Vietnam Journal of Mathematics **37**:2&3(2009) 225-236

Vietnam Journal
of
MATHEMATICS

A Note on Uniqueness Polynomials of Entire Functions*

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Received January 12, 2009
Revised May 10, 2009

Abstract. A complex polynomial P is a strong uniqueness polynomial for entire functions if one cannot find two distinct non-constant entire functions f and g and a none-zero constant c such that $P(f) = cP(g)$. It follows rather easily from Picard's theorem that $P(X)$ is a strong uniqueness polynomial for entire functions if and only if none of the two variable polynomials $P(X) - cP(Y)$ for all complex numbers $c \neq 0$ have linear or quadratic factors except for the linear factor $(X - Y)$ when $c = 1$ (cf. [6]). In this note, we show that if $P(X)$ is injective on the zeros of $P'(X)$, then $P(X)$ is a strong uniqueness polynomial for entire functions if and only if $\deg P \geq 4$ and none of the two variable polynomials $P(X) - cP(Y)$ for all complex numbers $c \neq 0$ have linear factors except for the linear factor $(X - Y)$ when $c = 1$.

2000 Mathematics Subject Classification: Primary 30D35, Secondary 30D05.
Key words: Uniqueness polynomials, entire functions.

1. Introduction

A polynomial P is called a *strong uniqueness polynomial* for a family of functions $\mathcal{F}$ if whenever there exist two non-constant functions $f, g \in \mathcal{F}$ and a complex

* Financial support provided to the first author by ICTP and by Alexander von Humboldt Foundation.

constant c such that $P(f) = cP(g)$, then we must have $c = 1$ and $f = g$. The problem of classifying strong uniqueness polynomials is often related to the study of unique range sets proposed by Gross [8] in the course of studying the factorization of meromorphic functions. On the other hand, this type of problem has also been studied by number theorists and their results are formulated in different ways; we refer to [4] for a more detailed discussion. In this paper, we will concentrate on the case of entire functions.

It follows rather easily from Picard's theorem that $P(X)$ is a strong uniqueness polynomial for entire functions if and only if none of the two-variable polynomials $P(X) - cP(Y)$ for all complex numbers $c \neq 0$ have linear or quadratic factors except for the linear factor $(X - Y)$ when $c = 1$ (cf. [6]). It is also easy to see that $P(X) - cP(Y)$ has a linear factor, say $Y - aX - b \neq Y - X$ if and only if $P(X) = cP(aX + b)$. However, the question of determining if any of the above two-variable polynomials has quadratic factors is more complicated and has recently been addressed by Bilu [5]. In this note, we will assume that P is injective on the roots of $P' = 0$ and show that $P(X)$ is a strong uniqueness polynomial for entire functions if and only if $\deg P \geq 4$ and none of the two-variable polynomials $P(X) - cP(Y)$, for all complex numbers $c \neq 0$, have linear factors except for the linear factor $(X - Y)$ when $c = 1$. Consequently, under the assumption that $P(X)$ is injective on the roots of $P'(X) = 0$, if $\deg P \geq 4$ and none of the polynomials $P(X) - cP(Y)$ for all complex numbers $c \neq 0$ has linear factor then none of them has irreducible quadratic factors.

From now we let $P(X)$ be a monic polynomial of degree n. We will use l to denote the number of distinct roots of $P'(X) = 0$, and we will denote those roots by $\alpha_1, \alpha_2, ..., \alpha_l$. We will use $n_1, n_2, ..., n_l$ to denote the multiplicities of the roots in $P'(X) = 0$. Thus,

$$P'(X) = n(X - \alpha_1)^{n_1}(X - \alpha_2)^{n_2}...(X - \alpha_l)^{n_l}. \tag{1}$$

We will continually assume what we call *Hypothesis I*:

$$P(\alpha_i) \neq P(\alpha_j) \quad \text{whenever} \quad i \neq j,$$

or in other words $P(X)$ is injective on the roots of $P'(X) = 0$. We note that Hypothesis I is a generic condition, and one can see later from our arguments that it makes the computation easier. We also recall that a subset U of $\mathbb{C}$ is said to be *affinely rigid* if it is not preserved by any linear transformation except the identity. We note that it is easy to check that the zero set of $P(X)$ is affinely rigid if and only if none of the polynomials $P(X) - cP(Y)$, for all complex numbers $c \neq 0$, have linear factors except for the linear factor $(X - Y)$ when $c = 1$ (cf. [12, Lemma 6]). The main results in this paper are as follows.

Theorem 1.1. *Let $P(X)$ be a polynomial in $\mathbb{C}[X]$ that satisfies Hypothesis I. Then $P(X)$ is a strong uniqueness polynomial for the family of entire functions if and only if the zero set of $P(X)$ is affinely rigid and* $\deg P \geq 4$.

We will first show that Theorem 1.1 is a consequence of Bilu's work [5]. However, the main purpose of our paper is to give an independent proof based on Nevanlinna theory that we think function theorists might find more accessible than Bilu's proof based on monodromy groups and Galois theory.

We begin by stating Bilu's result. For $a \in \mathbb{C}$, recall that the n-th Dickson polynomial $D_{n,a}(X)$ with parameter a is defined by

$$D_{n,a}(z + a/z) = z^n + (a/z)^n. \tag{2}$$

Theorem (Bilu [5, Theorem 1.3]) *Let $P(X), Q(X) \in \mathbb{C}[X]$ and let $q(X,Y) \in \mathbb{C}[X,Y]$ be an irreducible quadratic factor of $P(X) - Q(Y)$. Then, there exist polynomials $\phi(X), P_1(X), Q_1(X) \in \mathbb{C}[X]$ such that $P = \phi \circ P_1$, $Q = \phi \circ Q_1$ and such that one of the following holds:*
(a) $\max\{\deg P_1, \deg Q_1\} = 2$ *and* $q(X,Y) = P_1(X) - Q_1(Y)$.
(b) *There exists an integer $n > 2$ such that for some $\alpha, a \in \mathbb{C}^*$ and some $\beta, \gamma \in \mathbb{C}$, we have*

$$P_1(X) = D_{n,a}(X + \beta), \qquad Q_1(X) = -D_{n,a}(2\cos(\pi/n)(\alpha X + \gamma))$$

and $q(X,Y)$ is a quadratic factor of $P_1(X) - Q_1(Y)$.

Remark 1.2. Bilu works over an arbitrary field K of characteristic zero and also gives necessary and sufficient conditions for $P(X) - Q(Y)$ to have a quadratic factor in $K[X,Y]$.

Proof of Theorem 1.1 as a corollary of Bilu's Theorem. We prove that if P satisfies Hypothesis I, $\deg P \geq 4$, and the zeros of P are affinely rigid, then $P(X) - cP(Y)$ has no quadratic factors and hence is a strong uniqueness polynomial for entire functions.

Apply Bilu's Theorem with $Q(X) = cP(X)$. It is clear from the definition of Dickson polynomial (2) that $D_{n,a}(X)$ does not satisfy Hypothesis I when $a \neq 0$ and $n \geq 2$. Therefore, we may eliminate case (b) to conclude that $P = \phi \circ P_1$ with $\deg P_1 = 2$ and $\deg \phi \geq 2$. Without loss of generality we may assume P_1 is monic.

Next, we make the following observation that if $\phi'(w) = 0$, then $P_1(z) = w$ has only one zero z_0 and thus $P_1'(z) = z - z_0$. This is clear since the existence of two distinct roots z_1, and z_2 of $P_1(z) = w$ will yield that $P'(z_1) = P'(z_2) = 0$ and $P(z_1) = P(z_2)$ contradicting Hypothesis I.

Suppose ϕ' has at least two distinct zeros (not counting multiplicity), $w_0 \neq w_1$. The previous observation implies that $P_1'(z) = z - z_0$ where z_0 is the only zero of $P_1(z) = w_0$ and $P_1'(z) = z - z_1$ where z_1 is the only zero of $P_1(z) = w_1$. This is impossible since z_0 is certainly different from z_1. Therefore, ϕ' has only one zero, say w_0, then

$$\phi(X) = A(X - w_0)^r + B,$$

for some $A \in \mathbb{C}^*$ and $w_0, B \in \mathbb{C}$. Since $P_1(X) = w_0$ has only one zero from the previous observation, we may write

$$P_1(X) = w_0 + (X - a_0)^2.$$

Hence,

$$P(X) = A(X - a_0)^{2r} + B,$$

and so clearly the zeros of P would not be affinely rigid.

Thus, under the hypothesis of Theorem 1.1, neither case (a) nor (b) in Bilu's Theorem can occur, and so $P(X) - cP(Y)$ does not contain any quadratic factor. ■

We now proceed to prove Theorem 1.1 without using Bilu's result by applying Nevanlinna theory. These techniques can also be applied to the study of strong uniqueness polynomials for meromorphic functions and algebraic functions on algebraic curves of characteristic zero (or equivalently, meromorphic functions on compact Riemann surfaces) (cf. [2]). However, we do not attempt to do so for the case of meromorphic functions, since the proof in [3] through the construction of two linearly independent of regular 1-forms is quite satisfactory. We record the following results (cf. [1, 3, 7]) for later use.

Theorem M. (cf. [3, Theorem 3]) *Let $P(X)$ be a polynomial as* (1) *in* $\mathbb{C}[X]$ *satisfying Hypothesis I. Then $P(X)$ is a strong uniqueness polynomial for meromorphic functions if and only if the zero set of P is affinely rigid and P does not satisfy*
(1A) $l = 2$ *and* $\min\{n_1, n_2\} = 1$*;*
(1B) $l = 2$ *and* $n_1 = n_2 = 2$; *or*
(1C) $l = 3$ *and* $n_1 = n_2 = n_3 = 1$.

Theorem E. (cf. [6, Theorem 1.1]) *Let $P(X)$ be a polynomial of degree at least two in* $\mathbb{C}[X]$. *Then $P(X)$ is a strong uniqueness polynomial for entire functions if and only if none of the polynomials $P(X) - cP(Y)$, for all complex numbers $c \neq 0$, have linear or quadratic factors except for the linear factor $(X - Y)$ when $c = 1$.*

2. Preliminary on Nevanlinna's Theory

We recall some standard definitions and results in Nevanlinna theory (cf. [9, 11]). Let f be a meromorphic function. At each point $a \in \mathbf{D}_r$, the complex disk of radius r, we define a valuation by

$$v_{a,r}(f) := \begin{cases} (\mathrm{ord}_a f)\log\left|\frac{r}{a}\right| & \text{if } a \neq 0, \\ (\mathrm{ord}_a f)\log r & \text{if } a = 0. \end{cases}$$

The *counting function* of poles, *proximity function* and *characteristic function* are defined respectively as follows

$$N_f(r,\infty) := \sum_{a \in \mathbf{D}_r, f(a)=\infty} -v_{a,r}(f),$$
$$m_f(r) := \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi},$$
$$T_f(r) := m_f(r) + N_f(r,\infty).$$

The logarithmic derivative lemma can be stated as follows (cf. [11, Theorem A.1.2.5]).

Lemma 2.1. (Logarithmic Derivative Lemma) *Let f be a non-constant meromorphic function on $\mathbb{C}$. Then*

$$m_{\frac{f'}{f}}(r) = o(T_f(r)) \quad \text{as } r \to \infty$$

outside a subset of finite measure.

Let

$$\mathbf{f} : \mathbb{C} \to \mathbb{P}^n(\mathbb{C}), \qquad \mathbf{f} = (f_0, ..., f_n)$$

be a holomorphic map into projective space, with meromorphic coordinates $f_0, ..., f_n$. The characteristic function of the map $\mathbf{f}$ is defined by the formula

$$\begin{aligned} T_{\mathbf{f}}(r) :=& T(f_0, ..., f_n)(r) \\ :=& \int_0^{2\pi} \log \max_j |f_j(re^{i\theta})| \frac{d\theta}{2\pi} - \log \max_{j \in M} |c_{f_j}| - \sum_{a \in \mathbf{D}_r} \min_j v_{a,r}(f_j) \end{aligned}$$

with the following notation. Let k be the minimal order at 0 of the functions f_j. Then we let M be the set of indices j such that f_j has order k at 0 and c_{f_j} is the leading coefficient of f_j. By Jensen's formula, $T_{\mathbf{f}}(r)$ is independent of homogeneous coordinates. More precisely, if $\mathbf{f} = (f_0, ..., f_n) = (g_0, ..., g_n)$, i.e. there exists a meromorphic function h such that $f_j = hg_j$ for all $j = 0, ..., n$, then $T(f_0, ..., f_n)(r) = T(g_0, ..., g_n)(r)$. In particular, if f_0,...,f_n are chosen to be entire functions with no common zeros then

$$T(f_0, ..., f_n)(r) = \int_0^{2\pi} \log \max_j \left| f_j(re^{i\theta}) \right| \frac{d\theta}{2\pi} - \log \max_j |f_j(0)| .$$

3. Proof of Theorem 1.1 by Nevanlinna Theory

The idea of the proof of Theorem 1.1 is as follows: Suppose that f and g are two distinct non-constant entire functions such that $P(f) = cP(g)$ for some constant $c \neq 0$. Then, we try to derive a contradiction by establishing a lower bound and an upper bound on $T(P'(f), P'(g))(r)$ as r tends to ∞. The following lemma gives an upper bound.

Lemma 3.1. *Suppose that f and g are two distinct non-constant entire functions such that $P(f) = cP(g)$ for some complex number $c \neq 0$. Then as $r \to \infty$, $\log|f(re^{i\theta})| = \log|g(re^{i\theta})| + O(1)$ and consequently we have*
(i) $T_f(r) = T_g(r) + O(1)$;
(ii) $T(P'(f), P'(g))(r) \leq T_f(r) - \sum_{a\in\mathbf{D}_r} \min\{v_{a,r}(f'), v_{a,r}(g')\} + o(T_f(r))$ *as $r \to \infty$ outside an exceptional set of finite measure.*

Proof. Since f and g are non-constant entire functions, both $|f(re^{i\theta})|$ and $|g(re^{i\theta})|$ tend to infinity as $r \to \infty$. Therefore, if $P(f) = cP(g)$ for some $c \neq 0$, then $\log|f(re^{i\theta})| = \log|g(re^{i\theta})| + O(1)$.

Since f is an entire function, $N_f(r, \infty) = 0$. Therefore, $T_f(r) = m_f(r)$. Moreover, since $|f(re^{i\theta})|$ tends to infinity as $r \to \infty$,

$$\begin{aligned} m_f(r) &= \int_0^{2\pi} \log|f(re^{i\theta})| \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \log|g(re^{i\theta})| \frac{d\theta}{2\pi} + O(1) = m_g(r) + O(1). \end{aligned}$$

Therefore, $T_f(r) = T_g(r) + O(1)$ as $r \to \infty$.

For (ii), $P(f) = cP(g)$ implies that $f'P'(f) = cg'P'(g)$. Hence,

$$T(P'(f), P'(g))(r) = T(f', g')(r).$$

It follows from the definition that

$$\begin{aligned} T(f', g')(r) = &\int_0^{2\pi} \log\max\{|f'(re^{i\theta})|, |g'(re^{i\theta})|\} \frac{d\theta}{2\pi} \\ &- \sum_{a\in\mathbf{D}_r} \min\{v_{a,r}(f'), v_{a,r}(g')\} + O(1). \end{aligned}$$

Writing $f' = f(f'/f)$ and $g' = g(g'/g)$, then as $r \to \infty$

$$\begin{aligned} &\int_0^{2\pi} \log\max\left\{\left|f'(re^{i\theta})\right|, \left|g'(re^{i\theta})\right|\right\} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \left(\log\max\left\{\left|f(re^{i\theta})\right|, \left|g(re^{i\theta})\right|\right\} + \log\left|\frac{f'}{f}(re^{i\theta})\right| + \log\left|\frac{g'}{g}(re^{i\theta})\right|\right) \frac{d\theta}{2\pi} \\ &\quad + O(1) \\ &= \int_0^{2\pi} \left(\log\left|f(re^{i\theta})\right| + \log\left|\frac{f'}{f}(re^{i\theta})\right| + \log\left|\frac{g'}{g}(re^{i\theta})\right|\right) \frac{d\theta}{2\pi} + O(1) \quad \text{(by (i))} \\ &\leq m_f(r) + m_{f'/f}(r) + m_{g'/g}(r) + O(1) \\ &\leq T_f(r) + o(T_f(r)), \end{aligned}$$

where the last inequality follows from the definition of characteristic function and the Logarithmic Lemma. Putting all the equations together, we have (ii).

■

To obtain a lower bound for $T(P'(f), P'(g))(r)$, we will need to estimate the order of the common zeros of $P'(f)$ and $P'(g)$. To do so, we make the following observation. Suppose that $P(X)$ satisfies Hypothesis I. Then the common solutions of $(P(X) - P(Y))/(X - Y) = 0$ and $P'(X) = P'(Y) = 0$ are (α_i, α_i), $i = 1, ..., l$; and $P(X) - cP(Y) = 0$, $c \neq 0, 1$, and $P'(X) = P'(Y) = 0$ have at most l common solutions $(\alpha_i, \alpha_{\phi(i)})$, $i = 1, ..., l$, where ϕ is a permutation of $\{\alpha_1, ..., \alpha_l\}$ such that $\phi(\alpha_i) \neq \alpha_i$ and $\phi(i) = j$ if $P(\alpha_i) = cP(\alpha_j)$. Fix a permutation as above and let

$$L_{i,j} = L^{\phi}_{i,j}(f, g) := (g - \alpha_{\phi(i)}) - \frac{\alpha_{\phi(i)} - \alpha_{\phi(j)}}{\alpha_i - \alpha_j}(f - \alpha_i), \tag{3}$$

$$= (g - \alpha_{\phi(j)}) - \frac{\alpha_{\phi(i)} - \alpha_{\phi(j)}}{\alpha_i - \alpha_j}(f - \alpha_j) \tag{4}$$

for $1 \leq i \neq j \leq l$.

Proposition 3.2. *Let $P(X)$ be a polynomial in $\mathbb{C}[X]$ as* (1) *that satisfies Hypothesis I. Suppose f and g are two distinct non-constant entire functions such that $P(f) = cP(g)$ for some complex number $c \neq 0$.*

(I) *Suppose that $c = 1$.*

(i) $P'(g(z)) = P'(f(z)) = 0$ *only when $f(z) = \alpha_i$ and $g(z) = \alpha_i$.*

(ii) *If $f(a) = \alpha_i$ and $g(a) = \alpha_i$, then*

(a) $\text{ord}_a(f - \alpha_i) = \text{ord}_a(g - \alpha_i)$,

(b) $\text{ord}_a((f - \alpha_i)^{n_i+1} - (g - \alpha_i)^{n_i+1}) \geq (n_i + 2)\text{ord}_a(f - \alpha_i)$,

(c) $\text{ord}_a(f - g) \geq \text{ord}_a(f - \alpha_i)$,

(II) *Suppose that $c \neq 0, 1$. Let ϕ be a permutation of $\{\alpha_1, ..., \alpha_l\}$ such that $\phi(\alpha_i) \neq \alpha_i$ and $\phi(i) = j$ if $P(\alpha_i) = cP(\alpha_j)$.*

(i) $P'(g(z)) = P'(f(z)) = 0$ *only when $f(z) = \alpha_i$, $g(z) = \alpha_{\phi(i)}$, and $P(\alpha_i) = cP(\alpha_{\phi(i)})$.*

(ii) *If $f(a) = \alpha_i$, $g(a) = \alpha_{\phi(i)}$, and $P(\alpha_i) = cP(\alpha_{\phi(i)})$, then*

(a) $(n_i + 1)\text{ord}_a(f - \alpha_i) = (n_{\phi(i)} + 1)\text{ord}_a(g - \alpha_{\phi(i)})$,

(b) $\text{ord}_a(L_{i,j}) \geq \min\{\text{ord}_a(f - \alpha_i), \text{ord}_a(g - \alpha_{\phi(i)})\}$,

(c) $\text{ord}_a(L_{i,j}) \geq \min\{\text{ord}_a(f - \alpha_j), \text{ord}_a(g - \alpha_{\phi(j)})\}$,

Proof. The statements of (I.i) and (II.i) follow from the previous observation. (I.ii) and (II.ii) can be deduced easily by expressing P in terms of $X - \alpha_i$ and $X - \alpha_{\phi(i)}$ and basic properties of order functions. For example, (II.ii.a) follows from the following expression

$$\begin{aligned} &(f - \alpha_i)^{n_i+1} + \text{higher terms in } (f - \alpha_i) \\ &= \lambda(g - \alpha_{\phi(i)})^{n_{\phi(i)}+1} + \text{higher terms in } (g - \alpha_{\phi(i)}) \end{aligned}$$

where λ is a non-zero constant; and (I.ii.c) follows from the expression $f - g = (f-\alpha_i) - (g-\alpha_i)$. We will skip the proof for the other cases. ∎

Proof of Theorem 1.1. Suppose the zero set of $P(X)$ is affinely rigid. Then Theorem M implies that $P(X)$ is a strong uniqueness polynomial for meromorphic functions except when $P(X)$ satisfies
(1A) $l = 2$ and $\min\{n_1, n_2\} = 1$;
(1B) $l = 2$ and $n_1 = n_2 = 2$; or
(1C) $l = 3$ and $n_1 = n_2 = n_3 = 1$.
Therefore, to show $P(X)$ is a strong uniqueness polynomial for entire functions when $\deg P \geq 4$, it suffices to consider when $P(X)$ satisfies (1A′) $l = 2$, $\min\{n_1, n_2\} = 1$, and $\max\{n_1, n_2\} \geq 2$, (1B) or (1C).

First of all, we assume that f and g are two distinct non-constant entire functions such that $P(f) = P(g)$, and we will show that this is impossible when $P(X)$ satisfies (1A′), (1B) or (1C).

When $P(X)$ satisfies (1A′), we may assume that $n_2 = 1$ and $n_1 \geq 2$. Then $n = n_1 + 2$ and $P'(X) = (X-\alpha_1)^{n_1}(X-\alpha_2)$. Let

$$G := (f-g)^2((f-\alpha_1)^{n_1+1} - (g-\alpha_1)^{n_1+1})^{n_1-1}.$$

We will show that

$$\mathrm{ord}_a(G) \geq \min\left\{\mathrm{ord}_a(P'(f)^{n_1+1}), \mathrm{ord}_a(P'(g)^{n_1+1})\right\}, \qquad \forall\, a \in \mathbb{C}. \tag{5}$$

It implies that

$$\sum_{a\in\mathbf{D}_r} \min\left\{v_{a,r}\Big(\frac{P'(f)^{n_1+1}}{G}\Big), v_{a,r}\Big(\frac{P'(g)^{n_1+1}}{G}\Big)\right\} \leq 0.$$

Then,

$$\begin{aligned}
&(n_1+1)T(P'(f), P'(g))(r)\\
&= T(P'(f)^{n_1+1}, P'(g)^{n_1+1})(r)\\
&= T(P'(f)^{n_1+1}/G, P'(g)^{n_1+1}/G)(r)\\
&= \int_0^{2\pi} \log\max\left\{\left|\frac{P'(f)^{n_1+1}}{G}(re^{i\theta})\right|, \left|\frac{P'(g)^{n_1+1}}{G}(re^{i\theta})\right|\right\}\frac{d\theta}{2\pi}\\
&\quad - \sum_{a\in\mathbf{D}_r} \min\left\{v_{a,r}\left(\frac{P'(f)^{n_1+1}}{G}\right),\ v_{a,r}\left(\frac{P'(g)^{n_1+1}}{G}\right)\right\} + O(1)\\
&\geq \int_0^{2\pi} \log\max\left\{\left|\frac{P'(f)^{n_1+1}}{G}(re^{i\theta})\right|, \left|\frac{P'(g)^{n_1+1}}{G}(re^{i\theta})\right|\right\}\frac{d\theta}{2\pi} + O(1)\\
&= 2n_1 T_f(r) + O(1)
\end{aligned}$$

as $r \to \infty$. Here $2n_1$ equals $(n_1+1)\deg P'$ minus the homogeneous degree of G in f and g. On the other hand, by Lemma 3.1, we have as $r \to \infty$

$$T(P'(f), P'(g))(r) \le T_f(r) + o(T_f(r)), \tag{6}$$

except a subset of finite measure. These two inequalities imply that outside of a subset of finite measure

$$\frac{n_1-1}{n_1} T_f(r) \le o(T_f(r))$$

which is impossible since $n_1 \ge 2$. It remains to show (5). Since $P(f) = P(g)$, it follows from Proposition 3.2 (I.i) that $P'(f(a)) = P'(g(a)) = 0$ only when $f(a) = g(a) = \alpha_j$, $j = 1, 2$. Therefore, it suffices to show (5) for those a such that $f(a) = g(a) = \alpha_1$ or α_2. Proposition 3.2 (I.ii) implies that if $f(a) = g(a) = \alpha_1$ then

$$\mathrm{ord}_a(G) \ge n_1(n_1+1)\mathrm{ord}_a(f-\alpha_1) = \mathrm{ord}_a(P'(f)^{n_1+1}) = \mathrm{ord}_a(P'(g)^{n_1+1});$$

and if $f(a) = g(a) = \alpha_2$ then

$$\begin{aligned}\mathrm{ord}_a(G) &= (n_1+1)\mathrm{ord}_a(f-g) \ge (n_1+1)\mathrm{ord}_a(f-\alpha_2)\\ &= \mathrm{ord}_a(P'(f)^{n_1+1}) = \mathrm{ord}_a(P'(g)^{n_1+1}).\end{aligned}$$

This concludes (5).

When $P(X)$ satisfies (1B) or (1C), we let

$$G := (f-g)^{n_1}.$$

Similarly, it follows from Proposition 3.2 that

$$\mathrm{ord}_a(G) \ge \min\{\mathrm{ord}_a(P'(f)), \mathrm{ord}_a(P'(g))\}, \qquad \forall\, a \in \mathbb{C},$$

and

$$\begin{aligned}T(P'(f), P'(g))(r) &= T(P'(f)/G, P'(g)/G)(r)\\ &\ge \int_0^{2\pi} \log\max\left\{\left|\frac{P'(f)}{G}(re^{i\theta})\right|, \left|\frac{P'(g)}{G}(re^{i\theta})\right|\right\}\frac{d\theta}{2\pi} + O(1)\\ &\ge 2T_f(r) + O(1)\end{aligned}$$

as $r \to \infty$. Again by Lemma 3.1, we have (6) and a contradiction can be established by these two inequalities. Therefore, we have shown that there do not exist distinct non-constant entire functions f and g such that $P(f) = P(g)$.

Next, we assume that f and g are two non-constant entire functions such that $P(f) = cP(g)$, for some complex number $c \ne 0, 1$, and we'll show again that this is impossible when $P(X)$ satisfies (1A′), (1B) or (1C).

When $P(X)$ satisfies (1A′), we may assume that $n_2 = 1$ and $n_1 \geq 2$. Then $n = n_1 + 2$ and $P'(X) = (X - \alpha_1)^{n_1}(X - \alpha_2)$. By Proposition 3.2, $P'(f(a)) = P'(g(a)) = 0$ only if (i) $f(a) = \alpha_1$, $g(a) = \alpha_2$, and $P'(\alpha_1) = cP'(\alpha_2)$ or (ii) $f(a) = \alpha_2$, $g(a) = \alpha_1$, and $P'(\alpha_2) = cP'(\alpha_1)$. Moreover,

$$(n_1 + 1)\mathrm{ord}_a(f - \alpha_1) = 2\mathrm{ord}_a(g - \alpha_2) \qquad \text{if (i) holds;} \tag{7}$$

$$(n_1 + 1)\mathrm{ord}_a(g - \alpha_1) = 2\,\mathrm{ord}_a(f - \alpha_2) \qquad \text{if (ii) holds.} \tag{8}$$

We first consider when $n_1 \geq 3$, and let

$$G := (f - \alpha_2)(g - \alpha_2).$$

If a satisfies (i), then $2\mathrm{ord}_a(g - \alpha_2) = (n_1 + 1)\mathrm{ord}_a(f - \alpha_1)$ and

$$\min\{\mathrm{ord}_a(P'(f)),\, \mathrm{ord}_a(P'(g))\} = \min\{n_1\mathrm{ord}_a(f - \alpha_1),$$
$$\mathrm{ord}_a(g - \alpha_2)\} = \mathrm{ord}_a(g - \alpha_2).$$

Therefore, in this case we have

$$\mathrm{ord}_a(G) = \mathrm{ord}_a(g - \alpha_2) = \min\{\mathrm{ord}_a(P'(f)), \mathrm{ord}_a(P'(g))\}. \tag{9}$$

If a satisfies (ii), we may derive (9) similarly. Therefore, (9) holds for each complex number a. Hence,

$$\begin{aligned} T(P'(f), P'(g))(r) &= T(P'(f)/G, P'(g)/G)(r) \\ &\geq (n_1 - 1)T_f(r) + O(1). \end{aligned}$$

Similarly, when $n_1 \geq 3$, this leads to a contradiction with Lemma 3.1. It is left to consider when $n_1 = 2$. In this case, we let

$$G := L_{1,2} = (g - \alpha_2) + (f - \alpha_1)$$

as in (3), and will show that

$$\mathrm{ord}_a(G) + \min\{\mathrm{ord}_a(f'),\, \mathrm{ord}_a(g')\} \geq \min\{\mathrm{ord}_a(P'(f)), \mathrm{ord}_a(P'(g))\} \tag{10}$$

holds if a satisfies (i) or (ii). In this case,

$$3\mathrm{ord}_a(f - \alpha_1) = 2\mathrm{ord}_a(g - \alpha_2) \qquad \text{if (i) holds;} \tag{11}$$

$$3\mathrm{ord}_a(g - \alpha_1) = 2\,\mathrm{ord}_a(f - \alpha_2) \qquad \text{if (ii) holds.} \tag{12}$$

If (i) holds, then (11) implies that $\mathrm{ord}_a(f - \alpha_1) \geq 2$, $\mathrm{ord}_a G = \mathrm{ord}_a(f - \alpha_1)$, and $\min\{\mathrm{ord}_a(P'(f)),\, \mathrm{ord}_a(P'(g))\} = \frac{3}{2}\mathrm{ord}_a(f - \alpha_1)$. In conclusion, we have

$$\begin{aligned}\mathrm{ord}_a G + \min\{\mathrm{ord}_a(f'),\, \mathrm{ord}_a(g')\} &= 2\mathrm{ord}_a(f-\alpha_1) - 1\\ &= \frac{3}{2}\mathrm{ord}_a(f-\alpha_1) + \left[\frac{1}{2}\mathrm{ord}_a(f-\alpha_1) - 1\right]\\ &\geq \min\{\mathrm{ord}_a(P'(f)),\, \mathrm{ord}_a(P'(g))\}.\end{aligned}$$

If a satisfies (ii), we can proceed similarly. Therefore, (10) holds for every $a \in \mathbb{C}$. Then, as $r \to \infty$ we have

$$\begin{aligned}T(P'(f), P'(g))(r) &= T(P'(f)/G, P'(g)/G)(r)\\ &\geq 2T_f(r) - \sum_{a\in \mathbf{D}_r} \min\{v_{a,r}(f'), v_{a,r}(g')\} + O(1).\end{aligned}$$

On the other hand, Lemma 3.1 gives that

$$T(P'(f), P'(g))(r) \leq T_f(r) - \sum_{a\in \mathbf{D}_r} \min\{v_{a,r}(f'), v_{a,r}(g')\} + o(T_f(r)),$$

as $r \to \infty$ and outside of a subset of finite measure. Similarly, a contradiction can be obtained from these two inequalities.

When $P(X)$ satisfies (1B), we also let $G := L_{1,2}$. Since the proof for this case is similar to the previous one and indeed easier, we will omit the proof.

When $P(X)$ satisfies (1C), we let ϕ be a permutation of $\{1, 2, 3\}$ such that $\phi(i) \neq i$ and $\phi(i) = j$ if $P(\alpha_i) = P(\alpha_j)$. Let $L_{i,j}$ be as defined in (3), and let

$$G := L_{1,2}L_{2,3}L_{3,1}.$$

One can prove similarly that

$$\mathrm{ord}_a(G) \geq \min\{\mathrm{ord}_a(P'(f)^2), \mathrm{ord}_a(P'(g)^2)\}, \qquad \forall\, a \in \mathbb{C}.$$

Then,

$$T(P'(f), P'(g))(r) = \frac{1}{2}T(P'(f)^2/G, P'(g)^2/G)(r) \geq \frac{3}{2}T_f(r) + O(1)$$

as $r \to \infty$. Again, a contradiction can be derived from this and Lemma 3.1.

For the converse part of the theorem, we first note that if the zero set of $P(X)$ is not affinely rigid, then it is easy to see that $P(X)$ is not a strong uniqueness polynomial for entire functions (cf. [6]). When $\deg P \leq 2$, it is clear that $P(X)$ is not a strong uniqueness polynomial for entire functions. When $\deg P = 3$, either (i) $l = 1$ or (ii) $l = 2$ and $n_1 = n_2 = 1$. For case (i), we see that $P(X) = (X - \alpha_1)^3 + b$ for some $b \in \mathbb{C}$. Clearly, it is not a strong uniqueness polynomial for entire functions. For case (ii), this was done by explicit construction in [6, Prop. 2.2]. Therefore, the proof is complete. ■

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