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On Partially Elliptic and Coercive Boundary Problems

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Abstract. Applying iteration method, we prove fixed point theorems for operators, which may neither be continuous nor monotone. Using these results and some considerations in sub-supersolution methods, we can partially relax the coercivity, ellipticity and compactness in some boundary problems.

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1. Introduction

Let X be a non-empty set, $\leq$ and d be a partially order and a metric on X respectively. We call $(X, d, \leq)$ an ordered metric space if $(X, d, \leq)$ satisfies the following condition

(C) $x \leq y$ (resp. $y \leq x$) for any x and y in X such that x is the limit of an increasing (resp. decreasing) sequence $\{x_n\}$ and $x_n \leq y$ (resp. $y \leq x_n$) for any integer n.

We say $x \geq y$ (resp. $x < y$; $x > y$) if $y \leq x$ (resp. $x \leq y$ and $x \neq y$; $y \leq x$ and $x \neq y$).

The continuity and monotonicity of mappings and their modified versions play essential roles of fixed point theorems in ordered metric spaces (see [2, 3, 5-7, 10-13, 16-18]). The motivation of our paper is the following example: let $f(t) = t$ if t is a rational number in the interval $(0, 1]$ and $f(t) = \frac{1}{2} + \frac{1}{2}t$ if t is a irrational number in the interval $(0, 1]$. We see that f has many fixed points in $(0, 1]$, but it is neither continuous nor monotone in $(0, 1]$. We point out that the relation between x and $f(x)$ can give us the fixed points of f by using iteration methods. We obtain the following result.

Theorem 1.1. *Let A be a non-empty subset of an ordered metric space $(X, d, \leq)$, and f be an operator from X into itself. Suppose that*

(i) *$f(A) \subset A$ and $x \leq f(x)$ for any x in A,*

(ii) *each increasing sequence of A has a limit in X and an upper bound in A.*

Then f has a fixed point in A.

Applying this result we solve a class of elliptic equations in the last section.

2. Proof of Theorem 1.1

We will prove the theorem by using the lemmas, what follow.

Lemma 2.1. *Let W be a non-empty subset of an ordered metric space $(X, d, \leq)$, and g be a mapping from W into W. Suppose that*

(i) *$x \leq g(x)$ for any x in W, and*

(ii) *$\{g(x_n)\}$ has a limit in X and an upper bound in W for any increasing sequence $\{x_n\}$ in W.*

Then W has a maximal element y, i.e. $a = y$ whenever a is in W and $y \leq a$.

Proof. By Hausdorff's principle, there exists a maximal chain B of W. Now we prove that B has the greatest element. Let x_0 be an arbitrary element of B. We shall show that there is a sequence $\{x_n\}$ in B having the following property

$$x_n \geq x_{n-1} \text{ and } d(g(x), g(x_n)) < \frac{1}{n}, \forall\, x \in \{z \in B : z \geq x_n\}, n \in \mathbb{N}. \quad (1)$$

Suppose by contradiction that we only can find a finite family $\{x_0, \ldots, x_{m-1}\}$ satisfying (1), where m is a positive integer. In this case, for each x in $\{z \in B : z \geq x_{m-1}\}$, we can find y_x in B such that $y_x > x$ and $d(g(x), g(y_x)) \geq \frac{1}{m}$. Hence we can construct an increasing sequence $\{y_k\}$ such that $y_0 = x_{m-1}$ and $d(g(y_{k+1}), g(y_k)) \geq \frac{1}{m}$ for any non-negative integer k. Since $\{y_k\}$ is increasing, $\{g(y_k)\}$ has a limit. This is a contradiction and we get such a sequence $\{x_n\}$.

Since $\{x_n\}$ is increasing, then $\{g(x_n)\}$ has a limit x in X and an upper bound y in W. Because $x_n \leq g(x_n)$ for any non-negative integer n, y is also an upper bound of $\{x_n\}$. Since $(X, d, \leq)$ is an ordered metric space, we have $x \leq y$. Let z be in B, we prove that $z \leq y$. If $z \leq x_n$ for some positive integer n, then $z \leq y$. Otherwise, $z > x_n$ for any positive integer n. Hence $d(g(z), g(x_n)) < \frac{1}{n}$, for any positive integer n, which implies $z \leq g(z) = x \leq y$. Since B is a maximal chain, then $y \in B$ and y is the greatest element of B.

Finally, we show that y is a maximal element of W. Suppose by contradiction that there exists a in W such that $a > y$. Then $B \cup \{a\}$ is a chain containing B and B is not a maximal chain. This contradiction yields the lemma. ∎

Lemma 2.2. *Let W be a non-empty set in an ordered metric space $(X, d, \leq)$. Suppose that each increasing sequence of W has a limit in X and an upper bound in W. Then W has a maximal element.*

Proof. Apply Lemma 2.1 for the case $g(x) \equiv x$, we get the lemma. ∎

Lemma 2.3. *Let U be a non-empty ordered set and f be an operator from U into U such that $x \leq f(x)$ for any x in U. Suppose that α is a maximal element of U. Then α is a fixed point of f.*

Proof. We have $\alpha \leq f(\alpha)$ and $f(\alpha)$ is in U. Thus $\alpha = f(\alpha)$.

Combining Lemmas 2.2 and 2.3, we get the theorem. ∎

Remark 2.4. Our results relax the monotonicity in [2, 3, 5-7, 10-12, 16-18]. In next sections, using this idea, we can solve some equations involving with operators which may not be monotone.

3. Applications to Elliptic Equations with Discontinuity

Let N be a positive integer, Ω be a smooth bounded open subset of R^N and p and r be in $(1, \infty)$. We denote by $L^s(\Omega)$ and $W_0^{1,s}(\Omega)$ the usual Lebesgue space and Sobolev space as in [1] for any s in $[1, \infty)$. Let $a_1, \ldots, a_N$ be real functions on $\Omega \times \mathbb{R} \times \mathbb{R}^N$, f be a real function on $\Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ having the following properties.

(A0) The functions $a_1, \ldots, a_N$ satisfy the Caratheodory conditions on $\Omega \times \mathbb{R} \times \mathbb{R}^N$.

(A1) There exist $k_0 \in L^{p/p-1}(\Omega)$, a non-negative real number C_0, and $\underline{u}$ and $\overline{u}$ in $W_0^{1,p}(\Omega) \cap L^r(\Omega)$ such that for all (s, ζ) in $[\underline{u}(x), \overline{u}(x)] \times \mathbb{R}^N$ and for almost everywhere x in Ω, we have

$$|a_i(x, s, \zeta)| \leq k_0(x) + C_0(|s|^{\frac{r(p-1)}{p}} + |\zeta|^{p-1}) \quad \forall\, i = 0, \ldots, N.$$

(A2) For almost everywhere x in Ω, all s in $[\underline{u}(x), \overline{u}(x)]$ and any $\zeta \neq \zeta'$ in $\mathbb{R}^N$

$$\sum_{i=1}^{N} [a_i(x, s, \zeta) - a_i(x, s, \zeta')](\zeta_i - \zeta'_i) > 0.$$

(A3) There exist $C_1 > 0$ and $k_1 \in L^1(\Omega)$ such that for all (s, ζ) in $[\underline{u}(x), \overline{u}(x)] \times \mathbb{R}^N$ and for almost everywhere x in Ω

$$\sum_{i=1}^{N} a_i(x,s,\zeta)\zeta_i \geq C_1|\zeta|^p - k_1(x).$$

(F1) There exist a function $k_2 \in L^{p/p-1}(\Omega)$ and a constant $C_2 \geq 0$ such that

$$|f(x,t,s,\zeta)| \leq k_2(x)+C_2(|s|^{\frac{r(p-1)}{p}}+|\zeta|^{p-1}) \text{ a.e.} x \in \Omega, \forall\, \zeta \in R^N, t, s \in [\underline{u}(x), \overline{u}(x)]$$

(F2) The function f satisfies the Caratheodory conditions on $\Omega \times \mathbb{R}^{N+2}$, and there exist a continuous real function a on $\mathbb{R}$ and a non-negative real number C_3 such that: the function $f(x,.,s,\zeta)+a(.)$ is increasing on $[\underline{u}(x), \overline{u}(x)]$ for almost everywhere x in Ω and for any $(s,\zeta) \in [\underline{u}(x), \overline{u}(x)] \times \mathbb{R}^N$, and

$$|a(t)| \leq C_3(1+|t|^{\frac{r(p-1)}{p}}) \text{ and } [a(t_1)-a(t_2)](t_1-t_2) \geq 0 \text{ for any } t \in \mathbb{R}.$$

Remark 3.1. For almost everywhere x in Ω, we only need the conditions (A1), (A2), (A3), (F1) and (F2) for any s in $[\underline{u}(x), \overline{u}(x)]$ instead of in the whole $\mathbb{R}$, therefore our results can be applied to the cases that we partially have the ellipticity, coercivity and compactness.

In this section we consider the following equation

$$\begin{cases} -\sum\limits_{i=1}^{N} \dfrac{\partial}{\partial x_i} a_i(x,u,\nabla u) = f(x,u,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Let u be in $W_0^{1,p}(\Omega)$. Then u is called a solution (resp. subsolution, supersolution) of (2) if

$$\int_\Omega \sum_{i=1}^{N} a_i(x,u,\nabla u)\frac{\partial\varphi}{\partial x_i}dx + \int_\Omega f(x,u,u,\nabla u)\varphi dx = 0 \text{ (resp. } \leq, \geq)$$

for all $v \in W_0^{1,p}(\Omega), v \geq 0$.

The main result of this section is the following theorem.

Theorem 3.2. *Suppose that the conditions* (A0), (A1)-(A3), (F1) *and* (F2) *are satisfied,* $\underline{u}$ *and* $\overline{u}$ *are a subsolution and a supersolution of* (2) *respectively. Then* (2) *has a solution* u *in* $[\underline{u}, \overline{u}]$.

In order to prove the theorem we need following lemmas.

Lemma 3.3. *For any* u *in* $W_0^{1,p}(\Omega)$, *we put*

$$T(u(x)) = \begin{cases} \overline{u}(x) & \text{if } u(x) > \overline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x), \\ \underline{u}(x) & \text{if } u(x) < \underline{u}(x), \end{cases}$$

and we define $S_1(u)$ in $(W_0^{1,p}(\Omega))^*$ as follows

$$< S_1(u), \varphi >= \int_\Omega \sum_{i=1}^N a_i(x, T(u), \nabla u) \frac{\partial \varphi}{\partial x_i} dx \ \ \forall \varphi \in W^{1,p}(\Omega).$$

Then S_1 is a $(S)_+$ operator on $W^{1,p}(\Omega)$, i.e. it has the following properties.

(i) *$\{S_1(u_n)\}$ converges weakly to $S_1(u)$ in $(W_0^{1,p}(\Omega))^*$ for any sequence $\{u_n\}$ converging strongly to u in $W_0^{1,p}(\Omega)$.*

(ii) *Let $\{u_n\}$ be a sequence in $W_0^{1,p}(\Omega)$ such that $\{u_n\}$ converges weakly to u in $W_0^{1,p}(\Omega)$. Then $\{u_n\}$ converges strongly to x in $W_0^{1,p}(\Omega)$ if*

$$\limsup_{n\to\infty} < S_1(u_n), u_n - u >\leq 0.$$

Moreover S_1 is pseudomonotone, i.e.

(iii) *If $\{u_n\}$ weakly converges to x in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n\to\infty} < S_1(x_n), x_n - x >\leq 0,$$

then $\{S_1(x_n)\}$ weakly converges to $S_1(x)$ in $(W_0^{1,p}(\Omega))^*$ and

$$\lim_{n\to\infty} < S_1(x_n), x_n - x >= 0.$$

Proof. (i) We note that T is a bounded and continuous operator from $W_0^{1,p}(\Omega)$ into itself (see [8]). Let w be in $W_0^{1,p}(\Omega)$, we see that $|Tw(x)| \leq (|\overline{u}(x)| + |\underline{u}(x)|)$, therefore Tw belongs to $L^r(\Omega)$ by $(A1)$ and for all ζ in $\mathbb{R}^N$ and for almost everywhere x in Ω, we have

$$|a_i(x, Tw(x), \zeta)| \leq k_0(x) + C_0(|\overline{u}(x)| + |\underline{u}(x)|)^{\frac{r(p-1)}{p}} + C_0|\zeta|^{p-1} \, \forall\, i = 0, \dots, N.$$

Applying a result on superposition operators (see [14, p. 30]), we get the continuity of the map $w \mapsto a_i(x, Tw(x), \nabla w)$ from $W_0^{1,p}(\Omega)$ into $L^{p/p-1}(\Omega)$, and (i).

(ii) and (iii) Let $\{u_n\}$ be a sequence weakly converging to u in $W_0^{1,p}(\Omega)$ such that

$$\limsup_{n\to\infty} < S_1 u_n, u_n - u >\leq 0.$$

We shall prove (ii) and (iii) by the following steps.

Step 1. We show that $\{\nabla u_n\}$ converges pointwisely to ∇u almost everywhere in Ω.

Using (A2), we have

$$< S_1 u_n, u_n - u > = \int_\Omega \sum_{i=1}^N [a_i(x, T(u_n), \nabla u_n) - a_i(x, T(u_n), \nabla u)] \frac{\partial}{\partial x_i}(u_n - u) dx$$

$$+\int_{\Omega}\sum_{i=1}^{N}a_i(x,T(u_n),\nabla u)\frac{\partial}{\partial x_i}(u_n-u)dx$$
$$\geq\int_{\Omega}\sum_{i=1}^{N}a_i(x,T(u_n),\nabla u)\frac{\partial}{\partial x_i}(u_n-u)dx.$$

Note that the sequence $\left\{\frac{\partial}{\partial x_i}(u_n-u)\right\}$ converges weakly to 0 in $L^p(\Omega)$. By the Sobolev embedding theorem, $(A1)$ and the Lebesgue dominated convergence theorem, we see that $\{a_i(x,T(u_n),\nabla u)\}$ converges strongly to $a_i(x,T(u),\nabla u)$ in $L^q(\Omega)$. Therefore, we obtain

$$\lim_{n\to\infty}\int_{\Omega}\sum_{i=1}^{N}a_i(x,T(u_n),\nabla u)\frac{\partial}{\partial x_i}(u_n-u)dx=0.$$

Since $\limsup_{n\to\infty} < S_1u_n, u_n-u > \leq 0$, it follows that

$$\lim_{n\to\infty} < S_1u_n, u_n-u >= 0. \tag{3}$$

Thus

$$\lim_{n\to\infty}\int_{\Omega}\sum_{i=1}^{N}[a_i(x,T(u_n),\nabla u_n)-a_i(x,T(u_n),\nabla u)]\frac{\partial}{\partial x_i}(u_n-u)dx=0.$$

By (A2), it implies the convergence in $L^1(\Omega)$ of the sequence of non-negative functions

$$\left\{\sum_{i=1}^{N}[a_i(x,T(u_n),\nabla u_n)-a_i(x,T(u_n),\nabla u)]\frac{\partial}{\partial x_i}(u_n-u)\right\}.$$

By Theorem IV.9 in [4], we can assume that

$$\lim_{n\to\infty}\sum_{i=1}^{N}[a_i(x,T(u_n),\nabla u_n)-a_i(x,T(u_n),\nabla u)]\frac{\partial}{\partial x_i}(u_n-u)=0 \textit{ a.e. in } \Omega \tag{4}$$

and there is a non-negative integrable function h on Ω such that

$$\sum_{i=1}^{N}[a_i(x,T(u_n),\nabla u_n)-a_i(x,T(u_n),\nabla u)]\frac{\partial}{\partial x_i}(u_n-u)\leq h(x) \textit{ a.e. in } \Omega. \tag{5}$$

Denote by Ω_0 the set of all x in Ω such that

$$\lim_{n\to\infty}\sum_{i=1}^{N}[a_i(x,T(u_n)(x),\nabla u_n(x))-a_i(x,T(u_n)(x),\nabla u(x))]\frac{\partial(u_n-u)}{\partial x_i}(x)=0 \tag{6}$$

and

$$\lim_{n\to\infty}T(u_n)(x)=T(u)(x). \tag{7}$$

We see that the measure of $\Omega\setminus\Omega_0$ is null. Let x be in Ω_0, we shall prove that $\{\nabla u_n(x)\}$ converges to $\nabla u(x)$. Assume by contradiction that there is a subsequence $\{\nabla u_{n_m}(x)\}$ of $\{\nabla u_n(x)\}$ such that $|\nabla u_{n_m}(x)-\nabla u(x)|>\epsilon$ for some positive real number ϵ and for every integer m. Denote $\nabla u(x)$, $\nabla u_{n_m}(x)$, $T(u_{n_m}(x))$ and $T(u(x))$ by ρ, ρ_m, s_m and s respectively. We can suppose that $\left\{\dfrac{\rho_m-\rho}{|\rho_m-\rho|}\right\}$ converges to ρ^* in $\mathbb{R}^N$. Note that $|\rho^*|=1$. Using (A2), we have

$$\begin{aligned}&\sum_{i=1}^{N}[a_i(x,s_m,\rho_m)-a_i(x,s_m,\rho+\epsilon\tfrac{\rho_m-\rho}{|\rho_m-\rho|})](\rho_{mi}-\rho_i)\\&=\frac{|\rho_m-\rho|}{|\rho_m-\rho|-\epsilon}\sum_{i=1}^{N}\Big[a_i(x,s_m,\rho_m)-a_i(x,s_m,\rho+\epsilon\tfrac{\rho_m-\rho}{|\rho_m-\rho|})\Big]\times\\&\quad\times\left(1-\frac{\epsilon}{|\rho_m-\rho|}\right)(\rho_{mi}-\rho_i)\\&\geq 0,\end{aligned} \tag{8}$$

$$\begin{aligned}0\leq&\sum_{i=1}^{N}[a_i(x,s_m,\rho+\epsilon\tfrac{\rho_m-\rho}{|\rho_m-\rho|})-a_i(x,s_m,\rho)](\rho_{mi}-\rho_i) \qquad (9)\\=&\sum_{i=1}^{N}[a_i(x,s_m,\rho+\epsilon\tfrac{\rho_m-\rho}{|\rho_m-\rho|})-a_i(x,s_m,\rho_m)](\rho_{mi}-\rho_i)\\&+\sum_{i=1}^{N}[a_i(x,s_m,\rho_m)-a_i(x,s_m,\rho)](\rho_{mi}-\rho_i).\end{aligned}$$

Combining (8) and (9), we get

$$\begin{aligned}0&\leq\sum_{i=1}^{N}[a_i(x,s_m,\rho+\epsilon\frac{\rho_m-\rho}{|\rho_m-\rho|})-a_i(x,s_m,\rho)]\frac{\rho_{mi}-\rho_i}{|\rho_m-\rho|}\\&\leq\frac{1}{|\rho_m-\rho|}\sum_{i=1}^{N}[a_i(x,s_m,\rho_m)-a_i(x,s_m,\rho)](\rho_{mi}-\rho_i).\end{aligned} \tag{10}$$

Since $|\rho_m-\rho|>\epsilon$, by (6) and (A0), we have

$$\sum_{i=1}^{N}[a_i(x,s,\rho+\epsilon\rho^*)-a_i(x,s,\rho)]\rho_i^*=0.$$

Therefore, $\rho^* = 0$ by $(A2)$. This is a contradiction and the sequence $\{\nabla u_n(x)\}$ should converge to $\nabla u(x)$ and we get the first step.

Step 2. $\{u_n\}$ converges strongly to u in $W_0^{1,p}(\Omega)$.

Let E be a measurable subset of Ω, by (A1), (A3), we have

$$C_1 \int_E |\nabla u_n|^p dx \le \int_E k_1(x)dx + \int_E \sum_{i=1}^N a_i(x, T(u_n), \nabla u_n) \frac{\partial u_n}{\partial x_i} dx$$
$$= \int_E k_1(x)dx + \sum_{j=1}^4 I_j,$$

where

$$I_1 = \int_E \sum_{i=1}^N [a_i(x, T(u_n), \nabla u_n) - a_i(x, T(u_n), \nabla u)] \frac{\partial(u_n - u)}{\partial x_i} dx \le \int_E h(x)dx,$$

$$I_2 = \int_E \sum_{i=1}^N a_i(x, T(u_n), \nabla u_n) \frac{\partial u}{\partial x_i} dx$$
$$\le \sum_{i=1}^N \left(\int_E |a_i(x, T(u_n), \nabla u_n)|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_E |\frac{\partial u}{\partial x_i}|^p dx \right)^{1/p}$$
$$\le \sum_{i=1}^N \left\| k_0 + C_0 |T(u_n)|^{\frac{r(p-1)}{p}} + C_0 |\nabla u_n|^{p-1} \right\|_{L^{\frac{p}{p-1}}(E)} \left(\int_E |\frac{\partial u}{\partial x_i}|^p dx \right)^{1/p}$$
$$\le \sum_{i=1}^N \left\| k_0(x) + C_0 (|\underline{u}|^{\frac{r(p-1)}{p}} + |\overline{u}|^{\frac{r(p-1)}{p}}) + C_0 |\nabla u_n|^{p-1} \right\|_{L^{\frac{p}{p-1}}(E)} \times$$
$$\times \left(\int_E |\frac{\partial u}{\partial x_i}|^p dx \right)^{1/p}$$
$$\le \sum_{i=1}^N \left\{ ||k_0||_{L^q(E)} + C_0 ||\underline{u}||_{L^r(E)}^{\frac{r(p-1)}{p}} + C_0 ||\overline{u}||_{L^r(E)}^{\frac{r(p-1)}{p}} + C_0 ||\nabla u_n||_{L^p(E)}^{p-1} \right\} \times$$
$$\times \left(\int_E |\frac{\partial u}{\partial x_i}|^p dx \right)^{1/p},$$

$$I_3 = \int_E \sum_{i=1}^N a_i(x, T(u_n), \nabla u) \frac{\partial u_n}{\partial x_i} dx$$

$$\leq \sum_{i=1}^{N}\left[\int_E |a_i(x,T(u_n),\nabla u)|^{\frac{p}{p-1}}dx\right]^{\frac{p-1}{p}}\left(\int_E |\frac{\partial u_n}{\partial x_i}|^p dx\right)^{1/p}$$

$$\leq \sum_{i=1}^{N}\left\{\|k_0\|_{L^q(E)}+C_0\|\underline{u}\|_{L^r(E)}^{\frac{r(p-1)}{p}}+C_0\|\overline{u}\|_{L^r(E)}^{\frac{r(p-1)}{p}}+C_0\|\nabla u\|_{L^p(E)}^{p-1}\right\}\times$$

$$\times\left(\int_E |\frac{\partial u_n}{\partial x_i}|^p dx\right)^{1/p},$$

$$I_4=-\int_E \sum_{i=1}^{N} a_i(x,T(u_n),\nabla u)\frac{\partial u}{\partial x_i}dx$$

$$\leq \sum_{i=1}^{N}\left[\int_E |a_i(x,T(u_n),\nabla u)|^{\frac{p}{p-1}}dx\right]^{\frac{p-1}{p}}\left(\int_E |\frac{\partial u}{\partial x_i}|^p dx\right)^{1/p}$$

$$\leq \sum_{i=1}^{N}\left\{\|k_0\|_{L^q(E)}+C_0\|\underline{u}\|_{L^r(E)}^{\frac{r(p-1)}{p}}+C_0\|\overline{u}\|_{L^r(E)}^{\frac{r(p-1)}{p}}+C_0\|\nabla u\|_{L^p(E)}^{p-1}\right\}\times$$

$$\times\left(\int_E |\frac{\partial u}{\partial x_i}|^p dx\right)^{1/p}.$$

Let ε be a positive real number. By the boundedness of $\left\{\|\nabla u_n\|_{L^p(\Omega)}\right\}$, the r-integrability of $\overline{u}$ and $\underline{u}$, and conditions (A1) and (A3), there is a positive real number δ such that for any measurable subset E of Ω with Lebesgue measure $m(E)<\delta$, we have

$$\int_E |\nabla u_n|^p dx \leq \varepsilon \quad \forall\, n\in\mathbb{N}.$$

Thus the sequence $\{|\nabla u_n|^p\}$ is equi-integrable. It follows that $\{|\nabla u_n-\nabla u|^p\}$ is also equi-integrable. By Vitali's theorem (see [19]), $\{\nabla u_n\}$ converges to ∇u in $L^p(\Omega)$, which implies $\{u_n\}$ converges strongly to u in $W^{1,p}(\Omega)$.

Step 3. $\{S_1(u_n)\}$ weakly converges to $S_1(u)$ in $(W_0^{1,p}(\Omega))^*$.

By the previous steps, $\{T(u_n)\}$ and $\{\nabla u_n\}$ converge to $T(u)$ and ∇u in $L^p(\Omega)$ respectively. Thus we can find an integrable function k such that

$$|T(u_n)|^p+|\nabla u_n|^p\leq k \quad \forall\, n\in\mathbb{N}.$$

Therefore, by $(A1)$ and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega \sum_{i=1}^{N}\left[a_i(x,T(u_n),\nabla u_n)-a_i(x,T(u),\nabla u)\right]\frac{\partial\varphi}{\partial x_i}dx=0\ \forall\varphi\in W^{1,p}(\Omega).$$

Step 4. $\lim\limits_{n\to\infty} < S_1(u_n), u_n - u >= 0.$

It is just (3). Thus we get the lemma. ■

Lemma 3.4. *Let u, v and w be in $W^{1,p}(\Omega)$ such that $v \leq w$. We put*

$$\gamma_{v,w}(u)(x) = (u(x) - w(x))_+^{p-1} - (v(x) - u(x))_+^{p-1}.$$

We define an operator $B_{v,w}$ from $W_0^{1,p}(\Omega)$ into $(W_0^{1,p}(\Omega))^*$ as follows

$$< B_{v,w}u, \varphi >= \int_\Omega \gamma_{v,w}(u)\varphi dx \quad \forall\ u, \varphi \in W_0^{1,p}(\Omega).$$

Then we have

(i) *$B_{v,w}$ is bounded.*

(ii) *There exist two positive real numbers α and β such that*

$$\int_\Omega \gamma_{v,w}(u)udx \geq \alpha||u||_p^p - \beta \quad \forall\ u\ \in W_0^{1,p}(\Omega).$$

(iii) *$\{B_{v,w}u_n\}$ converges strongly to $B_{v,w}u$ in $(W_0^{1,p}(\Omega))^*$ for any sequence $\{u_n\}$ weakly converging to u in $W_0^{1,p}(\Omega)$.*

Proof. The proof of (i) and (ii) can be found in ([15, p. 791]). We prove (iii). Let $\{u_n\}$ be a sequence weakly converging to u in $W_0^{1,p}(\Omega)$. We can assume that $\{u_n\}$ converges strongly to u in $L^p(\Omega)$ and $\{u_n(x)\}$ converges to $u(x)$ for a.e. $x \in \Omega$, and there exists a nonnegative function h in $L^p(\Omega)$ such that $|u_n(x)| \leq h(x)$ for a.e. $x \in \Omega$. Hence $\{\gamma_{v,w}(u_n)(x)\}$ converges to $\gamma_{v,w}(u)(x)$ for a.e. $x \in \Omega$. We have

$$\begin{aligned}|\gamma_{v,w}(u_n)(x)| &\leq \left\{[|v(x)| + |u_n(x)|]^{p-1} + [|u_n(x)| + |w(x)|]^{p-1}\right\}\\ &\leq \left\{[|v(x)| + h(x)]^{p-1} + [|w(x)| + h(x)]^{p-1}\right\} \quad a.e.\ x \in \Omega.\end{aligned}$$

Since $[(|v| + h)^{p-1} + (|w| + h)^{p-1}]$ is in $L^q(\Omega)$, using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \|\gamma_{v,w}(u_n) - \gamma_{v,w}(u)\|_q = 0, \tag{11}$$

$$\begin{aligned}| < B_{v,w}u_n - B_{v,w}u, \varphi > | &= \left|\int_\Omega \gamma_{v,w}(u_n)\varphi - \gamma_{v,w}(u)\varphi dx\right| \\ &\leq \|\gamma_{v,w}(u_n) - \gamma_{v,w}(u)\|_q\|\varphi\|_{1,p}\ \forall\ \varphi \in W_0^{1,p}(\Omega).\end{aligned} \tag{12}$$

Combining (11) and (12), we get the lemma. ■

Lemma 3.5. *Let v be a subsolution of* (2) *such that $\underline{u} \leq v \leq \overline{u}$. We put*

$$a_v(x,u,\nabla u) = -f(x,v,u,\nabla u) + a(u(x)) - a(v(x)) \ \forall \ x \in \Omega,$$

Then the following equation has a solution w in $W_0^{1,p}(\Omega)$

$$\begin{cases} -\sum\limits_{i=1}^{N} \dfrac{\partial}{\partial x_i} a_i(x,u,\nabla u) + a_v(x,u,\nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

such that $v \leq w \leq \overline{u}$. Moreover w is also a subsolution of (2).

Proof. We define the operator S_2, S_3 and S as follows

$$< S_2 u, \varphi > = \int_\Omega a_0(x, Tu, \nabla Tu)\varphi dx,$$

$$< S_3 u, \varphi > = M \int_\Omega \gamma(x,u)\varphi dx$$

$$< Su, \varphi > =< (S_1 + S_2 + S_3)u, \varphi > \quad \forall u, \varphi \in W_0^{1,p}(\Omega).$$

We prove the lemma by the following steps.

Step 1. S is bounded.

By (A1), we have

$$\begin{aligned} | < S_1 u, \varphi > | &= \left| \int_\Omega \sum_{i=1}^{N} a_i(x, Tu, \nabla u) \frac{\partial \varphi}{\partial x_i} dx \right| \\ &\leq \int_\Omega \sum_{i=1}^{N} [k_0(x) + C_0(|Tu|^{\frac{r(p-1)}{p}} + |\nabla u|^{p-1})] |\frac{\partial \varphi}{\partial x_i}| dx \\ &\leq N \|\varphi||_{1,p} [||k_0||_q + C_0 ||\underline{u}||_r^{\frac{r(p-1)}{p}} + C_0 ||\overline{u}||_r^{\frac{r(p-1)}{p}} + C_0 ||\nabla u||_p^{p-1}], \\ | < S_2 u, \varphi > | &= | \int_\Omega a_0(x, Tu, \nabla Tu)\varphi dx| \\ &\leq \int_\Omega [k_0(x) + C_0 |Tu|^{\frac{r(p-1)}{p}} + C_0 |\nabla Tu|^{p-1}] |\varphi| dx \\ &\leq ||\varphi||_{1,p} [||k_0||_q + C_0 ||\nabla Tu||_p^{p-1} + C_0 ||\underline{u}||_r^{\frac{r(p-1)}{p}} + C_0 ||\overline{u}||_r^{\frac{r(p-1)}{p}}. \end{aligned}$$

According to Lemma 3.4, S_3 is bounded. Thus $S = S_1 + S_2 + S_3$ is bounded.

Step 2. S is pseudomonotone.

By Lemma 3.4, and Proposition 27.7 in [20], it is sufficient to prove that S_1+S_2 is a pseudomonotone operator on $W_0^{1,p}(\Omega)$. Let $\{u_n\}$ be a sequence converging weakly to u in $W_0^{1,p}(\Omega)$ such that $\limsup\limits_{n\to\infty} < S_1 u_n + S_2 u_n, u_n - u > \leq 0$. Note that

$$
\begin{aligned}
| < S_2 u_n, u_n - u > | &\le \int_\Omega |a_0(x, Tu_n, \nabla T u_n)(u_n - u)| dx \\
&\le ||u_n - u||_p ||a_0(x, T(u_n), \nabla T u_n)||_q,
\end{aligned}
$$

which implies

$$
\lim_{n\to\infty} < S_2 u_n, u_n - u > = \ 0. \tag{14}
$$

Since $\limsup_{n\to\infty} < (S_1 + S_2)u_n, u_n - u > \le \ 0$, then $\limsup_{n\to\infty} < S_1 u_n, u_n - u > \le \ 0$.

By Lemma 3.3, $\{S_1 u_n\}$ converges weakly to $S_1 u$ in $(W_0^{1,p}(\Omega))^*$, $\{u_n\}$ converges to u in $W_0^{1,p}(\Omega)$ and $\lim_{n\to\infty} < S_1 u_n, u_n > = < S_1 u, u >$. Hence $\{S_2 u_n\}$ weakly converges to $S_2 u$ in $(W_0^{1,p}(\Omega))^*$ and $\lim_{n\to\infty} < S_2 u_n, u_n >=< S_2 u, u >$. Consequently, $\{(S_1 + S_2)u_n\}$ weakly converges to $(S_1 + S_2)u$ in $(W_0^{1,p}(\Omega))^*$ and $\lim_{n\to\infty} < (S_1 + S_2)u_n, u_n >=< (S_1 + S_2)u, u >$. That means $S_1 + S_2$ is pseudomonotone. Therefore, S is pseudomonotone.

Step 3. S is coercive.

By (A3), we have

$$
\begin{aligned}
< S_1 u, u > &= \int_\Omega \sum_{i=1}^N a_i(x, T(u), \nabla u) \frac{\partial}{\partial x_i} u dx \\
&\ge \int_\Omega [C_1 |\nabla u|^p - k_1(x)] dx \qquad (15)\\
&= C_1 ||\nabla u||_p^p - ||k_1||_1,
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega |\nabla T u|^p dx &= \int_{\underline{u}\le u\le \overline{u}} |\nabla u|^p dx + \int_{u<\underline{u}} |\nabla \underline{u}|^p dx + \int_{u>\overline{u}} |\nabla \overline{u}|^p dx \qquad (16)\\
&\le ||\nabla u||_p^p + ||\nabla \underline{u}||_p^p + ||\nabla \overline{u}||_p^p,
\end{aligned}
$$

$$
\int_\Omega |Tu|^r dx \le \int_\Omega (|\underline{u}| + |\overline{u}|)^r dx = M_0. \tag{17}
$$

Combining (16), (17), using Young's inequality and the Sobolev embedding theorem, we can find a positive constant M_1 such that for any positive number ϵ

$$
\begin{aligned}
< S_2 u, u > &= \int_\Omega a_0(x, Tu, \nabla T u) u dx \\
&\ge \int_\Omega \left[-C_0 |Tu|^{r\frac{p-1}{p}} - C_0 |\nabla T u|^{p-1} - k_0(x) \right] |u| dx \\
&\ge -C_0 ||Tu||_r^{r\frac{p-1}{p}} ||u||_p - C_0 ||\nabla T u||_p^{p-1} ||u||_p - ||k_0||_q ||u||_p \\
&\ge -C_0 M_0^{\frac{p-1}{p}} ||u||_p - C_0 \left[\frac{||u||_p^p}{\epsilon^p p} + \frac{\epsilon^q ||\nabla T u||_p^p}{q} \right]
\end{aligned}
$$

$$\geq -C_0 M_0^{\frac{p-1}{p}} ||u||_p - C_0 \left[\frac{||u||_p^p}{\epsilon^p p} + \frac{\epsilon^q ||\nabla u||_p^p}{q} \right]$$
$$- C_0 \frac{\epsilon^q [||\nabla \underline{u}||_p^p + ||\nabla \overline{u}||_p^p]}{q} - ||k_0||_q ||u||_p. \tag{18}$$

Applying Lemma 3.4, we can find positive real numbers α, β such that

$$< S_3 u, u > \geq M(\alpha ||u||_p^p - \beta). \tag{19}$$

Combining (15), (18) and (19), we obtain

$$< Su, u > \geq C_1 \|\nabla u\|_p^p - \|k_1\|_1 - C_0 M_0^{\frac{p-1}{p}} \|u\|_p - C_0 \left[\frac{||u||_p^p}{\epsilon^p p} + \frac{\epsilon^q ||\nabla u||_p^p}{q} \right]$$
$$- C_0 \frac{\epsilon^q [||\nabla \underline{u}||_p^p + ||\nabla \overline{u}||_p^p]}{q} - ||k_0||_q ||u||_p + M(\alpha ||u||_p^p - \beta). \tag{20}$$

Choosing a sufficiently small positive real number ϵ and a sufficiently large positive real number M such that $C_1 > \dfrac{C_0 \epsilon^q}{q}$, $M\alpha > \dfrac{C_0}{\epsilon^p p}$, we see that

$$\lim_{||u||_{1,p} \to \infty} \frac{< Su, u >}{||u||_{1,p}} = \infty.$$

Therefore, S is coercive.

Step 4. There is a solution of (13) in $[v, \overline{u}]$.

By Theorem 27.A in [20], there is a solution w of $S(u, \varphi) = 0$ in $W_0^{1,p}(\Omega)$. We prove that w is in the interval $[v, \overline{u}]$. Choosing $\varphi = (w - \overline{u})_+$, we obtain

$$0 = \int_\Omega \sum_{i=1}^N a_i(x, Tw, \nabla w) \frac{\partial}{\partial x_i} (w - \overline{u})_+ dx + \int_\Omega a_0(x, T(w), \nabla T(w))(w - \overline{u})_+ dx$$
$$+ M \int_\Omega (w - \overline{u})_+^p dx$$
$$= \int_\Omega \sum_{i=1}^N a_i(x, \overline{u}, \nabla w) \frac{\partial}{\partial x_i} (w - \overline{u})_+ dx + \int_\Omega a_0(x, \overline{u}, \nabla \overline{u})(w - \overline{u})_+ dx$$
$$+ M \int_\Omega (w - \overline{u})_+^p dx. \tag{21}$$

Since $\overline{u}$ is a supersolution of (2) and $(w - \overline{u})_+ \geq 0$, then

$$\int_\Omega \sum_{i=1}^N a_i(x, \overline{u}, \nabla \overline{u}) \frac{\partial}{\partial x_i} (w - \overline{u})_+ dx + \int_\Omega a_0(x, \overline{u}, \nabla \overline{u})(w - \overline{u})_+ dx \geq 0 \tag{22}$$

Therefore,

$$\int_\Omega \sum_{i=1}^N [a_i(x,\overline{u},\nabla w)-a_i(x,\overline{u},\nabla\overline{u})]\frac{\partial}{\partial x_i}(w-\overline{u})_+dx+M\int_\Omega (w-\overline{u})_+^p dx \le 0. \quad (23)$$

It follows from (A2) that

$$\int_\Omega \sum_{i=1}^N [a_i(x,\overline{u},\nabla w)-a_i(x,\overline{u},\nabla\overline{u})]\frac{\partial}{\partial x_i}(w-\overline{u})_+dx \ge 0. \quad (24)$$

Combining (23) and (24), we have

$$M\int_\Omega (w-\overline{u})_+^p dx \le 0,$$

which implies that $(w-\overline{u})_+(x)=0$ for a.e. x in Ω. Thus $w(x)\le\overline{u}(x)$ for a.e. $x\in\Omega$. Similarly, we also have $w(x)\ge v(x)$ for a.e. $x\in\Omega$.

Step 5. w is a subsolution of (2).

By (F2), it follows that for any nonnegative function φ in $W_0^{1,p}(\Omega)$

$$\begin{aligned}\int_\Omega \sum_{i=1}^N a_i(x,u,\nabla u)\frac{\partial\varphi}{\partial x_i}dx &= \int_\Omega [f(x,v,w,\nabla w)+a(v)-a(w)]\varphi dx\\ &\le \int_\Omega f(x,w,w,\nabla w)\varphi dx. \quad (25)\end{aligned}$$

Thus w is also a subsolution of (2). ■

Lemma 3.6. *There exists a positive real number M independent of v such that $||w||_{W_0^{1,p}(\Omega)} \le M$ for any w in Lemma* 3.5.

Proof. Replacing φ by w in (25), by (A3), (F1) and (F2), we get

$$\begin{aligned}C_1\|\nabla w\|_p^p-\|k_1\|_1 &= \int_\Omega [C_1|\nabla w|^p-k_1(x)]dx\\ &\le \int_\Omega \sum_{i=1}^N a_i(x,u,\nabla w)\frac{\partial w}{\partial x_i}dx\\ &= \int_\Omega [f(x,v,w,\nabla w)+a(v)-a(w)]udx\\ &\le \int_\Omega (k_2+C_2|\nabla w|^{p-1}+C_2|w|^{\frac{r(p-1)}{p}}+C_3|v|^{\frac{r(p-1)}{p}}\\ &\quad +C_3|w|^{\frac{r(p-1)}{p}}+2C_3)|w|dx\end{aligned}$$

$$\begin{aligned}
&\leq \int_\Omega [k_2 + 2C_3 + C_2|\nabla w|^{p-1} + C_2(|\underline{u}| + |\overline{w}|)^{\frac{r(p-1)}{p}} \\
&\quad + 2C_3(|\underline{w}| + |\overline{w}|)^{\frac{r(p-1)}{p}}](|\underline{u}| + |\overline{u}|)dx \\
&\leq \|k_2\|_q\|(|\underline{u}| + |\overline{u}|)\|_p + 2C_3||(|\underline{u}| + |\overline{u}|)||_1 \\
&\quad + (C_2 + 2C_3)||(|\underline{u}| + |\overline{u}|)||_r^{\frac{r(p-1)}{p}}||(|\underline{u}| + |\overline{u}|)||_p \\
&\quad + C_2 \int_\Omega |\nabla u|^{p-1}(|\underline{u}| + |\overline{u}|) \\
&\leq M_4 + C_2||\nabla u||_p^{p-1}||(|\underline{u}| + |\overline{u}|)||_p,
\end{aligned}$$

Thus we have

$$C_1||\nabla u||_p^p - ||k_1||_1 \leq M_4 + M_5 + C_2||\nabla u||_p^{p-1}||(|\underline{u}| + |\overline{u}|)||_p,$$

which yields the lemma. ■

Proof of Theorem 3.2. Denote by $\mathfrak{S}_0$ the set of subsolutions u in $[\underline{u}, \overline{u}]$ of (2) such that there exists a subsolution v in $[\underline{u}, u]$ of (2) and u is a solution of (13). We see that $\mathfrak{S}_0$ is non-empty and bounded by Lemmas 3.5 and 3.6.

Let u be in $\mathfrak{S}_0$, by Lemma 3.5, there is a solution $u' \equiv H_0(u)$ in $[u, \overline{u}]$ of the following equation

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u', \nabla u') + a(u') = f(x, u, u', \nabla u') + a(u) & \text{in } \Omega, \\ u' = 0 & \text{on } \partial\Omega. \end{cases} \tag{26}$$

It is easy to see that $H_0(\mathfrak{S}_0) \subset \mathfrak{S}_0$. Let $\{w_n\}$ be an increasing sequence in $\mathfrak{S}_0$. Since $\mathfrak{S}_0$ is bounded, then $\{w_n\}$ converges weakly to w. Since $w_n \in \mathfrak{S}_0$, there exists v_n being a subsolution of (2) such that $\underline{u} \leq v_n \leq w_n \leq \overline{u}$ and for any nonnegative function φ in $W_0^{1,p}(\Omega)$ we have

$$\begin{aligned}
\int_\Omega \sum_{i=1}^N a_i(x, w_n, \nabla w_n)\frac{\partial \varphi}{\partial x_i}dx &= \int_\Omega [f(x, v_n, w_n, \nabla w_n) + a(v_n) - a(w_n)]\varphi dx \\
&\geq \int_\Omega [f(x, \underline{u}, w_n, \nabla w_n) + a(\underline{u}) - a(w_n)]\varphi dx
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^N a_i(x, w_n, \nabla w_n)\frac{\partial}{\partial x_i}(w_n - w)dx \\
&\leq \int_\Omega [f(x, \underline{u}, w_n, \nabla w_n) + a(\underline{u}) - a(w_n)](w_n - w)dx
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^N [a_i(x, w_n, \nabla w_n) - a_i(x, w_n, \nabla w)] \frac{\partial}{\partial x_i}(w_n - w)dx \\
&\leq \int_\Omega \sum_{i=1}^N a_i(x, w_n, \nabla w) \frac{\partial}{\partial x_i}(w_n - w)dx \\
&\quad + \int_\Omega [f(x, \underline{u}, w_n, \nabla w_n) + a(\underline{u}) - a(w_n)](w_n - w)dx.
\end{aligned}$$

Using the same argument as in Lemma 3.3, we see that $\{w_n\}$ converges strongly to w in $W_0^{1,p}(\Omega)$. We can suppose that $\{w_n(x)\}$ and $\{\nabla w_n(x)\}$ converge to $w(x)$ and $\nabla w(x)$ for almost everywhere x in Ω. Now, we prove that $\{w_n\}$ has an upper bound v in $\mathfrak{S}_0$. Since $v_n \leq w_n$ for any integer n, we have

$$v_n \leq w \quad \forall\, n \in \mathbb{N}. \tag{27}$$

By (F2) and (27), for any nonnegative function φ in $W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega \sum_{i=1}^N a_i(x, w_n, \nabla w_n) \frac{\partial \varphi}{\partial x_i} dx &= \int_\Omega [f(x, v_n, w_n, \nabla w_n) + a(v_n) - a(w_n)]\varphi dx \\
&\leq \int_\Omega [f(x, w, w_n, \nabla w_n) + a(w) - a(w_n)]\varphi dx.
\end{aligned}$$

By $(A0)$ and $(F2)$, it follows that

$$\int_\Omega \sum_{i=1}^N a_i(x, w, \nabla w) \frac{\partial \varphi}{\partial x_i} dx \leq \int_\Omega f(x, w, w, \nabla w)\varphi dx.$$

Thus w is a subsolution of (2). By Lemma 3.5, there exists v in $\mathfrak{S}_0$ such that $\underline{u} \leq w \leq v \leq \overline{u}$ and $\forall \varphi \in W_0^{1,p}(\Omega)$

$$\int_\Omega \sum_{i=1}^N a_i(x, v, \nabla v) \frac{\partial \varphi}{\partial x_i} dx = \int_\Omega [f(x, w, v, \nabla v) + a(w) - a(v)]\varphi dx.$$

Therefore, v is an upper bound of $\{w_n\}$ in $\mathfrak{S}_0$. By Theorem 1.1, the operator H_0 has a fixed point w^* in $\mathfrak{S}_0 \subset [\underline{u}, \overline{u}]$. It follows that for any φ in $W_0^{1,p}(\Omega)$

$$\int_\Omega \sum_{i=1}^N a_i(x, w^*, \nabla w^*) \frac{\partial \varphi}{\partial x_i} dx = \int_\Omega f(x, w^*, w^*, \nabla w^*)\varphi dx.$$

Let w^{**} be a solution of (13) in $[\underline{u}, \overline{u}]$ such that $w^* \leq w^{**}$, then $w^{**} \in \mathfrak{S}_0$. By Theorem 1.1, we have $w^* = w^{**}$ and get the theorem. ∎

Remark 3.7. Theorem 3.2 have been studied in [11] if $a_i(x, u, \nabla u) = A_i(x, \nabla u)$ and there is a positive real number c such that

$$[a(r_1) - a(r_2)](r_1 - r_2) \geq c|r_1 - r_2|^p \quad \forall r_1, r_2 \in \mathbb{R}. \tag{28}$$

In our results we only need the following condition (see (F2))

$$[a(r_1) - a(r_2)](r_1 - r_2) \geq 0 \quad \forall r_1, r_2 \in \mathbb{R}, r_1 \neq r_2.$$

Remark 3.8. If $1 < p < 2$, we show that the condition (28) is never satisfied by any a. Indeed, suppose that such a function exists. Put $x_n = \sum_1^n \frac{1}{m^{1/(p-1)}}$. We see that $\{x_n\}$ is an increasing sequence converging to a real number x, thus $a(x) \geq \sup_{n\in\mathbb{N}} a(x_n)$. Since $a(x_n) - a(x_{n-1}) \geq c(x_n - x_{n-1})^{p-1} = \frac{c}{n}$, then $a(x_n) - a(x_1) \geq \sum_2^n \frac{c}{m}$, which tends to infinity when n goes to infinity. Hence $a(x) = \infty$, which is a contradiction.

Moreover our result only partially needs conditions on compactness, ellipticity and coercivity.

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