Vietnam Journal of Mathematics **37**:2&3(2009) 237-253

Vietnam Journal
of
MATHEMATICS

Sensitivity Analysis for Weak and Strong Vector Quasiequilibrium Problems

Lam Quoc Anh[1] **and Phan Quoc Khanh**[2]

[1] *Department of Mathematics,*
Teacher College, Cantho University, Cantho City, Vietnam

[2] *Department of Mathematics, International University of Hochiminh City,*
Linh Trung, Thu Duc, Ho Chi Minh City, Vietnam

Received November 20, 2008
Revised May 16, 2009

Abstract. We present the main contributions of our recent works on various kinds of stability for two typical vector quasiequilibrium problems. They reflect the major results and are improvements or modifications, not repeated statements, of our recent sufficient conditions for semicontinuities, continuity and Hölder continuity of solution maps of the considered problems.

2000 Mathematics Subject Classification: 90C31, 49K40, 65K99.

Key words: Parametric vector quasiequilibrium problems, Solution maps, Semicontinuity, Hausdorff lower semicontinuity, Continuity, Hölder continuity.

1. Introduction

The aim of this paper is to present results on continuity of different types of solution maps of rather general parametric vector quasiequilibrium problems. These results reflect our recent works on stability of such problems but are not a survey with theorems repeated from the previous papers, since many of them are improvements and some of them are modifications of what presented in those papers. For this purpose we choose the following setting. Let X, Λ and M be topological spaces, Y a topological vector space, $A \subseteq X$ a nonempty subset and $C \subseteq Y$ a solid closed convex cone. Let $K : A \times \Lambda \to 2^A$ and $f : A \times A \times M \to Y$.

For $x, y \in Y$ we adopt the notations

$$x \geq y \Leftrightarrow x - y \in C;$$
$$x > y \Leftrightarrow x - y \in \text{int}C;$$
$$x \not\geq y \Leftrightarrow x - y \notin C,$$

and similarly for $\leq, <, \not\leq$, etc, where int$C$ stands for the interior of C. For $(\lambda, \mu) \in \Lambda \times M$ consider the following parametric quasiequilibrium problems

(WQEP) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, for all $y \in K(\bar{x}, \lambda)$,

$$f(\bar{x}, y, \mu) \not< 0.$$

(SQEP) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, for all $y \in K(\bar{x}, \lambda)$,

$$f(\bar{x}, y, \mu) \geq 0.$$

Note that quasiequilibrium problems encompass many optimization-related models like vector minimization, variational inequalities, Nash equilibria, fixed-point and coincidence-point problems, complementarity problems, minimax inequalities, traffic networks, etc. We investigated more general quasiequilibrium problems in [4, 8, 9, 14, 18, 19], but the above two problems are typical. Moreover, quasivariational inclusion problems and relation problems in [7, 10, 16, 17, 19, 20, 23, 24] are generalizations of quasiequilibirum problems, but it is not much hard to extend results of the latter to them. Note also that there is a large number of works on the subject, but we have to refer the reader to the references listed in our above papers, since we mention here mainly the latter due to the lack of space.

For a set-valued mapping $Q : X \to 2^Y$, recall that Q is called upper semicontinuous (usc in short; lower semicontinuous, lsc, respectively) at x_0 if for any open subset U of Y with $Q(x_0) \subseteq U$ ($Q(x_0) \cap U \neq \emptyset$), there is a neighborhood N of x_0 such that $Q(N) \subseteq U$ ($\forall x \in N, Q(x) \cap U \neq \emptyset$). When Y is a topological vector space, Q is said to be Hausdorff upper semicontinuous (H-usc in short; Hausdorff lower semicontinuous, H-lsc, respectively) at x_0 if for each neighborhood B of the origin in Y, there exists a neighborhood N of x_0 such that, $Q(x) \subseteq Q(x_0) + B, \forall x \in N$ ($Q(x_0) \subseteq Q(x) + B, \forall x \in N$). Q is said to be continuous at x_0 if it is both lsc and usc at x_0. When X and Y are vector spaces, Q is said to be concave on $A \subseteq X$ if, for each $x, z \in A$ and $t \in [0, 1]$,

$$Q((1 - t)x + tz) \subseteq (1 - t)Q(x) + tQ(z).$$

There are several notions of semicontinuity of vector (single-valued) functions in the literature, but they may appear under slightly different terms. In this paper, $h : A \to Y$ is said to be C-lower semicontinuous, C-lsc in short (C-upper semicontinuous, C-usc, respectively) at x_0 if for any neighborhood U of $h(x_0)$, there is a neighborhood N of x_0 such that $h(N) \subseteq U + C$ ($h(N) \subseteq U - C$, respectively). For $b \in Y$, we use the following notations for the level sets of h

with respect to different orderings $\leq$ (by the context, no confusion threat),

$$\mathrm{L}_{\leq b}h = \{x \in A : h(x) \leq b\}$$

and similarly for other level sets $\mathrm{L}_{\not< b}h$, $\mathrm{L}_{\geq b}h$, etc.

Proposition 1.1. *Let $h : A \to Y$.*

i) *h is C-lsc at x_0 if and only if $\mathrm{L}_{\not> b}h$ is closed at x_0 for all $b \in Y$.*

ii) *h is C-usc at x_0 if and only if $\mathrm{L}_{\not< b}h$ is closed at x_0 for all $b \in Y$.*

In Sec. 2 we establish sufficient conditions for various kinds of semicontinuity of solution maps. Sec. 3 is devoted to continuity conditions. Sec. 4 deals with Hölder continuity of solution maps. Consequences for quasivariational inequalities are derived in Sec. 5.

2. Semicontinuities of Solution Maps

2.1. Quasi-semicontinuities of Vector Functions

The equivalence stated in Proposition 1.1 suggests the following weaker notions.

Definition 2.1. Let $h : A \to Y$ and $b \in Y$ be given.

i) h is called b-level C-lower semicontinuous ((b, C)-lsc) at x_0 if $\mathrm{L}_{\not> b}h$ is closed at x_0.

ii) h is called b-level C-upper semicontinuous ((b, C)-usc) at x_0 if $\mathrm{L}_{\not< b}h$ is closed at x_0.

iii) h is called b-level C-quasilower semicontinuous ((b, C)-qlsc) at x_0 if $\mathrm{L}_{\leq b}h$ is closed at x_0.

iv) h is called b-level C-quasiupper semicontinuous ((b, C)-qusc) at x_0 if $\mathrm{L}_{\geq b}h$ is closed at x_0.

Remark 2.2. It is not difficult to equivalently rewrite the above notions in terms of neighborhoods. We rewrite only ii) and iii) as examples.

ii) h is (b, C)-usc at x_0 if and only if there is a neighborhood N of x_0 with $h(N) \subseteq b - \mathrm{int}C$ whenever $h(x_0) \in b - \mathrm{int}C$.

iii) h is (b, C)-qlsc at x_0 if and only if there exists a neighborhood N of x_0 such that $h(N) \cap (b - C) = \emptyset$ whenever $h(x_0) \notin b - C$.

The following relationships between C-semicontinuity and C- quasisemicontinuity resemble those for the scalar counterparts.

Proposition 2.3. *Let $h : A \to Y$.*

i) *If h is C-lsc at x_0 then h is C-qlsc at x_0.*

ii) *If h is C-usc at x_0 then h is C-qusc at x_0.*

Proof. For i) suppose that h is C-lsc at x_0 and $x_\alpha \to x_0$ with $h(x_\alpha) \leq b$ for some $b \in Y$, but $h(x_0) \in Y \setminus (b - C)$. Since h is C-lsc at x_0 and $Y \setminus (b - C)$ is open, one can assume that $h(x_\alpha) \in Y \setminus (b - C) + C$. So there are $t_\alpha \in Y \setminus (b - C)$ and $c_\alpha \in C$, such that $h(x_\alpha) = t_\alpha + c_\alpha$. As $h(x_\alpha) \in b - C$, there is $c'_\alpha \in C$ such that $h(x_\alpha) = b - c'_\alpha$. Therefore, $t_\alpha = b - c'_\alpha - c_\alpha \in b - C$, which is impossible as $t_\alpha \in Y \setminus (b - C)$.

ii) It may be checked similarly. ■

Examples 2.1 and 2.2 in [13] show that the converse of Proposition 2.3 is not true and the relations asserted in Proposition 2.3 cannot be similarly stated for a particular level b (unlike the scalar case). The following results are sum rules.

Proposition 2.4. *Let X, Z be topological spaces, $A \subseteq X$ and $B \subseteq Z$ be nonempty open sets, $C \subseteq Y$ be a solid closed convex cone and $f : A \to Y, g : B \to Y$, $A \subseteq X, B \subseteq Z$. If f and g are C-lsc (or C-usc) at x_0 and z_0, respectively, then $h = f + g$ is C-lsc (or C-usc) at (x_0, z_0).*

Proof. By similarity and symmetry, it suffices to check the C-upper semicontinuity. Let U be any open subset containing $f(x_0)+g(z_0)$. There are open neighborhoods U_1 and U_2 of $f(x_0)$ and $g(z_0)$, respectively, such that $U_1 + U_2 \subseteq U$. Since f and g are C-lsc, there are open neighborhoods N_1 and N_2 of x_0 and z_0, respectively such that $f(x) \in U_1 - C$ and $g(z) \in U_2 - C$, for all $(x, z) \in N_1 \times N_2$. Hence $h(x, z) = f(x) + g(z) \in U_1 + U_2 - C \subseteq U - C$, for all $(x, z) \in N_1 \times N_2$, i.e., h is C-usc at (x_0, z_0). ■

Unfortunately, Proposition 2.4 cannot be extended for C-quasi semicontinuous functions as shown by Example 2.3 in [13].

Proposition 2.5. *Let X, Y, Z, A, B, C, f and g be as in Proposition 2.4.*

i) *If f is C-qlsc at x_0 and g is C-lsc at z_0, then $h = f+g$ is C-qlsc at (x_0, z_0).*

ii) *If f is C-qusc at x_0 and g is C-usc at z_0, then h is C-qusc at (x_0, z_0).*

Proof. We demonstrate only i). For arbitrary $b \in Y$, assume that $(x_\alpha, z_\alpha) \to (x_0, z_0)$ and $h(x_\alpha, z_\alpha) \leq b$. For each $e \in \text{int}C$, as g is C-lsc at z_0, there is a subnet z_β such that $g(z_\beta) > g(z_0) - e$, for all β. Hence

$$f(x_\beta) + g(z_0) - e \leq b.$$

As f is C-qlsc at x_0,

$$f(x_0) + g(z_0) - e \leq b.$$

Consequently, $f(x_0) + g(z_0) \leq b$. Thus, h is (b, C)-qlsc at (x_0, z_0). By the arbitrariness of b, h is $C-$qlsc at (x_0, z_0). ■

Examples 2.4-2.6 in [13] prevent us to state results similar to Propositions 2.4 and 2.5 for a given fixed level b.

2.2. Upper Semicontinuity of Solution Maps

In the sequel, for problems (WQEP) and (SQEP) and for $\lambda \in \Lambda$, let $E(\lambda) = \{x \in A \mid x \in K(x,\lambda)\}$ and $S^w(\lambda,\mu)$, $S^s(\lambda,\mu)$ be the solutions sets of these problems, respectively, corresponding to (λ,μ). Since the existence of solutions for quasiequilibrium problems have been intensively studied, see e.g. our recent papers [5, 14-19, 23, 24], we focus on the stability study, assuming always the needed existence. In the subsections of Sec. 2 assume additionally that the spaces X, Y, Λ and M involved in our problems are Hausdorff.

Theorem 2.6. *S^w is usc at (λ_0,μ_0) if*

i) *f is $(0,C)$-usc in $K(A,\Lambda) \times K(A,\Lambda) \times \{\mu_0\}$;*

ii) *one of the following two conditions holds*

ii$_1$) *K is continuous in $A \times \{\lambda_0\}$ and A is compact;*

ii$_2$) *E is usc at λ_0 and $E(\lambda_0)$ is compact, and K is lsc in $E(\lambda_0) \times \{\lambda_0\}$.*

Proof. It suffices to consider the case ii$_2$), since the other case is similar and simpler. Suppose that S^w is not usc at (λ_0,μ_0), i.e., there is an open superset U of $S^w(\lambda_0,\mu_0)$ such that there are nets $(\lambda_\alpha,\mu_\alpha)$ tending to (λ_0,μ_0) and $x_\alpha \in S^w(\lambda_\alpha,\mu_\alpha)$ with $x_\alpha \notin U$, for all α. By the upper semicontinuity of E and the compactness of $E(\lambda_0)$ one can assume that $x_\alpha \to x_0$, for some $x_0 \in E(\lambda_0)$. Suppose there is $y_0 \in K(x_0,\lambda_0)$ such that $f(x_0,y_0,\mu_0) < 0$. The lower semicontinuity of K in turn shows the existence of $y_\alpha \in K(x_\alpha,\lambda_\alpha)$ such that $y_\alpha \to y_0$. Condition i) allows one to assume that $f(x_\alpha,y_\alpha,\mu_\alpha) < 0$, which is impossible as $x_\alpha \in S^w(\lambda_\alpha,\mu_\alpha)$, for all α. Thus, $x_0 \in S^w(\lambda_0,\mu_0) \subseteq U$, which is again a contradiction, since $x_\alpha \notin U$ for each α. ■

The assumptions of Theorem 2.6 are essential (see Examples 2.2 and 2.3 in [12]). Similarly, for (SQEP) we have

Theorem 2.7. *Impose assumption ii) of Theorem 2.6 and assume further that*

i) *f is $(0,C)$-qusc in $K(A,\Lambda) \times K(A,\Lambda) \times \{\mu_0\}$.*

Then, S^s is usc at (λ_0,μ_0).

2.3. Quasiconvexity of Vector Functions

Now we discuss several generalized quasiconvexity notions for vector functions in order to use them in the next subsection. In this subsection let X be a vector space, $A \subseteq X$ be a convex set, Y be a topological vector space, $C \subseteq Y$ be a solid convex cone and $h : A \to Y$. The following definition is known. (The relations $\leq$, $<$ are defined by C.)

Definition 2.8. i) h is said to be strictly convex if, for every $x_1, x_2 \in A$ and $t \in (0,1)$,

$$h(tx_1 + (1-t)x_2) < th(x_1) + (1-t)h(x_2).$$

ii) h is called quasiconvex if, for every $x_1, x_2 \in A$ and $t \in [0,1]$, either $h(tx_1 + (1-t)x_2) \leq h(x_1)$ or $h(tx_1 + (1-t)x_2) \leq h(x_2)$.

iii) h is said to be strictly quasiconvex if, for every $x_1, x_2 \in A$ and $t \in (0,1)$,

either $h(tx_1 + (1-t)x_2) < h(x_1)$ or $h(tx_1 + (1-t)x_2) < h(x_2)$.

Remark 2.9. If $Y = C \cup (-C)$, like for scalar functions, convexity (strict convexity, respectively) implies quasiconvexity (strict quasiconvexity). However, this is not true in general as shown by Examples 4.1 and 4.2 in [13].

We propose the following relaxed convexity for vector functions.

Definition 2.10. Let $b \in Y$ be fixed. If for all $x_1, x_2 \in A$, $t \in (0,1)$ and $x_t = (1-t)x_1 + tx_2$,

i) $h(x_1) \not< b, h(x_2) \not\leq b$ implies $h(x_t) \not\leq b$, then h is said to be $\mathrm{lev}_{\not\leq b}$-convex;

ii) $h(x_1) \geq b, h(x_2) > b$ implies $h(x_t) > b$, then h is called $\mathrm{lev}_{>b}$-convex.

Note that, if $Y = C \cup (-C)$, these two properties coincide. Furthermore, if h is $\mathrm{lev}_{\not\leq b}$-convex ($\mathrm{lev}_{>b}$-convex, respectively), then $L_{\not\leq b}h$ ($L_{>b}h$, respectively) is convex, but not vice versa as shown by Example 4.3 in [13].

Remark 2.11. If h is strictly quasiconcave, then h is $\mathrm{lev}_{\not\leq b}$-convex, for all $b \in Y$. Indeed, let $x_1, x_2 \in A$ be such that $h(x_1) \not< b, h(x_2) \not\leq b$. Suppose the existence of $c_1 \in C$ such that $h(x_t) - b = -c_1$. Since h is strictly quasiconcave, $h(x_t) > h(x_1)$ or $h(x_t) > h(x_2)$, for all $t \in (0,1)$. If $h(x_t) > h(x_1)$, there is $c_2 \in \mathrm{int}C$ with $h(x_t) = h(x_1) + c_2$. Hence, $h(x_1) - b = -c_1 - c_2 \in -\mathrm{int}C$, which is impossible as $h(x_1) \not< b$. If $h(x_t) > h(x_2)$ we have a similar contradiction.

Examples 4.2 and 4.4 in [13] assert that $\mathrm{lev}_{\not\leq b}$-convexity does not follow neither from strict concavity nor from strict convexity and indicates that neither quasiconcavity nor quasiconvexity implies $\mathrm{lev}_{\not\leq b}$-convexity.

Remark 2.12. If h is strictly quasiconcave or concave, then h is $\mathrm{lev}_{>b}$-convex for all $b \in Y$. Indeed, assume that h is strictly quasiconcave and $x_1, x_2 \in A$ such that $h(x_1) \geq b, h(x_2) > b$. For any $t \in (0,1)$, if $h(x_t) > h(x_1)$, there is $c_1 \in \mathrm{int}C$ with $h(x_t) - h(x_1) = c_1$. Since $h(x_1) \geq b$, there is $c_2 \in C$ with $h(x_1) = b + c_2$. Consequently, $h(x_t) - b = c_1 + c_2 \in \mathrm{int}C$, i.e., $h(x_t) > b$. Similarly, $h(x_t) > b$ if $h(x_t) > h(x_2)$. Thus, h is $\mathrm{lev}_{>b}$-convex. The case, where h is concave, is analogous.

The mentioned Example 4.4 shows also that neither quasiconcavity nor quasiconvexity yields $\mathrm{lev}_{>b}$-convexity.

2.4. Lower Semicontinuity of Solution Maps

In this subsection assume additionally that X is a Hausdorff topological vector space and A is convex. We consider the following problems (WQEP_1) and (SQEP_1) as auxiliary problems to (WQEP) and (SQEP), respectively:

(WQEP_1) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, for all $y \in K(\bar{x}, \lambda)$,

$$f(\bar{x}, y, \mu) \nleq 0;$$

(SQEP$_1$) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, for all $y \in K(\bar{x}, \lambda)$,

$$f(\bar{x}, y, \mu) > 0.$$

Let $S_1^w(\lambda, \mu)$ and $S_1^s(\lambda, \mu)$ be the solution sets of (WQEP$_1$) and (SQEP$_1$), respectively, corresponding to (λ, μ).

Theorem 2.13. *The map S^w of* (WQEP) *is lsc at (λ_0, μ_0) if*

i) *f is $(0, C)$-qlsc in $K(A, \Lambda) \times K(A, \Lambda) \times \{\mu_0\}$ and $f(., ., \mu_0)$ is* $\mathrm{lev}_{\nleq 0}$*-convex in $E(\lambda_0) \times K(A, \lambda_0)$;*

ii) *E is lsc at λ_0 and $E(\lambda_0)$ is convex; K is usc and compact-valued in $E(\lambda_0) \times \{\lambda_0\}$; $K(., \lambda_0)$ is concave in $E(\lambda_0)$.*

Proof. We start by proving that S_1^w is lsc at (λ_0, μ_0). Suppose to the contrary that S_1^w is not lsc at (λ_0, μ_0), i.e., there are $x_0 \in S_1^w(\lambda_0, \mu_0)$, and a net $(\lambda_\alpha, \mu_\alpha)$ tending to (λ_0, μ_0), such that for any choice of $x_\alpha \in S_1^w(\lambda_\alpha, \mu_\alpha)$, $\{x_\alpha\}$ does not converge to x_0. Since E is lsc at λ_0, there is $\bar{x}_\alpha \in E(\lambda_\alpha)$ with $\bar{x}_\alpha \to x_0$. By our assumption, there must be a subnet $\bar{x}_\beta$ such that $\bar{x}_\beta \notin S_1^w(\lambda_\beta, \mu_\beta)$, for all β, i.e., for some $y_\beta \in K(\bar{x}_\beta, \lambda_\beta)$,

$$f(\bar{x}_\beta, y_\beta, \mu_\beta) \leq 0. \tag{1}$$

As K is usc at (x_0, λ_0) and $K(x_0, \lambda_0)$ is compact one has $y_0 \in K(x_0, \lambda_0)$ such that $y_\beta \to y_0$ (taking a subnet if necessary). By assumption ii), (1) yields that $f(x_0, y_0, \mu_0) \leq 0$, which is impossible since $x_0 \in S_1^w(\lambda_0, \mu_0)$.

Now let us check that

$$S^w(\lambda_0, \mu_0) \subseteq \mathrm{cl} S_1^w(\lambda_0, \mu_0). \tag{2}$$

Let $\bar{x} \in S^w(\lambda_0, \mu_0), \bar{x}^1 \in S_1^w(\lambda_0, \mu_0)$ and $x_t = (1-t)\bar{x} + t\bar{x}^1$ with $t \in (0, 1)$. Since $K(., \lambda_0)$ is concave, for all $y \in K(x_t, \lambda_0)$, there exist $\bar{y} \in K(\bar{x}, \lambda_0)$ and $\bar{y}_1 \in K(\bar{x}^1, \lambda_0)$ such that $y = (1-t)\bar{y} + t\bar{y}_1$. Since $f(., ., \mu_0)$ is $\mathrm{lev}_{\nleq 0}$-convex, $f(x_t, y, \mu_0) \nleq 0$, i.e., $x_t \in S_1^w(\lambda_0, \mu_0)$. Therefore (2) holds. By the lower semicontinuity of S_1^w at (λ_0, μ_0), S^w is lsc at (λ_0, μ_0) since

$$S^w(\lambda_0, \mu_0) \subseteq \mathrm{cl} S_1^w(\lambda_0, \mu_0) \subseteq \liminf S_1^w(\lambda_\alpha, \mu_\alpha) \subseteq \liminf S^w(\lambda_\alpha, \mu_\alpha).$$

■

Example 3.1 in [12] shows the essentialness of the above assumptions.

Theorem 2.14. *The solution map S^s is lsc at (λ_0, μ_0) if*

i) *f is $(0, C)$-lsc in $K(A, \Lambda) \times K(A, \Lambda) \times \{\mu_0\}$ and $f(., ., \mu_0)$ is* $\mathrm{lev}_{>0}$*-convex in $E(\lambda_0) \times K(A, \lambda_0)$;*

ii) *E is lsc at λ_0 and $E(\lambda_0)$ is convex; K is usc and compact-valued in $E(\lambda_0) \times \{\lambda_0\}$; $K(., \lambda_0)$ is concave in $E(\lambda_0)$.*

We omit the proof since its technique is similar to that for Theorem 2.13. We now proceed to Hausdorff lower semicontinuity.

Theorem 2.15. *Impose the assumptions of Theorem 2.13 and*

iii) $f(.,.,\mu_0)$ *is* $(0,C)$*-usc in* $K(A,\Lambda)\times K(A,\Lambda)$;

iv) $K(.,\lambda_0)$ *is lsc in* $E(\lambda_0)$ *and* $E(\lambda_0)$ *is compact.*

Then S^w *is Hausdorff lower semicontinuous at* (λ_0,μ_0).

Proof. We first show that $S^w(\lambda_0,\mu_0)$ is closed. Assume that $x_\alpha\in S^w(\lambda_0,\mu_0)$ and $x_\alpha\to x_0$. Since $E(\lambda_0)$ is compact, $x_0\in E(\lambda_0)$. Suppose there exists $y_0\in K(x_0,\lambda_0)$ such that

$$f(x_0,y_0,\mu_0)<0. \tag{3}$$

Since $K(.,\lambda_0)$ is lsc at x_0, there is $y_\alpha\in K(x_\alpha,\lambda_0)$ with $y_\alpha\to y_0$. As $x_\alpha\in S(\lambda_0)$,

$$f(x_\alpha,y_\alpha,\mu_0)\not<0. \tag{4}$$

Assumption iii) shows a contradiction between (3) and (4). Thus, $S^w(\lambda_0,\mu_0)$ is closed and then compact.

Now suppose S^w is not Hlsc at (λ_0,μ_0), i.e., there are B (a neighborhood of the origin in X) and $(\lambda_\alpha,\mu_\alpha)\to(\lambda_0,\mu_0)$ such that, for all α, there exists $x_{0\alpha}\in S^w(\lambda_0,\mu_0)\backslash(S^w(\lambda_\alpha,\mu_\alpha)+B)$. Since $S^w(\lambda_0,\mu_0)$ is compact, we can assume that $x_{0\alpha}\to x_0\in S^w(\lambda_0,\mu_0)$. Then there are α_1, a neighborhood B_1 of 0 in X with $B_1+B_1\subseteq B$ and $b_\alpha\in B_1$ such that, for each $\alpha\ge\alpha_1$, $x_{0\alpha}=x_0+b_\alpha$. Since S^w is lsc at λ_0, there is $z_\alpha\in S^w(\lambda_\alpha,\mu_\alpha)$ with $z_\alpha\to x_0$ and then there is α_2 such that, for all $\alpha\ge\alpha_2$, $b'_\alpha\in B_1$ exists with $z_\alpha=x_0-b'_\alpha$. Consequently, for all $\alpha\ge\alpha_0:=\max\{\alpha_1,\alpha_2\}$,

$$x_{0\alpha}=x_0+b_\alpha=z_\alpha+b'_\alpha+b_\alpha\in z_\alpha+B.$$

As $x_{0\alpha}\notin S^w(\lambda_\alpha,\mu_\alpha)+B$, this is impossible. Thus, S^w is Hlsc at λ_0. ■

Example 3.2 in [12] confirms the essentialness of the added assumptions.

Theorem 2.16. *Impose the assumptions of Theorem 2.14, and the following additional conditions:*

iii) $f(.,.,\mu_0)$ *is* $(0,C)$*-qusc in* $K(A,\Lambda)\times K(A,\Lambda)$;

iv) $K(.,\lambda_0)$ *is lsc in* $E(\lambda_0)$ *and* $E(\lambda_0)$ *is compact.*

Then S^s *is Hausdorff lower semicontinuous at* (λ_0,μ_0).

3. Continuity of Solution Maps

In this section we avoid concavity assumptions and use relaxed monotonicity conditions. The following monotonicity notions are extensions of the counterparts for the scalar case.

Definition 3.1. [2] Let X be a set, Y be a linear space and $C \subseteq Y$ be a convex cone. Let $y_1 \leq y_2$ mean $y_2 - y_1 \in C$. Let $g : X \times X \to Y$.
i) g is called quasimonotone I in X if, for all $x, z \in X$ and $x \neq z$,

$$[g(x,z) \nleq 0] \Longrightarrow [g(z,x) \leq 0].$$

ii) g is called quasimonotone II in X if, for all $x, z \in X$ and $x \neq z$,

$$[g(x,z) > 0] \Longrightarrow [g(z,x) \ngtr 0].$$

iii) g is termed pseudomonotone I in X if, for all $x, z \in X$ and $x \neq z$,

$$[g(x,z) \nless 0] \Longrightarrow [g(z,x) \leq 0].$$

iv) g is termed pseudomonotone II in X if, for all $x, z \in X$ and $x \neq z$,

$$[g(x,z) \geq 0] \Longrightarrow [g(z,x) \ngtr 0].$$

Remark 3.2. Note that if g is quasimonotone I (pseudomonotone I), then g is quasimonotone II (pseudomonotone II, respectively). Indeed, when g is quasimonotone I, if $g(x,z) > 0$ (i.e. $g(x,z) \nleq 0$) then $g(z,x) \leq 0$, i.e. $g(z,x) \ngtr 0$. So g is quasimonotone II. The relation between the pseudomonotonicities I and II is checked similarly.

The following example ensures that the converse is not true.

Example 3.3. Let $X = R, Y = R^2, C = R^2_+$ and $g : R \times R \to R \times R$ is defined by

$$g(x,z) = \begin{cases} (1,1), & \text{if} \quad x \leq z, \\ (-1,1), & \text{if} \quad x > z. \end{cases}$$

Then g is both quasimonotone II and pseudomonotone II in R, but is neither quasimonotone I nor pseudomonotone I.

Theorem 3.4. *Impose for* (WQEP) *the assumptions of Theorem 2.6 and*

a) *$f(.,.,\mu_0)$ is quasimonotone I in $K(A,\lambda_0) \times K(A,\lambda_0)$;*

b) *for all $x \in S^w(\lambda_0,\mu_0)$ and all $y \in S^w(\lambda_0,\mu_0) \setminus \{x\}$, one has $f(x,y,\mu_0) \nleq 0$.*

Then S^w is continuous at (λ_0,μ_0).

Proof. It suffices to prove that S^w is lsc at (λ_0,μ_0). Suppose to the contrary that there are a net $(\lambda_\alpha,\mu_\alpha)$ tending to (λ_0,μ_0) and a point $x_0 \in S^w(\lambda_0,\mu_0)$ such that, for any choice of $x_\alpha \in S^w(\lambda_\alpha,\mu_\alpha)$, the net $\{x_\alpha\}$ does not converge to x_0. Since E is usc and $E(\lambda_0)$ is compact, we can assume that $x_\alpha \to \bar{x}_0$ for some $\bar{x}_0 \in E(\lambda_0)$.

From the proof of Theorem 2.6, we see that $\bar{x}_0 \in S^w(\lambda_0,\mu_0)$. By the contradiction assumption one has $\bar{x}_0 \neq x_0$. Due to assumption b) one has

$$f(\bar{x}_0, x_0, \mu_0) \not\leq 0 \quad \text{and} \quad f(x_0, \bar{x}_0, \mu_0) \not\leq 0,$$

which is impossible since $f(.,.,\mu_0)$ is quasimonotone I. ■

Theorem 3.5. *Impose the assumptions of Theorem 2.7. Assume further that*

a) $f(.,.,\mu_0)$ *is quasimonotone II in* $K(A, \lambda_0) \times K(A, \lambda_0)$;

b) *for all* $x \in S^s(\lambda_0, \mu_0)$ *and all* $y \in S^s(\lambda_0, \mu_0) \setminus \{x\}$, *one has* $f(x, y, \mu_0) > 0$.

Then S^s *is continuous at* (λ_0, μ_0).

We can modify the assumptions and use pseudomonotonicity as follows.

Theorem 3.6. *Impose the assumptions of Theorem 2.6 and*

a') $f(.,.,\mu_0)$ *is pseudomonotone I in* $K(A, \lambda_0) \times K(A, \lambda_0)$;

b') *if* $f(x, y, \mu_0) \in \text{bd}(-C)$ *then* $x = y$ (bd(.) *denotes the boundary);*

c') *for all* x *and* $\bar{x}$ *in* $S(\lambda_0, \mu_0)$, $f(x, \bar{x}, \mu_0) \not< 0$.

Then S^w *is continuous at* (λ_0, μ_0).

Proof. We retain the first part of the proof of Theorem 2.6 to obtain $\bar{x}_0 \neq x_0$. By c') one has $f(x_0, \bar{x}_0, \mu_0) \not< 0$ and $f(\bar{x}_0, x_0, \mu_0) \not< 0$. Clearly, a') implies that $f(\bar{x}_0, x_0, \mu_0) \leq 0$. Hence $f(\bar{x}_0, x_0, \mu_0) \in \text{bd}(-C)$. Assumption b') now yields a contradiction that $\bar{x}_0 = x_0$. ■

Theorem 3.7. *Assume as in Theorem 2.7 and that*

a') $f(.,.,\mu_0)$ *is pseudomonotone II in* $K(A, \lambda_0) \times K(A, \lambda_0)$;

b') *if* $f(x, y, \mu_0) \in \text{bd}(C)$ *then* $x = y$;

c') *for all* x *and* $\bar{x}$ *in* $S(\lambda_0, \mu_0)$, $f(x, \bar{x}, \mu_0) \geq 0$.

Then S^s *is continuous at* (λ_0, μ_0).

4. Hölder Continuity of Solution Maps

4.1. Hölder-related Notions

Throughout this section let X, Λ and M be metric spaces and Y be a metric linear space. For convenience we use $d(.,.)$ to denote the metric in any of these spaces (the context makes it clear which space is considered).

Definition 4.1. i) (Classical) A multifunction $K : X \times \Lambda \to 2^X$ is said to be $(l_1.\alpha_1, l_2.\,\alpha_2)-$Hölder at (x_0, λ_0) if there exist neighborhoods N of x_0 and U of λ_0 such that, for all $x_1, x_2 \in N$ and all $\lambda_1, \lambda_2 \in U$,

$$K(x_1, \lambda_1) \subseteq \{x \in X \mid \exists z \in K(x_2, \lambda_2), d(x, z) \leq l_1 d^{\alpha_1}(x_1, x_2) + l_2 d^{\alpha_2}(\lambda_1, \lambda_2)\}.$$

ii) For m, β, $\theta > 0$ and $f : X \times X \times M \to R$, f is termed $m.\beta$-Hölder at $\mu_0 \in M$ θ-relative to $A \subseteq X$ if there is a neighborhood V of μ_0 such that for all $\mu_1, \mu_2 \in V$ and all $x \neq y$ in A,

$$d(f(x,y,\mu_1), f(x,y,\mu_2)) \leq md^{\beta}(\mu_1,\mu_2)d^{\theta}(x,y).$$

Definition 4.2. Let $f : X \times X \to R$.
i) For $h \geq 0$ and $\beta > 0$, f is called $h.\beta$-Hölder-strongly pseudomonotone in $S \subseteq X$ if, for all $x \neq y$ in S,

$$[f(x,y) \geq 0] \Longrightarrow [f(y,x) + hd^{\beta}(x,y) \leq 0].$$

ii) f is called quasimonotone in $S \subseteq X$ if, for all $x \neq y$ in S,

$$[f(x,y) < 0] \Longrightarrow [f(y,x) \geq 0].$$

f is called $h.\beta$-Hölder-strongly monotone in S if, for all $x \neq y$ in S,

$$f(x,y) + f(y,x) + hd^{\beta}(x,y) \leq 0.$$

It is easy to see that if f is $h.\beta$-Hölder-strongly monotone in S, then f is $h.\beta$-Hölder-strongly pseudomonotone in S. The following Hölder-related assumptions (cf. [3]) will be crucial for considering problems (WQEP) and (SQEP). For the reference point $(\lambda_0, \mu_0) \in \Lambda \times M$, there are neighborhoods $U(\lambda_0)$ and $V(\mu_0)$ of λ_0 and μ_0, respectively, such that

(W) for fixed $h > 0$, $\beta > \theta > 0$, all $\mu \in V(\mu_0)$ and all $x \neq y$ in $E\big(U(\lambda_0)\big)$,

$$hd^{\beta}(x,y) \leq d(f(x,y,\mu), Y \setminus -\mathrm{int}C) + d(f(y,x,\mu), Y \setminus -\mathrm{int}C). \tag{5}$$

(S) It is similar to (W), but with (5) replaced by

$$hd^{\beta}(x,y) \leq d(f(x,y,\mu), C) + d(f(y,x,\mu), C).$$

Remark 4.3. These assumptions look complicated. But they are not hard to be checked. To make their meanings clearer we consider the simplest case where $f : X \times X \to R$. Then (W) and (S) coincide and collapse to the following: for all $x \neq y$ in S,

$$hd^{\beta}(x,y) \leq d\big(f(x,y), R_+\big) + d\big(f(y,x), R_+\big). \tag{6}$$

Proposition 4.4. i) *If $f : X \times X \to R$ satisfies* (6) *then f is $h.\beta$-Hölder-strongly pseudomonotone in S* (*the two types defined in Definition 4.2* ii) *coincide in this case*). *Conversely, if f is $h.\beta$-Hölder-strongly pseudomonotone in S and quasimonotone in S, then f satisfies* (6).

ii) *If $f : X \times X \to R$ is $h.\beta-$Hölder-strongly monotone in $K \subseteq X$, then f satisfies* (6).

Examples 1.1 and 1.2 in [3] show that the converse of Proposition 4.4 is not true.

4.2. Hölder Continuity of Solution Maps

Theorem 4.5. *Assume that*
i) *there are neighborhoods* $U(\lambda_0)$ *of* λ_0 *and* $V(\mu_0)$ *of* μ_0 *such that* f *is* $n_1\delta_1$*-Hölder at* μ_0 θ*-relative to* $E(U(\lambda_0))$ *and, for all* $x \in E(U(\lambda_0))$ *and all* $\mu \in V(\mu_0)$, $f(x,.,\mu)$ *is* $n_2.\delta_2$*-Hölder in* $E(U(\lambda_0))$;
ii) *for* (WQEP) (*for* (SQEP)) *assumption* (W) (*assumption* (S)) *is satisfied, respectively.*
iii) $K(.,.)$ *is* $(l_1.\alpha_1, l_2.\alpha_2)$*-Hölder in* $E(U(\lambda_0)) \times \lambda_0$;
iv) $\alpha_1\delta_2 = \beta > \theta$ *and* $h > 2n_2 l_1^{\delta_2}$.
Then, for each (λ,μ) *in a neighborhood of* (λ_0,μ_0), (WQEP) ((SQEP)*, respectively*) *has a unique solution* $x(\lambda,\mu)$ *which satisfies the Hölder condition*

$$d\big(x(\lambda_1,\mu_1), x(\lambda_2,\mu_2)\big) \le k_1 d^{\alpha_2\delta_2/\beta}(\lambda_1,\lambda_2) + k_2 d^{\delta_1/(\beta-\theta)}(\mu_1,\mu_2),$$

where k_1 *and* k_2 *are positive constants depending on* $h, \beta, n_1, n_2, \delta_1, \delta_2, \theta$*, etc.*

Proof. We demonstrate the assertion only for (SQEP). Let $\lambda_1,\lambda_2 \in U(\lambda_0)$ and $\mu_1,\mu_2 \in V(\mu_0)$.

Step 1. We prove that, for all $x(\lambda_1,\mu_1) \in S^s(\lambda_1,\mu_1)$ and all $x(\lambda_1,\mu_2) \in S^s(\lambda_1,\mu_2)$,

$$d_1 := d\big(x(\lambda_1,\mu_1), x(\lambda_1,\mu_2)\big) \le \left(\frac{n_1}{h - 2n_2 l_1^{\delta_2}}\right)^{1/(\beta-\theta)} d^{\delta_1/(\beta-\theta)}(\mu_1,\mu_2). \tag{7}$$

Let $x(\lambda_1,\mu_1) \ne x(\lambda_1,\mu_2)$ (if the equality holds then we are done). Since $x(\lambda_1,\mu_1) \in K(x(\lambda_1,\mu_1),\lambda_1)$, $x(\lambda_1,\mu_2) \in K(x(\lambda_1,\mu_2),\lambda_1)$ and by the Hölder continuity of $K(.,\lambda_1)$, there are $x_1 \in K(x(\lambda_1,\mu_1),\lambda_1)$ and $x_2 \in K(x(\lambda_1,\mu_2),\lambda_1)$ such that

$$d\big(x(\lambda_1,\mu_1), x_2\big) \le l_1 d^{\alpha_1}\big(x(\lambda_1,\mu_1), x(\lambda_1,\mu_2)\big), \tag{8}$$

$$d\big(x(\lambda_1,\mu_2), x_1\big) \le l_1 d^{\alpha_1}\big(x(\lambda_1,\mu_1), x(\lambda_1,\mu_2)\big). \tag{9}$$

As $x(\lambda_1,\mu_1)$ and $x(\lambda_1,\mu_2)$ are solutions of (SQEP), we have

$$f(x(\lambda_1,\mu_1), x_1, \mu_1) \ge 0, \tag{10}$$

$$f(x(\lambda_1,\mu_2), x_2, \mu_2) \ge 0. \tag{11}$$

On the other hand, assumption ii) implies that

$$d(f(x(\lambda_1,\mu_1), x(\lambda_1,\mu_2), \mu_1), C) + d(f(x(\lambda_1,\mu_2), x(\lambda_1,\mu_1), \mu_1), C) \ge h d_1^{\beta}.$$

Hence, by (10) and (11),

$$\begin{aligned} &d(f(x(\lambda_1,\mu_1), x(\lambda_1,\mu_2), \mu_1), f(x(\lambda_1,\mu_1), x_1, \mu_1)) \\ &\quad + d(f(x(\lambda_1,\mu_2), x(\lambda_1,\mu_1), \mu_1), f(x(\lambda_1,\mu_2), x(\lambda_1,\mu_1), \mu_2)) \\ &\quad + d(f(x(\lambda_1,\mu_2), x(\lambda_1,\mu_1), \mu_2), f(x(\lambda_1,\mu_2), x_2, \mu_2)) \ge h d_1^{\beta}. \end{aligned}$$

This together with assumption i) yield

$$n_2 d^{\delta_2}(x(\lambda_1,\mu_2),x_1) + n_1 d_1^{\theta} d^{\delta_1}(\mu_1,\mu_2) + n_2 d^{\delta_2}(x(\lambda_1,\mu_1),x_2) \geq h d_1^{\beta},$$

which, by (8) and (9), implies that

$$n_2 l_1^{\delta_2} d_1^{\alpha_1\delta_2} + n_1 d_1^{\theta} d^{\delta_1}(\mu_1,\mu_2) + n_2 l_1^{\delta_2} d_1^{\alpha_1\delta_2} \geq h d_1^{\beta}.$$

Assumption iv) now yields that

$$d_1^{\beta-\theta} \leq \left(\frac{n_1}{h - 2n_2 l_1^{\delta_2}}\right) d^{\delta_1}(\mu_1,\mu_2).$$

Step 2. We show that, for each $x(\lambda_1,\mu_2) \in S^s(\lambda_1,\mu_2)$ and $x(\lambda_2,\mu_2) \in S^s(\lambda_2,\mu_2)$,

$$d_2 := d\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big) \leq \left(\frac{2n_2 l_2^{\delta_2}}{h - 2n_2 l_1^{\delta_2}}\right)^{1/\beta} d^{\alpha_2\delta_2/\beta}(\lambda_1,\lambda_2). \tag{12}$$

Let $x(\lambda_1,\mu_2) \neq x(\lambda_2,\mu_2)$. Thanks to iii) we have $x_1' \in K(x(\lambda_2,\mu_2),\lambda_1)$ and $x_2' \in K(x(\lambda_1,\mu_2),\lambda_2)$ such that

$$d\big(x(\lambda_1,\mu_2),x_2'\big) \leq l_2 d^{\alpha_2}(\lambda_1,\lambda_2), \tag{13}$$

$$d\big(x(\lambda_2,\mu_2),x_1'\big) \leq l_2 d^{\alpha_2}(\lambda_1,\lambda_2). \tag{14}$$

By the Hölder continuity of $K(.,.)$ there are $x_1'' \in K(x(\lambda_1,\mu_2),\lambda_1)$ and $x_2'' \in K(x(\lambda_2,\mu_2),\lambda_2)$ such that

$$d(x_1',x_1'') \leq l_1 d^{\alpha_1}\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big), \tag{15}$$

$$d(x_2',x_2'') \leq l_1 d^{\alpha_1}\big(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2)\big). \tag{16}$$

By the definition of (SQEP), we have

$$f(x(\lambda_1,\mu_2),x_1'',\mu_2) \geq 0, \tag{17}$$

$$f(x(\lambda_2,\mu_2),x_2'',\mu_2) \geq 0. \tag{18}$$

It follows from assumption ii) that

$$d(f(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2),C) + d(f(x(\lambda_2,\mu_2),x(\lambda_1,\mu_2),\mu_2),C) \geq h d_2^{\beta}.$$

Due to (17) and (18), one has

$$\begin{aligned}
&d(f(x(\lambda_1,\mu_2),x(\lambda_2,\mu_2),\mu_2),f(x(\lambda_1,\mu_2),x_1',\mu_2))\\
&\quad + d(f(x(\lambda_1,\mu_2),x_1',\mu_2),f(x(\lambda_1,\mu_2),x_1'',\mu_2))\\
&\quad + d(f(x(\lambda_2,\mu_2),x(\lambda_1,\mu_2),\mu_2),f(x(\lambda_2,\mu_2),x_2',\mu_2))
\end{aligned}$$

$$+ d(f(x(\lambda_2, \mu_2), x'_2, \mu_2), f(x(\lambda_2, \mu_2), x''_2, \mu_2))$$
$$\geq hd_2^{\beta}.$$

Hence, the Hölder continuity assumptions in i) of f imply that

$$n_2 d^{\delta_2}(x(\lambda_2, \mu_2), x'_1) + n_2 d^{\delta_2}(x'_1, x''_1) + n_2 d^{\delta_2}(x(\lambda_1, \mu_2), x'_2) + n_2 d^{\delta_2}(x'_2, x''_2) \geq hd_2^{\beta}.$$

From (13), (14), (15) and (16) we have

$$n_2 l_2^{\delta_2} d^{\alpha_2 \delta_2}(\lambda_1, \lambda_2) + n_2 l_1^{\delta_2} d_2^{\alpha_1 \delta_2} + n_2 l_2^{\delta_2} d^{\alpha_2 \delta_2}(\lambda_1, \lambda_2) + n_2 l_1^{\delta_2} d_2^{\alpha_1 \delta_2} + \geq hd_2^{\beta}.$$

Then (12) follows from assumption iv), since

$$d_2^{\beta} \leq \left(\frac{2n_2 l_2^{\delta_2}}{h - 2n_2 l_1^{\delta_2}} \right) d^{\alpha_2 \delta_2}(\lambda_1, \lambda_2).$$

Step 3. Finally, for all $x(\lambda_1, \mu_1) \in S^s(\lambda_1, \mu_1)$ and all $x(\lambda_2, \mu_2) \in S^s(\lambda_2, \mu_2)$, from

$$d\big(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)\big) \leq d_1 + d_2,$$

it follows that

$$\rho\big(S(\lambda_1, \mu_1), S(\lambda_2, \mu_2)\big) \leq k_1 d^{\alpha_2 \delta_2/\beta}(\lambda_1, \lambda_2) + k_2 d^{\delta_1/(\beta - \theta)}(\mu_1, \mu_2),$$

where $k_1 = \left(\frac{2n_2 l_2^{\delta_2}}{h - 2n_2 l_1^{\delta_2}} \right)^{\frac{1}{\beta}}$ and $k_2 = \left(\frac{n_1}{h - 2n_2 l_1^{\delta_2}} \right)^{\frac{1}{\beta - \theta}}$.

Taking $\lambda_2 = \lambda_1$ and $\mu_2 = \mu_1$ we see that the diameter of $S^s(\lambda_1, \mu_1)$ is 0, i.e., this set is a singleton $\{x(\lambda_1, \mu_1)\}$. Similarly, $S^s(\lambda_2, \mu_2)$ is also a singleton. Thus the solution is unique and the required Hölder condition is checked. ■

In the special case where $Y = R$ and $C = R_+$, (WQEP) and (SQEP) become (QEP) studied in [11]. From Proposition 4.4 we see that Theorem 4.5 not only generalizes but also improves Theorems 2.1 and 2.2 of [11].

5. Particular Cases

Since equilibrium problems contain - as special cases - many problems as mentioned in Sec. 1, we can derive from the results of Sec. 2-4 consequences for such special cases. Now we discuss only some corollaries for quasivariational inequalities as examples.

Let X be a normed space, $A \subseteq X$ be nonempty and convex, Λ be a metric space and $K : A \times \Lambda \to 2^A$. Let X^* be the dual space of X and $T : X \times \Lambda \to X^*$. We consider the following parametric quasivariational inequality, for each $\lambda \in \Lambda$,

(QVI) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that, for all $y \in K(\bar{x}, \lambda)$,

$$\langle T(\bar{x},\lambda), y-\bar{x}\rangle \geq 0,$$

where $\langle .,. \rangle$ denotes the pairing between X and X^*.

Setting $Y = R, C = R_+$ and $f(x,y,\mu) = \langle T(x,\mu), y-x\rangle$, (QVI) becomes a case of (WQEP) and (SQEP) (in this special case the two problem coincide). Consequently, the following result is immediate from Theorem 2.6.

Corollary 5.1. *The solution map S of* (QVI) *is usc at λ_0 if*

i) $\{(x,y,\lambda) \mid \langle T(y,\lambda), y-x\rangle \geq 0\}$ *is closed in* $K(A,\Lambda)\times K(A,\Lambda)\times\{\lambda_0\}$;

ii) *E is usc at λ_0, $E(\lambda_0)$ is compact and K is lsc in $K(A,\Lambda)\times K(A,\Lambda)$.*

Corollary 5.1 includes and improves Theorems 2.2 and 2.3 of [21], Theorems 4.1 and 4.3 of [22]. Similarly, we can obtain direct corollaries of Theorems 2.13, 2.15, 3.4, 3.6 and these results are new for (QVI).

Corollary 5.2. *For problem* (QVI) *assume that*

a) *there are neighborhoods $U(\lambda_0)$ of λ_0 and $V(\mu_0)$ of μ_0 such that, for all $x \in E(U(\lambda_0))$, $T(x,.)$ is $n_3.\delta_3$-Hölder at μ_0 and $T(.,.)$ is bounded in $E\big(U(\lambda_0)\big)\times V(\mu_0)$, and $E\big(U(\lambda_0)\big)$ is bounded;*

b) *for all $\mu \in V(\mu_0)$, $T(.,\mu)$ is $h.\beta$-Hölder strongly monotone in $E(U(\lambda_0))$, i.e., $\forall x,y \in E(U(\lambda_0))$,*

$$\langle T(x,\mu) - T(y,\mu), y-x\rangle + h\|x-y\|^\beta \leq 0;$$

c) *K is $(l_1.\alpha_1, l_2.\alpha_2)$-Hölder in $E\big(U(\lambda_0)\big)\times\{\lambda_0\}$;*

d) *$\alpha_1 = \beta$ and $h > 2n_2 l_1$.*

Then, in a neighborhood of (λ_0,μ_0), the solution $x(\lambda,\mu)$ of (QVI) *is unique and has the following Hölder continuity property*

$$d\big(x(\lambda_1,\mu_1), x(\lambda_2,\mu_2)\big) \leq k_1 d^{\alpha_2/\beta}(\lambda_1,\lambda_2) + k_2 d^{\delta_3/\beta}(\mu_1,\mu_2),$$

where k_1 and k_2 are positive constants depending on h,β,n_3,δ_3, etc.

Proof. We verify the assumptions of Theorem 4.5. i) is fulfilled with $n_1 = Nn_3, \delta_1 = \delta_3, \theta = 0, n_2 = M$ and $\delta_2 = 1$, where $N, M > 0$ are such that $\|T(x,\mu)\| \leq M$ for each $(x,\mu) \in E(U(\lambda_0))\times V(\mu_0)$, and $\|x-y\| \leq N$ for each $x,y \in E(U(\lambda_0))$.

For ii) one has, by b),

$$\begin{aligned} 0 &\geq \langle T(x,\mu) - T(y,\mu), y-x\rangle + h\|y-x\|^\beta \\ &= f(x,y,\mu) + f(y,x,\mu) + h\|y-x\|^\beta. \end{aligned}$$

Applying Proposition 4.4 ii) we see that assumption ii) of Theorem 4.5 is satisfied.

iii) is the same as c). Finally, for iv) one has from d) that $\alpha_1\delta_2 = \beta > \theta$ and $h > 2n_2 l_1$, as $\delta_2 = 1$ and $\theta = 0$. ∎

References

1. L. Q. Anh and P. Q. Khanh, Semicontinuity of the solution sets of parametric multivalued vector quasiequilibrium problems, *J. Math. Anal. Appl.* **294** (2004), 699–711.
2. L. Q. Anh and P. Q. Khanh, On the Hölder continuity of solutions to parametric multivalued vector equilibrium problems, *J. Math. Anal. Appl.* **321** (2006), 308–315.
3. L. Q. Anh and P. Q. Khanh, Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces, *J. Glob. Optim.* **37** (2007), 449–465.
4. L. Q. Anh and P. Q. Khanh, On the stability of the solution sets of general multivalued vector quasiequilibrium problems, *J. Optim. Theory Appl.* **135** (2007), 271–284.
5. L. Q. Anh and P. Q. Khanh, Existence conditions in symmetric multivalued vector quasiequilibrium problems, *Control Cyber.* **36** (2007), 518–530.
6. L. Q. Anh and P. Q. Khanh, Sensitivity analysis for multivalued quasiequilibrium problems in metric spaces: Hölder continuity of solutions, *J. Glob. Optim.* **42** (2008), 515–531
7. L. Q. Anh and P. Q. Khanh, Semicontinuity of solution sets to parametric quasivariational inclusions with applications to traffic networks I: Upper semicontinuities, *Set-valued Anal.* **16** (2008), 267–279.
8. L. Q. Anh and P. Q. Khanh, Semicontinuity of the approximate solution sets of multivalued quasiequilibrium problems, *Numerical Funct. Anal. Optim.* **29** (2008), 24–42.
9. L. Q. Anh and P. Q. Khanh, Various kinds of semicontinuity and the solution sets of parametric multivalued symmetric vector quasiequilibrium problems, *J. Glob. Optim.* **41** (2008), 539–558.
10. L. Q. Anh and P. Q. Khanh, Semicontinuity of solution sets to parametric quasivariational inclusions with applications to traffic networks II: Lower semicontinuities, *Set-valued Anal.* **16** (2008), 943–960.
11. L. Q. Anh and P. Q. Khanh, Hölder continuity of the unique solution to quasiequilibrium problems in metric spaces, *J. Optim. Theory Appl.* **141** (2009), 37–54.
12. L. Q. Anh and P. Q. Khanh, Continuity of solution maps of parametric quasiequilibrium problems, *J. Glob. Optim.* (2009), online first.
13. L. Q. Anh, P. Q. Khanh, D. T. M. Van, and J.-C. Yao, Well-posedness under perturbations for vector quasiequilibria with applications to bilevel problems, *Taiwanese J. Math.* **13** (2009), 713-737.
14. N. X. Hai and P. Q. Khanh, Systems of multivalued quasiequilibrium problems, *Adv. Nonlinear Var. Inequal.* **9** (2006), 97–108.
15. N. X. Hai and P. Q. Khanh, Existence of solutions to general quasiequilibrium problems and applications, *J. Optim. Theory Appl.* **133** (2007), 317–327.
16. N. X. Hai and P. Q. Khanh, The solution existence of general variational inclusion problems, *J. Math. Anal. Appl.* **328** (2007), 1268–1277.
17. N. X. Hai and P. Q. Khanh, Systems of set-valued quasivariational inclusion problems, *J. Optim. Theory Appl.* **135** (2007), 55–67.
18. N. X. Hai and P. Q. Khanh, The existence of ε-solutions to general quasiequilibrium problems, *Vietnam J. Math.* **35** (2007), 563–572.
19. N. X. Hai, P. Q. Khanh, and N. H. Quan, On the existence of solutions to quasivariational inclusion problems, *J. Glob. Optim.* (2009), to appear.
20. P. Q. Khanh and D. T. Luc, Stability of solutions in parametric variational relation problems, *Set-valued Anal.* (2008), online first.
21. P. Q. Khanh and L. M. Luu, Upper semicontinuity of the solution set of parametric multivalued vector quasivariational inequalities and applications, *J. Glob. Optim.* **32** (2005), 551–568.

22. P. Q. Khanh and L. M. Luu, Lower and upper semicontinuity of the solution sets and approximate solution sets to parametric multivalued quasivariational inequalities, *J. Optim. Theory Appl.* **133** (2007), 329–339.
23. P. Q. Khanh and N. H. Quan, Existence conditions for quasivariational inclusion problems in G-convex spaces, *Acta Math. Vietnam.* (2009), to appear.
24. P. Q. Khanh and N. H. Quan, The solution existence of general inclusions using generalized KKM theorems with applications to minimax problems, *submitted.*
25. P. Q. Khanh, N. H. Quan, and J.-C. Yao, Generalized KKM-type theorems in GFC-spaces and applications, *Nonlinear Anal.* (2008), online first.