The concept of VVI was introduced by Giannessi [19].

Let $f_i : \mathbb{R}^n \to \mathbb{R} \quad (i = 1, 2, \ldots, m)$ be m *linear fractional functions*, that is

$$f_i(x) = \frac{a_i^T x + \alpha_i}{b_i^T x + \beta_i}$$

for some $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}$, and $\beta_i \in \mathbb{R}$. Let $\Delta = \{x \in \mathbb{R}^n : Cx \leq d\}$, where $C \in \mathbb{R}^{r\times n}$ and $d \in \mathbb{R}^r$, be a nonempty polyhedral convex set. We assume that $b_i^T x + \beta_i > 0$ for all $i \in \{1, \cdots, m\}$ and for all $x \in \Delta$. Define

$$f(x) = (f_1(x), \ldots, f_m(x)), \quad A = (a_1, \ldots, a_m), \quad B = (b_1, \ldots, b_m),$$

$$\alpha = (\alpha_1, \ldots, \alpha_m), \quad \beta = (\beta_1, \ldots, \beta_m), \quad \omega = (A, B, \alpha, \beta).$$

Thus A and B are $n \times m$-matrices, α and β are vectors of $\mathbb{R}^m$, and ω is a parameter containing all the data related to the vector function f.

Consider the following vector optimization problem

(VP) Minimize $f(x)$ subject to $x \in \Delta$.

Definition 3.14. One says that $x \in \Delta$ is an *efficient solution* of (VP) if there exists no $y \in \Delta$ such that $f(y) \leq f(x)$ and $f(y) \neq f(x)$. If there exists no $y \in \Delta$ such that $f(y) < f(x)$, then one says that $x \in \Delta$ is a *weakly efficient solution* of (VP).

Denote the *efficient solution set* and the *weakly efficient solution set* of (VP) by Sol(VP) and Sol^w(VP), respectively.

Let

$$\Lambda = \{\lambda \in \mathbb{R}^m_+ : \sum_{i=1}^m \lambda_i = 1\}.$$

Then the relative interior of Δ is given by

$$\mathrm{ri}\Lambda = \{\lambda \in \mathbb{R}^m_+ : \sum_{i=1}^m \lambda_i = 1, \ \lambda_i > 0 \ \text{ for all } \ i\}.$$

Theorem 3.15. (See [43]) *Let $x \in \Delta$. The following assertions hold:*

(i) $x \in \mathrm{Sol(VP)}$ *if and only if there exists* $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathrm{ri}\Lambda$ *such that*

$$\left\langle \sum_{i=1}^{m} \lambda_i \left[\left(b_i^T x + \beta_i\right) a_i - \left(a_i^T x + \alpha_i\right) b_i \right], y - x \right\rangle \geq 0, \quad \forall y \in \Delta. \tag{38}$$

(ii) $x \in \mathrm{Sol}^w(\mathrm{VP})$ *if and only if there exists* $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda$ *such that* (38) *holds.*

(iii) *Condition* (38) *is satisfied if and only if there exists* $\mu = (\mu_1, \ldots, \mu_r)$, $\mu_j \geq 0$ *for all* $j = 1, \ldots, r$, *such that*

$$\sum_{i=1}^{m} \lambda_i \left[\left(b_i^T x + \beta_i\right) a_i - \left(a_i^T x + \alpha_i\right) b_i \right] + \sum_{j \in I(x)} \mu_j C_j^T = 0, \tag{39}$$

where C_j *denotes the* j*-th row of the matrix* C *and* $I(x) = \{j : C_j x = d_j\}$.

In fact, (39) is a Fermat equation for (VP) in the Lagrange form. (The Lagrange multipliers λ_i are introduced because we have deal with the *vector-valued* objective function $f(x)$, while the Lagrange multipliers μ_j help us to take account of the inequality constraint $Cx \leq d$.) Theorem 3.15 is known as the *Lagrange Multiplier Rule* for (VP). It has origin in the Fermat Rule (see Theorem 2.2).

Condition (38) can be rewritten in the form of a parametric affine variational inequality

$$(\mathrm{VI})_\lambda \qquad \langle M(\lambda)x + q(\lambda), y - x \rangle \geq 0 \quad \forall y \in \Delta,$$

where

$$M(\lambda) = \left(M_{kj}(\lambda)\right),$$

$$M_{kj}(\lambda) = \sum_{i=1}^{m} \lambda_i \left(b_{i,j} a_{i,k} - a_{i,j} b_{i,k}\right), \quad 1 \leq k \leq n, \ 1 \leq j \leq n,$$

$$q(\lambda) = \left(q_k(\lambda)\right), \quad q_k(\lambda) = \sum_{i=1}^{m} \lambda_i \left(\beta_i a_{i,k} - \alpha_i b_{i,k}\right), \quad 1 \leq k \leq n,$$

$a_{i,k}$ and $b_{i,k}$ are the $k-$th components of a_i and b_i, respectively.

It is clear that $\left(M(\lambda)\right)^T = -M(\lambda)$. Therefore, $\langle M(\lambda)v, v \rangle = 0$ for every $v \in R^n$. In particular, $M(\lambda)$ is a positive semidefinite matrix; hence $(\mathrm{VI})_\lambda$ is a monotone AVI for every $\lambda \in \Lambda$. Denote by $F(\lambda)$ the solution set of the problem $(\mathrm{VI})_\lambda$. Consider the multifunction $F : \Lambda \rightrightarrows \mathbb{R}^n, \ \lambda \mapsto F(\lambda)$. According to Theorem 3.15,

$$\mathrm{Sol}(\mathrm{VP}) = \bigcup_{\lambda \in \mathrm{ri}\Lambda} F(\lambda) = F(\mathrm{ri}\Lambda), \tag{40}$$

$$\mathrm{Sol}^w(\mathrm{VP}) = \bigcup_{\lambda \in \Lambda} F(\lambda) = F(\Lambda). \tag{41}$$

By the results and the terminology in [34, 41] we can say that solving problem (VP) is equivalent to solve the monotone affine VVI defined by Δ and the affine functions

$$g_i(x) = \big(b_i^T x + \beta_i\big)a_i - \big(a_i^T x + \alpha_i\big)b_i \quad (i = 1, 2, \ldots, m).$$

Thus the first-order optimality condition of a LFVO problem can be treated as a special vector variational inequality.

According to the Minty lemma [33], $F(\lambda)$ is closed and convex. If Δ is compact, then $F(\lambda)$ is nonempty by the Hartman-Stampacchia theorem [22]. Using formulae (40), (41), and Robinson's stability theorem for monotone AVIs [56], we can establish the following results.

Theorem 3.16. (Benoist [7], Yen and Phuong [77]) *If Δ is compact, then* Sol(VP) *is a connected set.*

Theorem 3.17. (Choo and Atkins [11]) *If Δ is compact then* Sol^w(VP) *is a connected set.*

In fact, Choo and Atkins [11] established a stronger result: *If Δ is compact then* Sol^w(VP) *is connected by line segments.* The latter means that for any points $x, y \in \mathrm{Sol}^w(\mathrm{VP})$ there exists a finite sequence of points $x_0 = x, x_1, \ldots, x_k = y$ such that each line segment $[x_j, x_{j+1}]$ $(j = 0, 1, \ldots, k-1)$ is a subset of $\mathrm{Sol}^w(\mathrm{VP})$.

If Δ is unbounded, then Sol(VP) and $\mathrm{Sol(VP)}^w$ may be disconnected.

Due to (40) and (41), solution stability of (VP) can be studied by applying suitable stability results on monotone AVIs.

3.5. Stability of AVIs

We denote the problem

$$\text{Find } \bar{x} \in \Delta \text{ such that } \langle M\bar{x} + q, y - \bar{x}\rangle \geq 0 \ \ \forall y \in \Delta \tag{42}$$

by AVI(M, q, Δ). Here $M \in \mathbb{R}^{n\times n}$, $q \in \mathbb{R}^n$, and Δ is a nonempty polyhedral convex set in $\mathbb{R}^n$. Let $m \in \mathbb{N}$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ be such that

$$\Delta = \{x \in \mathbb{R}^n \,:\, Ax \geq b\}. \tag{43}$$

If (43) is valid, we can denote AVI(M, q, Δ) by AVI(M, A, q, b) to stress the fact that the AVI under consideration is a *parametric problem* depending on the parameter quadruplet $\{M, A, q, b\}$. As for other models, solution existence theorems are vital for stability analysis of AVIs. The proof of the forthcoming theorem can be found in [20] and also in [35].

Theorem 3.18. (Gowda and Pang [20, p. 432]) *If the following two conditions are satisfied*

(i) there exists $\bar{x} \in \Delta$ such that $(M\bar{x} + q)^T v \geq 0$ for every $v \in 0^+\Delta$;

(ii) $(y - x)^T M(y - x) \geq 0$ for all $x \in \Delta$ and $y \in \Delta$;

then the solution set Sol(AVI(M, q, Δ)) *is nonempty.*

Definition 3.19. (cf. [12, p. 176]) By abuse of terminology, we say that a matrix $M \in \mathbb{R}^{n\times n}$ is *monotone* on a closed convex set $\Delta \subset \mathbb{R}^n$ if the linear operator corresponding to M is monotone on Δ, that is

$$(y-x)^T M(y-x) \geq 0 \;\; \forall x \in \Delta, \; \forall y \in \Delta.$$

Matrix M is said to be *copositive* on Δ if

$$v^T Mv \geq 0 \;\; \forall v \in 0^+\Delta,$$

where $0^+\Delta := \{v \in \mathbb{R}^n : \exists x \in \Delta, x + tv \in \Delta \; \forall t > 0\}$ denotes the recession cone of the convex set Δ. If M is copositive on $\mathbb{R}^n_+$ then one simply says that M is a *copositive matrix*. Matrix M is said to be *strictly copositive* on Δ if

$$v^T Mv > 0 \;\; \forall v \in 0^+\Delta \setminus \{0\}.$$

Theorem 3.20. ([35, Theorem 6.3]) *If M is strictly copositive on a nonempty polyhedral convex set Δ, then for any $q \in \mathbb{R}^n$, problem* AVI(M, q, Δ) *has a solution.*

In the case where Δ is a cone, we have

Theorem 3.21. ([35, Theorem 6.5]) *Assume that Δ is a polyhedral convex cone. If M is copositive on Δ and*

$$q \in \mathrm{int}\big([\mathrm{Sol}(\mathrm{AVI}(M, 0, \Delta))]^+\big),$$

where $K^+ = \{\xi \in \mathbb{R}^n : \langle \xi, v\rangle \geq 0 \; \forall v \in K\}$ is the positive dual cone of a set $K \subset \mathbb{R}^n$ and intΩ *denotes the interior of Ω, then problem* AVI(M, q, Δ) *has a solution.*

We now describe some results on local *upper-Lipschitz continuity* of the multifunction $\Phi : \mathbb{R}^{n\times n} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by the formula

$$\Phi(M, q) = \mathrm{Sol}(\mathrm{AVI}(M, q, \Delta)).$$

First we consider the case where Δ is a polyhedral convex cone. The next theorem is a generalization of Theorem 7.5.1 in [12], where the case $\Delta = \mathbb{R}^n_+$ (that is, the AVI (42) is a parametric linear complementarity problem) was considered.

Theorem 3.22. ([35, Theorem 7.4]) *Suppose that $\Delta \subset \mathbb{R}^n$ is a polyhedral convex cone. Suppose that $M \in \mathbb{R}^{n\times n}$ is a given matrix and $q \in \mathbb{R}^n$ is a given vector. If M is copositive on Δ and*

$$q \in \mathrm{int}\left([\mathrm{Sol}(\mathrm{AVI}(M, 0, \Delta))]^+\right),$$

then there exist constants $\varepsilon > 0$, $\delta > 0$ and $\ell > 0$ such that if $(\widetilde{M}, \widetilde{q}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$, $\widetilde{M}$ is copositive on Δ, and if

$$\max\{\|\widetilde{M} - M\|,\ \|\widetilde{q} - q\|\} < \varepsilon,$$

then the set $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta))$ *is nonempty,*

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) \subset \bar{B}(0, \delta),$$

and

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) \subset \mathrm{Sol}(\mathrm{AVI}(M, q, \Delta)) + \ell(\|\widetilde{M} - M\| + \|\widetilde{q} - q\|)\bar{B}(0, 1).$$

The following theorem on AVI problems with positive semidefinite matrices is one of the best-known stability results for generalized equations. Unlike Theorem 3.22, here Δ is not assumed to be a cone. One can observe that Theorem 3.23 and Theorem 3.22 are independent results.

Theorem 3.23. ([56, Theorem 2]) *Let* $M \in \mathbb{R}^{n\times n}$ *be a positive semidefinite matrix,* Δ *a nonempty polyhedral convex set in* $\mathbb{R}^n$*, and* $q \in \mathbb{R}^n$*. Then the following two properties are equivalent:*

(i) *The solution set* $\mathrm{Sol}(\mathrm{AVI}(M, q, \Delta))$ *is nonempty and bounded;*

(ii) *There exists* $\varepsilon > 0$ *such that for each* $\widetilde{M} \in \mathbb{R}^{n\times n}$ *and each* $\widetilde{q} \in \mathbb{R}^n$ *with*

$$\max\{\|\widetilde{M} - M\|,\ \|\widetilde{q} - q\|\} < \varepsilon,$$

the set $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta))$ *is nonempty.*

When (i) *is valid, there exist constants* $\varepsilon > 0,\ \delta > 0$ *and* $\ell > 0$ *such that if* $(\widetilde{M}, \widetilde{q}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$*,* $\widetilde{M}$ *is positive semidefinite, and if*

$$\max\{\|\widetilde{M} - M\|,\ \|\widetilde{q} - q\|\} < \varepsilon,$$

then the set $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta))$ *is nonempty,*

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) \subset \bar{B}(0, \delta),$$

and

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, \Delta)) \subset \mathrm{Sol}(\mathrm{AVI}(M, q, \Delta)) + \ell(\|\widetilde{M} - M\| + \|\widetilde{q} - q\|)\bar{B}(0, 1).$$

An 'elementary' proof (in comparison with the original one in [56]) of Theorem 3.23 was given by Lee, Tam and Yen [35, Chap. 7].

Besides the continuity properties discussed above (Hölder continuity, Lipschitz continuity, pseudo-Lipschitz continuity, upper Lipschitz continuity), in set-valued and variational analysis there are two other important continuity concepts of multifunctions called upper semicontinuity and lower semicontinuity.

Definition 3.24. a) (Upper semicontinuity of multifunctions) A multifunction $F : W \rightrightarrows \mathbb{R}^n$, where W is a subset of an Euclidean space, is said to be upper semicontinuous (usc) at $\omega \in W$ if for each open set $\Omega \subset \mathbb{R}^n$ satisfying $F(\omega) \subset \Omega$, there exists $\delta > 0$ such that $F(\omega') \subset \Omega$ for every $\omega' \in W$ with the property that $\|\omega' - \omega\| < \delta$.

b) (Lower semicontinuity of multifunctions) We say that F is lower semicontinuous (lsc) at $\omega \in W$ if for each open set $\Omega \subset \mathbb{R}^n$ satisfying $F(\omega) \cap \Omega \neq \emptyset$, there exists $\delta > 0$ such that $F(\omega') \cap \Omega \neq \emptyset$ for every $\omega' \in W$ with the property that $\|\omega' - \omega\| < \delta$.

c) (Continuity of multifunctions) If F is simultaneously usc and lsc at ω, we say that it is continuous at ω.

Upper semicontinuity of F at ω indicates that the value sets of the restriction of F to a neighborhood of ω have an *external stability* (they do not 'explode'). The presence of the lower semicontinuity of F at ω assures us that the value sets possess an *internal stability* (they do not 'disappear'). Discussions of the various implications of the lsc, usc properties, and other types of continuity properties of multifunctions can be found in [74].

The multifunction $(M, A, q, b) \mapsto \mathrm{Sol}(M, A, q, b)$ is called the *solution map* of (42)–(43) and is abbreviated to $\mathrm{Sol}(\cdot)$. For a fixed pair $(q, b) \in \mathbb{R}^n \times \mathbb{R}^m$, the symbol $\mathrm{Sol}(\cdot, \cdot, q, b)$ stands for the multifunction $(M, A) \mapsto \mathrm{Sol}(M, A, q, b)$. Similarly, for a fixed pair $(M, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{m\times n}$, the symbol $\mathrm{Sol}(M, A, \cdot, \cdot)$ stands for the multifunction $(q, b) \mapsto \mathrm{Sol}(M, A, q, b)$.

Solutions of an AVI problem can be characterized via Lagrange multipliers (see [35, Theorem 5.3]). Namely, $x \in \mathbb{R}^n$ is a solution of $\mathrm{AVI}(M, A, q, b)$ if and only if there exists $\lambda \in \mathbb{R}^m$ such that

$$Mx - A^T\lambda + q = 0, \quad Ax \geq b, \quad \lambda \geq 0, \quad \langle \lambda, Ax - b\rangle = 0. \tag{44}$$

Vector $\lambda \in \mathbb{R}^m$ satisfying (44) is called a *Lagrange multiplier* corresponding to x.

The next theorem gives a necessary condition for the upper semicontinuity of the multifunction $\mathrm{Sol}(\cdot, \cdot, q, b)$ and the solution map $\mathrm{Sol}(\cdot)$.

Theorem 3.25. ([39]) *Let* $(M, A, q, b) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{m\times n} \times \mathbb{R}^n \times \mathbb{R}^m$. *Suppose that the solution set* $\mathrm{Sol}(M, A, q, b)$ *is bounded. Then the following statements are valid:*

(i) *If* $\mathrm{Sol}(\cdot, \cdot, q, b)$ *is usc at* (M, A)*, then*

$$\mathrm{Sol(M, A, 0, 0)} = \{0\}. \tag{45}$$

(ii) *If* $\mathrm{Sol}(\cdot)$ *is usc at* (M, A, q, b)*, then* (45) *is valid.*

We can characterize condition (45) as follows.

Proposition 3.26. ([39], cf. [20, Proposition 3]) *Condition* (45) *holds if and only if for every* $(q, b) \in \mathbb{R}^n \times \mathbb{R}^m$ *the set* $\mathrm{Sol}(M, A, q, b)$ *is bounded.*

The following proposition shows that, for a given pair $(M, A) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{m\times n}$, for almost all $(q, b) \in \mathbb{R}^n\times\mathbb{R}^m$ the set $\mathrm{Sol}(M, A, q, b)$ is bounded (may be empty).

Proposition 3.27. (cf. [48, Lemma 1]) *Let* $(M, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{m\times n}$*. The set*

$$W = \big\{(q, b) \in \mathbb{R}^n \times \mathbb{R}^m : \ \mathrm{Sol}(M, A, q, b) \text{ is bounded}\big\} \tag{46}$$

is of full Lebesgue measure in $\mathbb{R}^n \times \mathbb{R}^m$.

The forthcoming theorem provides sufficient conditions for the upper semicontinuity of $\mathrm{Sol}(\cdot)$.

Theorem 3.28. ([39]) *Let* $(M, A, q, b) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{m\times n} \times \mathbb{R}^n \times \mathbb{R}^m$,

$$K^- := \{v \in \mathbb{R}^n \ : \ \langle Mv, v\rangle \le 0, \quad Av \ge 0\},$$

$$K^+ := \{v \in \mathbb{R}^n \ : \ \langle Mv, v\rangle \ge 0, \quad Av \ge 0\}.$$

The following assertions are valid:

(i) *If the system* $Ax \ge 0$ *is regular (i.e., there exists* $x^0 \in \mathbb{R}^n$ *such that* $Ax^0 > 0$*), then* (45) *holds if and only if for every* $(q, b) \in \mathbb{R}^n \times \mathbb{R}^m$ *the solution map* $\mathrm{Sol}(\cdot)$ *is usc at* (M, A, q, b)*;*

(ii) *If* $K^- = \{0\}$ *and the system* $Ax \ge b$ *is regular (i.e., there exists* $x^0 \in \mathbb{R}^n$ *such that* $Ax^0 > b$*), then for any* $q \in \mathbb{R}^n$*, the solution map* $\mathrm{Sol}(\cdot)$ *is usc at* (M, A, q, b)*;*

(iii) *If* $K^+ = \{0\}$ *and the system* $Ax \ge -b$ *is regular, then for any* $q \in \mathbb{R}^n$*, the solution map* $\mathrm{Sol}(\cdot)$ *is usc at* (M, A, q, b).

We are going to describe necessary and sufficient conditions for the lower semicontinuity of the solution map $\mathrm{Sol}(M, A, \cdot, \cdot)$ in parametric AVI problems.

Theorem 3.29. ([39]) *Let* $(M, A, c, b) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{m\times n} \times \mathbb{R}^n \times \mathbb{R}^m$*. If the multifunction* $\mathrm{Sol}(M, A, \cdot, \cdot)$ *is lower semicontinuous at* (q, b) *then*

(a) *the set* $\mathrm{Sol}(M, A, q, b)$ *is finite, and*

(b) *the system* $Ax \ge b$ *is regular.*

By definition [12], a square matrix is called a *P-matrix* if the determinant of each of its principal submatrices is positive. For any subset $J \subset I := \{1, 2, \dots, m\}$, by A_J we denote the submatrix of A consisting of the rows A_j with $j \in J$.

Theorem 3.30. ([39]) *Let* $(M, A, q, b) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{m\times n} \times \mathbb{R}^n \times \mathbb{R}^m$*. Suppose that*

(i) *the set* $\mathrm{Sol}(M, A, q, b)$ *is finite, nonempty,*

(ii) *the system* $Ax \ge b$ *is regular,* *and suppose that for every* $x \in \mathrm{Sol}(M, A, q, b)$ *there exists a Lagrange multiplier* λ *corresponding to* x *such that at least one of the following conditions holds:*

(c1) $v^T M v \geq 0$ *for every* $v \in \mathbb{R}^n$ *with* $A_{I_0} v \geq 0$ *and* $(Mx + q)^T v = 0$,

(c2) $J = K = \emptyset$,

(c3) $J = \emptyset,\ K \neq \emptyset$, *and the system* $\{A_i : i \in K\}$ *is linearly independent,*

(c4) $J \neq \emptyset,\ K = \emptyset$, M *is nonsingular and* $A_J M^{-1} A_J^T$ *is a* P*-matrix,*

where K, J, I_0 *are defined via* (x, λ) *by setting* $K = \{i \in I : A_i x = b_i,\ \lambda_i > 0\}$, $J = \{i \in I : A_i x = b_i,\ \lambda_i = 0\}$ *and* $I_0 = \{i \in I : A_i x = b_i\} = K \cup J$. *Then the multifunction* $\mathrm{Sol}(M, A, \cdot, \cdot)$ *is lsc at* (q, b).

We refer to [39] for examples showing how (c1)–(c4) can be realized for parametric AVIs.

3.6. Stability of Homogeneous Equilibrium Problems

A very general concept of equilibrium problem was suggested by Blum and Oettli [8]. Oettli and Yen have studied the continuity of the solution set of homogeneous equilibrium problems and linear complementarity problems in [48, 49, 50]. The interested reader is referred to these papers for the formulation of the obtained results. Here we just mention one fact related to the upper semicontinuity of the solution map in LCPs.

Consider the solution map $\mathrm{Sol}(\cdot)$ of (6). As in [12], if $\mathrm{Sol}(M, 0) = \{0\}$ then we say that M is an R_0-matrix. By a result of Jansen and Tijs [25], if M is an R_0-matrix, then for any $q \in \mathbb{R}^n$, the solution map $\mathrm{Sol}(\cdot)$ of (6) is usc at (M, q). A natural question arises: *Does the converse statement hold true?* Taking $M = [0] \in \mathbb{R}^{n \times n},\ q = 0 \in \mathbb{R}^n,\ n \geq 2$, we check at once that $\mathrm{Sol}(\cdot)$ is usc at (M, q), but M is not an R_0-matrix. However, the following proposition is valid.

Proposition 3.31. (see [48]) *Let* $M \in \mathbb{R}^{n \times n},\ q \in \mathbb{R}^n$ *be such that the solution set* $\mathrm{Sol}(M, q)$ *of* (6) *is bounded. If the solution map* $\mathrm{Sol}(\cdot, q)$ *is usc at* M, *then* M *is an* R_0*-matrix.*

Using the Baire lemma, Oettli and Yen [48, Lemma 1] showed that for any $M \in \mathbb{R}^{n \times n}$ there exists $q \in \mathbb{R}^n$ such that $\mathrm{Sol}(M, q)$ is bounded (may be empty). Thus, from Proposition 3.31 it follows that if $\mathrm{Sol}(\cdot, q)$ is usc at M for all $q \in \mathbb{R}^n$ then M is an R_0-matrix.

The method of proving Proposition 3.31 has been used in [48, 49] to obtain *necessary conditions* for the upper semicontinuity of the solution map of homogeneous nonlinear equilibrium problems, and homogeneous quasi-equilibrium problems. It can be applied for studying stability of parametric AVIs (see [39] and the above Subsection 3.4). Besides, this method has enabled Tam and Yen [68] to perform a detailed stability analysis of the Karush-Kuhn-Tucker map of canonical (indefinite, nonconvex) quadratic programming problem. The latter, in its turn, has paved a way for a comprehensive study of various continuity properties of the local solution map and the global solution map, continuity and directional differentiability of the optimal value function in (indefinite, nonconvex) quadratic programming; see [35, 37, 38, 39, 54, 66, 67] and the next section for more details. Due to the specific structure of the problems under consideration, one can establish not only sufficient stability conditions (as usual), but also necessary conditions for stability. The work of Fang and Huang [17, 18] showed that the method of [48] is useful for studying the solution map of the vertical implicit homogeneous complementarity problem, homogeneous vector quasi-equilibrium problems.

4. Parametric Optimization Problems

Due to the limited space we will restrict ourselves to quadratic programming (QP for brevity) under linear constraints. The interested reader is referred to [40] for stability results on (indefinite, nonconvex) QPs under a quadratic constraint, to [26, 46, 73] for stability analysis of the solutions sets of generalized inequality systems and of the optimal value functions in general nonlinear programming problems.

Here we study QP problems of the canonical form:

$$\begin{cases} \text{Minimize} \quad f(x) := \dfrac{1}{2}x^T Dx + c^T x \\ \text{subject to} \quad x \in \Delta(A, b) := \{x \in R^n \,:\, Ax \geq b,\ x \geq 0\}, \end{cases} \tag{47}$$

where $D \in \mathbb{R}^{n\times n}_S$, $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are given data. Problem (47) will be also referred to as $QP(D, A, c, b)$.

Recall that $\bar{x} \in \mathbb{R}^n$ is a *Karush-Kuhn-Tucker point* of (47) if there exists a vector $\bar{\lambda} \in \mathbb{R}^m$ such that

$$\begin{cases} D\bar{x} - A^T\bar{\lambda} + c \geq 0, \quad A\bar{x} - b \geq 0, \\ \bar{x} \geq 0, \quad \bar{\lambda} \geq 0, \\ \bar{x}^T(D\bar{x} - A^T\bar{\lambda} + c) + \bar{\lambda}^T(A\bar{x} - b) = 0. \end{cases} \tag{48}$$

The set of all the Karush-Kuhn-Tucker points of (47) is denoted by $S(D, A, c, b)$. We may consider (48) as the Fermat equation for $QP(D, A, c, b)$ in the Lagrange form (a Lagrange multiplier is introduced to deal with the inequality constraint $Ax \geq b$).

If $\bar{x}$ is a local solution of (47) then $\bar{x} \in S(D, A, c, b)$. This fact leads to the following standard way to solve (47): *Find first the set* $S(D, A, c, b)$ *then compare the values* $f(x)$ *among the points* $x \in S(D, A, c, b)$. Hence, it is of interest to have some criteria for the (semi)continuity of the multifunction $(D, A, c, b) \mapsto S(D, A, c, b)$.

The next statement gives a necessary condition for $S(\cdot, \cdot, c, b)$ to be upper semicontinuous at a given pair $(D, A) \in \mathbb{R}^{n\times n}_S \times \mathbb{R}^{m\times n}$.

Theorem 4.1. (Tam and Yen [68]) *Assume that the set $S(D,A,c,b)$ is bounded. If the multifunction $S(\cdot,\cdot,c,b)$ is usc at (D,A), then*

$$S(D,A,0,0)=\{0\}. \tag{49}$$

In general, (49) is not a sufficient condition for the upper semicontinuity of $S(\cdot)$ at (D,A,c,b).

Example 4.2. Consider the problem $QP(D,A,c,b)$ where

$$D=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},\quad A=[0,\ -1],\quad b=(-1),\quad c=(0,0).$$

For each $t\in(0,1)$, let $A_t=[-t,\ -1]$. By direct computation using (48) we obtain

$$\begin{aligned}&S(D,A,0,0)=\{0\},\quad S(D,A,c,b)=\{(0,0),\ (0,1)\},\\ &S(D,A_t,c,b)=\left\{(0,0),\ (0,1),\ \left(\frac{1}{t},0\right),\ \left(\frac{t}{t^2+1},\frac{1}{t^2+1}\right)\right\}.\end{aligned}$$

Thus, for any bounded open set $\Omega\subset\mathbb{R}^2$ containing $S(D,A,c,b)$, the inclusion $S(D,A_t,c,b)\subset\Omega$ fails to hold for $t>0$ small enough. Since $A_t\to A$ as $t\to 0$, $S(\cdot)$ cannot be usc at (D,A,c,b).

We are going to present some sufficient conditions for the usc property of the multifunction $S(\cdot)$. For a matrix $A\in\mathbb{R}^{m\times n}$, the dual of the cone

$$\Lambda[A]:=\{\lambda\in\mathbb{R}^m\ :\ -A^T\lambda\geq 0,\ \lambda\geq 0\}$$

is denoted by $(\Lambda[A])^*$. By definition, $(\Lambda[A])^*=\{\xi\in\mathbb{R}^m\ :\ \lambda^T\xi\leq 0\ \ \forall\lambda\in\Lambda[A]\}$. The interior of $(\Lambda[A])^*$ is denoted by $\operatorname{int}(\Lambda[A])^*$. Note that

$$\operatorname{int}(\Lambda[A])^*=\{\xi\in\mathbb{R}^m\ :\ \lambda^T\xi<0\quad\forall\lambda\in\Lambda[A]\setminus\{0\}\}.$$

The desired usc property depends greatly on the behavior of the quadratic form x^TDx on the recession cone of $\Delta(A,b)$ and also on the position of b with respect to the set $\operatorname{int}(\Lambda[A])^*$.

Theorem 4.3. (Tam [65]) *The multifunction $S(\cdot)$ is usc at (D,A,c,b) if one of the following conditions holds:*

(i) $\operatorname{Sol}(D,A,0,0)=\{0\}$, $b\in\operatorname{int}(\Lambda[A])^*$, *and $c\in\mathbb{R}^n$ is given arbitrarily;*

(ii) $\operatorname{Sol}(-D,A,0,0)=\{0\}$, $b\in-\operatorname{int}(\Lambda[A])^*$, *and $c\in\mathbb{R}^n$ is given arbitrarily;*

(iii) $S(D,A,0,0)=\{0\}$, $\Lambda[A]=\{0\}$, *and $(c,b)\in\mathbb{R}^n\times\mathbb{R}^m$ is given arbitrarily.*

Consider the canonical QP problem of the form (47). The next statement gives a necessary condition for the lower semicontinuity of the multifunction $S(\cdot)$.

Theorem 4.4. (Tam and Yen [68]) *Let $D \in \mathbb{R}^{n\times n}_S$ and $A \in \mathbb{R}^{m\times n}$ be given. If the multifunction $S(D, A, \cdot, \cdot)$ is lsc at $(c, b) \in \mathbb{R}^n \times \mathbb{R}^m$, then the set $S(D, A, c, b)$ is finite.*

The following examples show that the finiteness of $S(D, A, c, b)$ may not be sufficient for the multifunction $S(\cdot)$ to be lsc at (D, A, c, b).

Example 4.5. Consider the problem (P_ε) of minimizing the function

$$f_\varepsilon(x) = -\frac{1}{2}x_1^2 - x_2^2 + x_1 - \varepsilon x_2$$

on the set $\Delta = \{x \in \mathbb{R}^2 : x \geq 0, -x_1 - x_2 \geq -2\}$. Note that Δ is a compact set with nonempty interior. Denote by $S(\varepsilon)$ the KKT point set of (P_ε). A direct computation using (48) gives

$$S(0) = \left\{(0,0), (1,0), (2,0), \left(\frac{5}{3}, \frac{1}{3}\right), (0,2)\right\},$$

and

$$S(\varepsilon) = \left\{(2,0), \left(\frac{5+\varepsilon}{3}, \frac{1-\varepsilon}{3}\right), (0,2)\right\}$$

for $\varepsilon > 0$ small enough. For $U := \{x \in \mathbb{R}^2 : \frac{1}{2} < x_1 < \frac{3}{2},\ -1 < x_2 < 1\}$ we have $S(\varepsilon) \cap U = \emptyset$ for every $\varepsilon > 0$ small enough. Meanwhile, $S(0) \cap U = \{(1,0)\}$. Hence the multifunction $\varepsilon \mapsto S(\varepsilon)$ is not lsc at $\varepsilon = 0$.

Example 4.6. Consider the problem $(\widetilde{P}_\varepsilon)$ of minimizing the function

$$\widetilde{f}_\varepsilon(x) = \frac{1}{2}x_1^2 - x_2^2 - x_1 - \varepsilon x_2$$

on the set $\Delta = \{x \in R^2 : x \geq 0, -x_1 - x_2 \geq -2\}$. Denote by $\widetilde{S}(\varepsilon)$ the KKT point set of $(\widetilde{P}_\varepsilon)$. Using (48) we can show that $\widetilde{S}(0) = \{(1,0), (0,2)\}$, and $\widetilde{S}(\varepsilon) = \{(0,2)\}$ for every $\varepsilon > 0$. For $U := \{x \in \mathbb{R}^2 : \frac{1}{2} < x_1 < \frac{3}{2},\ -1 < x_2 < 1\}$ we have $\widetilde{S}(0) \cap U = \{(1,0)\}$, but $\widetilde{S}(\varepsilon) \cap U = \emptyset$ for every $\varepsilon > 0$. Hence the multifunction $\varepsilon \mapsto \widetilde{S}(\varepsilon)$ is not lsc at $\varepsilon = 0$.

In the KKT point set $S(D, A, c, b)$ of (47) we distinguish three types of elements:
(1) Local solutions of $QP(D, A, c, b)$;
(2) Local solutions of $QP(-D, A, -c, b)$ which are not local solutions of $QP(D, A, c, b)$;
(3) Points of $S(D, A, c, b)$ which do not belong to the first two classes.

Elements of the first type (of the second type, of the third type) are called, respectively, the *local minima*, the *local maxima*, and the *saddle points* of (47).

In Example 4.5, $(1, 0) \in S(0)$ is a local maximum of (P_0) which lies on the boundary of Δ. Similarly, in Example 4.6, $(1, 0) \in \widetilde{S}(0)$ is a saddle point of $\widetilde{P}_0$

which lies on the boundary of Δ. If such situations do not happen, then the KKT point set is lower semicontinuous at the given parameter.

Theorem 4.7. (Tam and Yen [68]) *Assume that the inequality system $Ax \geq b,\ x \geq 0$ is regular. If the set $S(D, A, c, b)$ is nonempty, finite, and in $S(D, A, c, b)$ there exist no local maxima and no saddle points of* (47) *which are on the boundary of $\Delta(A, b)$, then the multifunction $S(\cdot)$ is lsc at (D, A, c, b).*

Being unable to formulate more results on stability of (indefinite, nonconvex) QPs under linear constraints, we observe that

- continuity of the solution map and directional differentiability of the optimal value function have been studied by Tam in [64] and [66], respectively;

- continuity properties of the local solution set were obtained by Phu and Yen [54];

- directional differentiability and piecewise linear-quadratic property of the optimal value function in a linearly perturbed QP problem were investigated by Lee, Tam and Yen [36];

- lower semicontinuity of the KKT point set in quadratic programs under linear perturbations has been discussed in [37];

- continuity of the solution map in QP problems under linear perturbations was studied in [38].

5. Some Recent Research Work on Variational Inequalities, Optimization Problems, Set-Valued and Variational Analysis

A brief information about recent results of the author and his coauthors on optimization problems, variational inequalities, set-valued and variational analysis is given in this section. The titles quoted below are not included in the list of references of this paper.

Parametric variational inequalities:

1. J.-C. Yao and N. D. Yen, Coderivative calculation related to a parametric affine variational inequality. Part 1: Basic calculations, *Acta Mathematica Vietnamica* **34** (1) (2009), 157–172.

2. J.-C. Yao and N. D. Yen, Coderivative calculation related to a parametric affine variational inequality. Part 2: Applications. (Accepted for publication in *Pacific Journal of Optimization.*)

Parametric optimization problems:

3. G. M. Lee, N. N. Tam and N. D. Yen, Stability of a class of quadratic programs with a conic constraint. (Submitted)

4. T. D. Chuong, J.-C. Yao and N. D. Yen, Further results on the lower semicontinuity of efficient point multifunctions, Institute of Mathematics, Vietnamese Academy of Science and Technology, E-Preprint No. 2008/03/02. (Accepted for publication in *Pacific Journal of Optimization.*)

Numerical methods for variational inequalities and optimization problems:

5. N. N. Tam, J.-C. Yao, and N. D. Yen, Solution methods for pseudomonotone variational inequalities, *J. Optim. Theory Appl.* **138** (2008), 253–273.

6. H. A. Le Thi, T. Pham Dinh, and N. D. Yen, Behavior of DCA sequences for solving the trust-region subproblem, Institute of Mathematics, Vietnamese Academy of Science and Technology, E-Preprint No. 2008/03/01. (Submitted)

7. H. A. Le Thi, T. Pham Dinh, and N. D. Yen, Properties of two DC algorithms in quadratic programming. (Submitted)

Set-valued and variational analysis:

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