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Integer Partitions in Discrete Dynamical Models and ECO Method*

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Abstract. In this paper, we study general types of integer partitions as configurations of discrete dynamical models with two transition rules and with the initial configuration being the singleton partition. This allows us to characterize its lattice structure, fixed point, and the recursive structure of the infinite extension of the lattice of these partitions. Besides, we use ECO method (Enumeration combinatorial objects) independently to study generating trees for integer partitions. By means of an operator satisfying two special conditions, we give some recursive structures which are exactly the same to those studied from the point of view of discrete dynamical systems. We also calculate their generating functions and present the bijection between the strict partitions and odd partitions.

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1. Introduction

Sand piles model (SPM) and related models, such as Chip Firing Game or

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Brylawski's model, have been introduced and studied in different contexts [6, 16, 2, 11]. A configuration in this model is a sequence of sand piles such that their heights are decreasing from left to right. For a given total number of sand grains n, each configuration can be represented by a partition of n. From the physics point of view, the initial evolution rule is the following: at each moment, one sand grain can fall down from one pile to its right neighbor with the condition that the difference of height of these two columns is greater than a threshold [2]. This threshold is defined more in detail in each study. However, most studies are only done on threshold 2 and some other cases were independently studies. In this paper, we study the case with a general threshold $\delta \geq 2$.

The ECO method have been described occasionally in the literature. A notable contribution in this field is explained by Barcucci, Del Lungo, Pergola, and Pinzani on two major papers [4, 5].

The idea of this method is the following: Let $\mathcal{O}$ be a class of combinatorial object. Let p be a *parameter* on $\mathcal{O}$ (i.e., $p : \mathcal{O} \to \mathbb{N}$ and $O_n = \{x \in \mathcal{O} : p(x) = n\}$ such that $|O_n|$ is finite. An operator ϑ on the class $\mathcal{O}$ is a function from O_n to the power set of O_{n+1}. Authors defined an *ECO operator* ϑ, i.e. ϑ satisfies the two following conditions:

1. for each $Y \in O_{n+1}$, there exists $X \in O_n$, such that $Y \in \vartheta(X)$,
2. for each $X_1, X_2 \in O_n$ and $X_1 \neq X_2$, then $\vartheta(X_1) \cap \vartheta(X_2) = \emptyset$,

which give a construction of each object $Y \in O_{n+1}$ from (exactly)one object $X \in O_n$. This implies that the family sets $\{\vartheta(X) : X \in O_n\}$ is a partition of O_{n+1}. Therefore, one has a recursive description of $\mathcal{O}'s$ element. In this cases, from this recursive description we can deduce a functional equation satisfied by $\mathcal{O}'s$ generating function. By solving this function equation, we determine $\mathcal{O}'s$ generating functions according to various parameters. The construction performed by ϑ can be described by a *generating tree,* i.e. a rooted tree whose nodes correspond to the objects of $\mathcal{O}$. The root, placed at level 0 of the tree, is the object with minimum size. Objects with the same size lie at the same level and the children of an object O are those produced by O through ϑ. Let $\{|O_n|\}_n$ be the sequence determined by the number of objects with size n. Then $f_{\mathcal{O}}(x) = \sum_{n \geq m} |O_n| x^n$ is its *generating function.*

ECO has been used for many combinatorial structure as: Fibonacci number, Bell number, Mottzkin number, etc [3]. In most of cases, the authors prove that operator ϑ satisfies a succession rule, which gives a recursion for the corresponding tree.

In this paper, we study ECO method for integer partitions and some special cases. Partitions have been investigated widely and the enumerative problems over partitions give many surprising results. These results have been proved by different approaches by using different structures. We believe that ECO method will give a new approach to solve these problems in a pure combinatorial way. Moreover, with ECO method, we prove different results by using a unique structure: the generating tree.

We will point out the strictly relation between ECO method and discrete dynamic systems, especially in Partition Theory.

In Sec. 2, we present integer partitions and some special cases from the point of view of discrete dynamical systems and prove the lattice structure of these sets. These results are generalized from [20] and [17]. Sec. 3 illustrates some results on ECO method for the set of integer partitions and show the strongly recursive structure of this set. By using ECO operators, we achieve the corresponding generating trees which coincide the spanning trees of the lattices in respective discrete dynamical systems. In this section, we also give a proof of Sylvester's bijection by using ECO method. Finally, Sec. 4 gives some computations on infinite trees and bijective proofs of classical partitions identities.

2. Integer Partitions and Discrete Dynamical Systems

In this section, we prove that the set of partitions with difference d ($d \geq 0$) between parts of a given positive integer n under dominance ordering can be considered as a configuration space of a discrete dynamical model with two transition rules and with initial configuration being the singleton partition. Many results presented in this section are obtained initially in the case of normal partitions ($d = 0$) [20] and in the case of strict partitions ($d = 1$) [17].

2.1. Notation and Definition

We defined a partition λ to be an integer sequence $(\lambda_1, \lambda_2, .., \lambda_l)$ such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l > 0$, (by convention $\lambda_i = 0$, $\forall i \geq l+1$). We say that λ is a partition of n (or λ is of *weight* n), and writing as $\lambda \vdash n$ or $|\lambda| = n$, if $\sum_i \lambda_i = n$. Denote by $\mathcal{P}(n) = \{\lambda \vdash n\}$ the set of all partitions of n. The set $\mathcal{P}(n)$ is equipped with a partial order, called *dominance ordering,* defined as follows. Given two partitions a and b of n, we have $a \geq b$ if the suffix sums of a are smaller than that of b, i.e. $\sum_{i>j} a_i \leq \sum_{i>j} b_i$ for all $j \geq 0$ [6]. Note that this definition still make sense even if a and b are partitions of different integers m and n respectively, where $m \geq n$ [20].

Let d be an integer. A *d-strict partition* $a = (a_1, \ldots, a_l)$ is a partition such that $a_i - a_{i+1} \geq d$, for all $1 \leq i \leq l-1$. Note that, especially, $a_l - a_{l+1} = a_l$ may be smaller than d. The set of all *d-strict partition* of n is denoted by d-$\mathcal{P}(n)$. In the case $d = 0$ this set is nothing but the set $\mathcal{P}(n)$ of all partitions of n while in the case $d = 1$ it is exactly the set $\mathcal{SP}(n)$ of all strict partitions (i.e. partitions with distinct parts) of n.

In the theory of discrete dynamical models, one model is defined by its *configurations* and by its *transition rule*(s). A *chain* is a sequence of transitions. A configuration b is called *reachable* from another configuration a if b is obtained from a by a chain. Usually, one starts with a distinguished configuration, also called the *initial configuration,* and the set of all reachable configurations from it is called the *configuration space* of the model under study. A *fixed point* of this model is a reachable configuration on which no transition can be applied.

Now, for each integer d, we consider the discrete dynamical models whose configurations are d-$\mathcal{P}(n)$ and whose transition rules are defined as follows: For any *d-strict partition* a of n, one can apply on a the following transition rules so that the resulting partition is also d-strict partition: (See Fig. 1)
- Vertical transition (V-transition):

$$(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) \to (a_1, \ldots, a_i - 1, a_{i+1} + 1, \ldots, a_n) \text{ if } a_i - a_{i+1} \geq d+2$$

- Horizontal transition (H-transition) with length l:
$(a_1, \ldots, p+l+1, p+l-d, p+l-2d, \ldots, p+l-(l-1)d, p+l-ld-1, \ldots, a_n) \to$
$(a_1, \ldots, p+l, p+l-d, p+l-2d, \ldots, p+l-(l-1)d, p+l-ld, \ldots, a_n)$
and Horizontal transition with length 1:

$$(a_1, \ldots, p+2, p-d, \ldots, a_n) \to (a_1, \ldots, p+1, p-d+1, \ldots, a_n).$$

Note that an H-transition of length 1 is also a V-transition. We define the cover relation as follows. A d-strict partition a covers another d-strict partition b if b can be obtained from a by applying a transition rule (write $a \succ_d b$). It is evident that the reflexive and transitive closure of this relation is an order relation. We denote it by $\leq_d$.

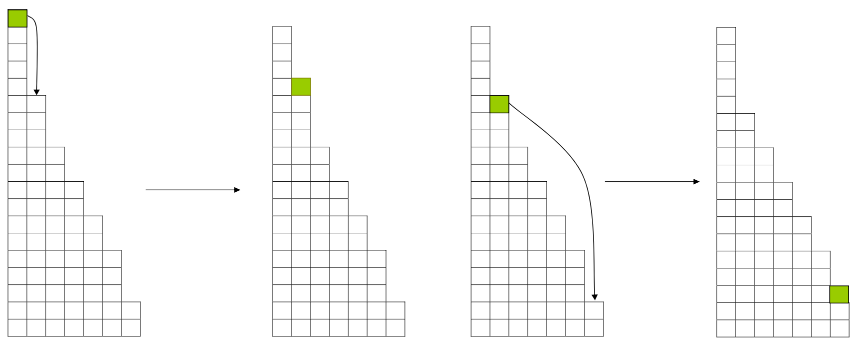

Fig. 1 Vertical transition and horizontal transition in the case $d = 2$.

2.2. Structure of d-$\mathcal{P}(n)$

Our purpose for this section is to study the structure of classes d-$\mathcal{P}(n), (d \geq 0)$, of partitions of integer n. To achieve this, we will mainly use order theory. A *partially ordered set* (or poset) is a set P equipped with a reflexive ($x \leq x$), transitive ($x \leq y$ and $y \leq z$ imply $x \leq z$) and antisymmetric ($x \leq y$ and $y \leq x$ imply $x = y$) binary relation $\leq$. A *lattice* is a poset such that two elements a and b admit a least upper bound (call *supremum* of a and b and denoted by $\sup(a, b)$) and a greatest lower bound (call *infimum* of a and b and denoted by $\inf(a, b)$). For more details see e.g [8]. The fact that the set of configurations of a naturally ordered dynamical system is a lattice implies some important properties such as convergence. Moreover, this convergence is very strong in the following sense:

for two configurations of the system, there exists a unique first configuration obtained from them and every configuration which can be obtained from both of them can be obtained from this first one.

Let us first show that all d-strict partitions can be obtained from the initial configuration (n) by applying transition rules. This proof is completely similar to [17].

Lemma 2.1. *The set d-$\mathcal{P}(n)$ is exactly the set of all d-strict partitions reachable from (n) by applying two transition rules V and H.*

From [6], it is known that the set $\mathcal{P}(n)$ of all partitions of an integer n with the dominance ordering is a lattice, denoted by $L_B(n)$, in which $\inf(a, b) = c$ if and only if, for all $j \geq 1$, one has $\sum_{i=1}^{j} c_i = \min(\sum_{i=1}^{j} a_i, \sum_{i=1}^{j} b_i)$.

Recall that 0-$\mathcal{P}(n)$ is exactly $\mathcal{P}(n)$ and also is $L_B(n)$ when we mention to its structure. By using the proof technics in [17], we can show that d-$\mathcal{P}(n)(d \geq 1)$ is a subposet of $\mathcal{P}(n)$ and it is also a lattice. Because of the above result, we can now write $b \leq a$ instead of $b \leq_d a$ for any two d-strict partitions a and b. We also have the similar result.

Theorem 2.2. *d-$\mathcal{P}(n)$ is a lattice. Moreover, the meet operation in d-$\mathcal{P}(n)$ is the same as that in $\mathcal{P}(n)$.*

Remark 2.3. d-$\mathcal{P}(n)$ is not a sublattice of $\mathcal{P}(n)$. In fact, the joint operations in d-$\mathcal{P}(n)$ and $\mathcal{P}(n)$ are different. For example, in $\mathcal{SP}(n)$, $(8, 4, 3, 1) \vee (7, 5, 4) = (8, 4, 4)$ which is not a strict partition. Nevertheless, we still have $a \vee_d b \geq a \vee b$ for any a and b.

While studying discrete dynamical systems, one important question is whether it has (a unique) fixed point (configurations on which no transition is possible). In the case of d-$\mathcal{P}(n)$, because it is a lattice, it has a unique minimal element and this is its unique fixed point. We now give an explicit formula for this fixed point. Let p be the unique number such that

$$\frac{1}{2}p((p-1)d+2) \leq n < \frac{1}{2}(p+1)(pd+2).$$

Then there exist $r < p$ and q uniquely satisfying

$$n - \frac{1}{2}p(2 + (p-1)k) = qp + r.$$

Now let Π be the following partition:

$$\Pi(k) = (1 + (p-1)d + q + 1, 1 + (p-2)d + q + 1, \ldots,$$

$$1 + (p-r)d + q + 1, 1 + (p-r-1)d + q, \ldots, 1 + d + q, 1 + q)$$

It is evident that Π is a d-strict partition on which no transition can be applied.

Proposition 2.4. *Π is the fixed point of d-$\mathcal{P}(n)$.*

2.3. The Relation Between d-$\mathcal{P}(n+1)$ and d-$\mathcal{P}(n)$

Our aim in this section is to investigate the relationship between lattices d-$\mathcal{P}(n+1)$ and d-$\mathcal{P}(n)$. First of all, we present here some preliminary definitions.

If $a = (a_1, a_2, \dots, a_m)$ is a d-strict partition, then the partition obtained from a by adding one grain on its j-th column is denoted by $a^{\downarrow_j}$. Notice that $a^{\downarrow_j}$ is not necessarily a d-strict partition. If S is a set of d-strict partitions, then $S^{\downarrow_j}$ denotes the set $\{a^{\downarrow_j} | a \in S\}$. We denote $a \xrightarrow{j} b$ if b is obtained from a by applying a transition at position j and by Succ(a) the set of configurations directly reachable from a.

Write $d_i(a) = a_i - a_{i+1}$ with the convention that $a_{m+1} = 0$. We say that a has a *cliff* at position i if $d_i(a) \geq d+2$. If there exists an $\ell \geq i$ such that $d_j(a) = d$ for all $i \leq j < \ell$ and $d_\ell(a) = d+1$, then we say that a has a *slippery plateau* at i. Likewise, a has a *non-slippery plateau* at i if $d_j(a) = d$ for all $i \leq j < \ell$ and it has a cliff at ℓ. The integer $\ell - i$ is called the *length* of the plateau at i. Note that in the special case $\ell = i$, the plateau is of length 0.

The set of elements of d-$\mathcal{P}(n)$ that begin with a cliff, a slippery plateau of length ℓ and a non-slippery plateau of length ℓ are denoted by $C, d\text{-}P_\ell, nd\text{-}P_\ell$ respectively.

Let $a = (a_1, a_2, \dots, a_k)$ be a d-strict partition. It is clear that $a^{\downarrow_1}$ is again a d-strict partition. This defines an embedding $\pi : d\text{-}\mathcal{P}(n) \to d\text{-}\mathcal{P}(n)^{\downarrow_1} \subset d\text{-}\mathcal{P}(n+1)$ which can be proved, by using infimum formula of d-$\mathcal{P}(n)$ and d-$\mathcal{P}(n+1)$, as a lattice map.

Lemma 2.5. *d-$\mathcal{P}(n)^{\downarrow_1}$ is a sublattice of d-$\mathcal{P}(n+1)$.*

Our next result characterizes the remaining elements of d-$\mathcal{P}(n+1)$ that are not in d-$\mathcal{P}(n)^{\downarrow_1}$.

Theorem 2.6. *For all $n \geq 1$, we have $d\text{-}\mathcal{P}(n+1) = d\text{-}\mathcal{P}(n)^{\downarrow_1} \sqcup_{\ell \geq 1} d\text{-}P_\ell^{\downarrow_{\ell+1}}$.*

Proof. It is easy to check that each element in one of the sets d-$\mathcal{P}(n)^{\downarrow_1}$ and $d\text{-}P_\ell^{\downarrow_{\ell+1}}$ is an element of d-$\mathcal{P}(n+1)$, and that these sets are disjoint. Now let us consider an element b of d-$\mathcal{P}(n+1)$. If b begins with a cliff or a step then b is in d-$\mathcal{P}(n)^{\downarrow_1}$. If b begins with a plateau of length $\ell+1, \ell \geq 1$, then b is in $d\text{-}P_\ell^{\downarrow_{\ell+1}}$. ■

2.4. The Infinite Lattice d-$\mathcal{P}(\infty)$

We will now define d-$\mathcal{P}(\infty)$ as the set of all the configurations reachable from (∞) (this is the configuration where the first column contains infinitely many grains and all the other columns contain no grain). Therefore, each element a of d-$\mathcal{P}(\infty)$ has the form $(\infty, a_2, a_3, \dots, a_k)$. The partial order is defined by declaring that $a \geq_\infty b$ if $\sum_{i \geq j} a_i \leq \sum_{i \geq j} b_i$ for all $j \geq 2$.

Imagine that (∞) is the initial configuration where the first column contains infinitely many grains and all the other columns contain no grain. Then the

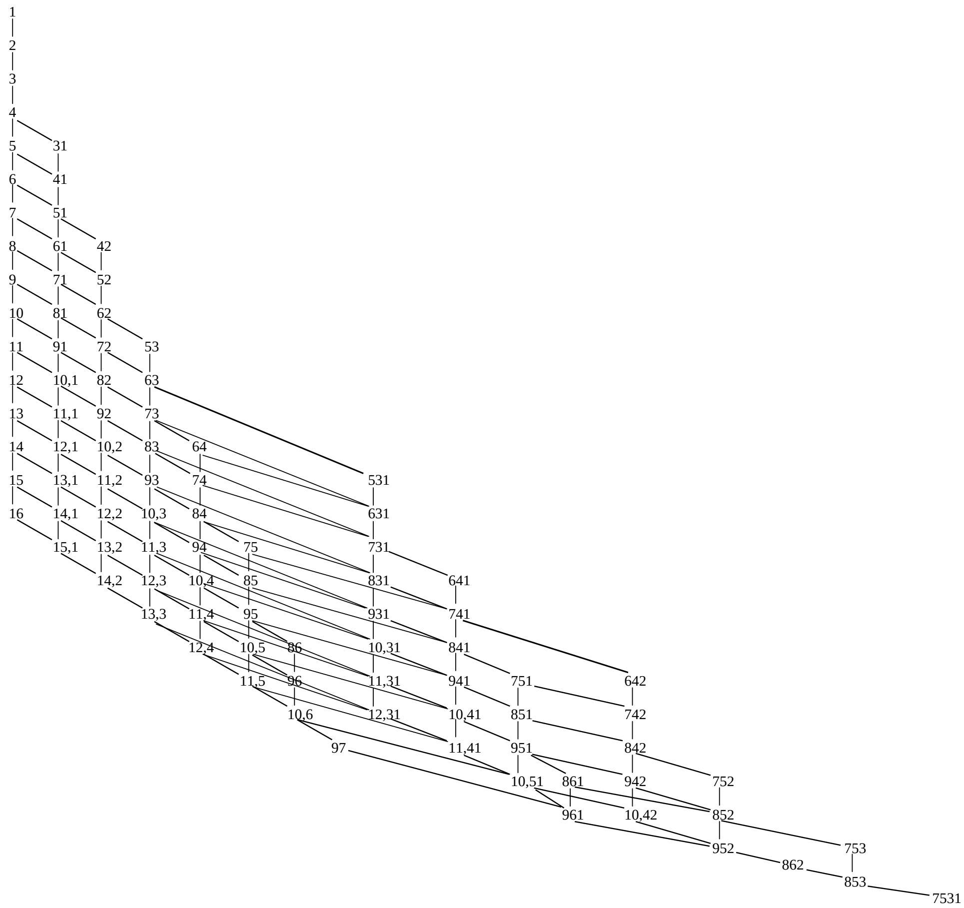

Fig. 2 First elements and transitions of 2-$\mathcal{P}(\infty)$. As shown on this figure for $n = 16$, we will see two ways of finding parts of 2-$\mathcal{P}(\infty)$ isomorphic to 2-$\mathcal{P}(n)$ for any n.

transitions V and H defined in the first section can be performed on (∞) just as if it is finite, and we call d-$\mathcal{P}(\infty)$ as the set of all the configurations reachable from (∞). A typical element a of d-$\mathcal{P}(\infty)$ has the form $(\infty, a_2, a_3, \ldots, a_k)$. As in the previous section, we find that the dominance ordering on d-$\mathcal{P}(\infty)$ (when the first component is ignored) is equivalent to the order induced by the dynamical model. The first partitions in 2-$\mathcal{P}(\infty)$ are given in Fig. 2 along with their covering relations (the first component, equal to ∞, is not represented on this diagram). Note that, we denote by $a_1 a_2 \ldots a_k$ instead of $(a_1, a_2, \ldots, a_k)$ the partitions of n on figures in this paper.

For any two elements $a = (\infty, a_2, \ldots, a_k)$ and $b = (\infty, b_2, \ldots, b_\ell)$ of d-$\mathcal{P}(\infty)$, we define c by: $c_i = \max(\sum_{j \geq i} a_j, \sum_{j \geq i} b_j) - \sum_{j > i} c_j$ for all i such that $2 \leq i \leq \max(k, \ell)$ and $c_i = 0$ if $i > \max(k, \ell)$. One can check that c is an element of

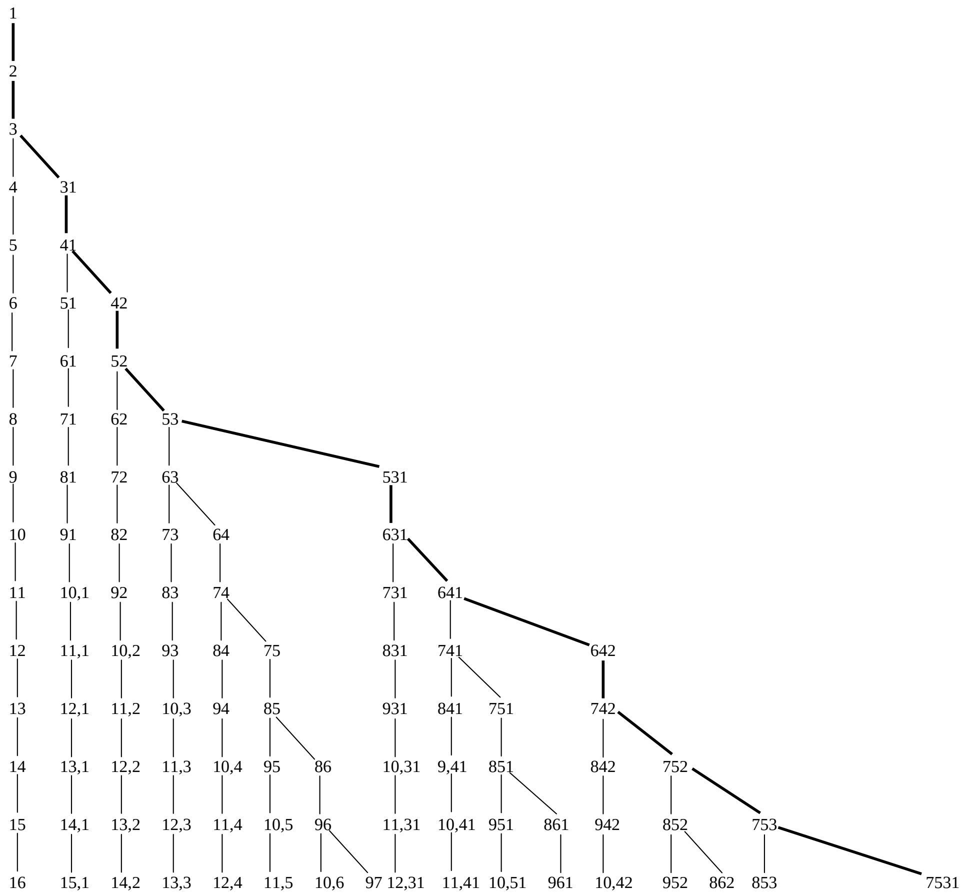

Fig. 3 The tree of 2-partitions

d-$\mathcal{P}(\infty)$, i.e. $c_1 = \infty$ and $c_i - c_{i+1} \geq d$ for all $i > 1$, and then $c = a \wedge b$. This implies the following

Theorem 2.7. *The set* d-$\mathcal{P}(\infty)$ *is a lattice.*

Now for any $n > 1$, there are two canonical embeddings of d-$\mathcal{P}(n)$ in d-$\mathcal{P}(\infty)$, defined by

$$\begin{array}{rccc} \pi : & d\text{-}\mathcal{P}(n) & \longrightarrow & d\text{-}\mathcal{P}(\infty) \\ & a = (a_1, a_2, \ldots, a_k) & \mapsto & \pi(a) = (\infty, a_2, \ldots, a_k) \end{array}$$

$$\begin{array}{rccc} \chi : & d\text{-}\mathcal{P}(n) & \longrightarrow & d\text{-}\mathcal{P}(\infty) \\ & a = (a_1, a_2, \ldots, a_k) & \mapsto & \chi(a) = (\infty, a_1, a_2, \ldots, a_k) \end{array}$$

The following result is straightforward:

Proposition 2.8. *Both π and χ are embeddings of lattices.*

By using the embedding χ, one can consider $d\text{-}\mathcal{P}(\infty)$ as the union disjoint of $d\text{-}\mathcal{P}(n)$ for all n, $d\text{-}\mathcal{P}(\infty) = \bigsqcup_{n\geq 0} d\text{-}\mathcal{P}(n)$.

2.5. Self-reference Property: the Infinite Binary Tree $T_{d\text{-}P}(\infty)$

Our purpose is to introduce an infinite tree $T_{d\text{-}P}(\infty)$ which generates all d-strict partitions by using the infinite lattice $d\text{-}\mathcal{P}(\infty)$. From Theorem 2.6, one can notice that each element of $d\text{-}\mathcal{P}(n+1)$ is obtained from a unique element of $d\text{-}\mathcal{P}(n)$ by adding one grain at some position i. Using this relation, we define our tree as follows: $T_{d\text{-}P}(\infty)$ is an infinite binary tree with the root is (0); and for each element a in $T_{d\text{-}P}(\infty)$, a always has a *left child* $a^{\downarrow 1}$, furthermore, if a begins with a slippery plateau of length ℓ, then a has a *right child* $a^{\downarrow \ell+1}$. The tree $T_{2\text{-}P}$ is illustrated in Fig. 3.

From Proposition 2.8, we have

Proposition 2.9. *The level n of $T_{d\text{-}P}(\infty)$ is exactly the set of all elements of $d\text{-}\mathcal{P}(n)$.*

3. ECO Method and Integer Partition

In this section, we study ECO method for integer partitions and some special cases. Partitions have been investigated widely and the enumerative problems over partitions give many surprising results. These results have been proved by approaching the point of view of infinite dynamical systems in the previous section. By using ECO method we are going to receive the same results. Moreover, we also prove the bijection between the set of strict partitions of a given number n and the set of partitions of n with odd parts, especially the Silveter's bijection, by using the structure of infinite generating trees.

We recall that a partition of an integer n is a sequence of non increasing positive integers $\lambda = (\lambda_1, \ldots, \lambda_l)$ such that $\lambda_1 + \ldots + \lambda_l = n$. We refer to the integers λ_i as the parts of the partition. Let $a(\lambda) = \lambda_1$ and $s(\lambda) = \lambda_l$ denote the largest and the smallest parts of the partition λ. The number of parts of λ we denote by $l(\lambda) = l$ and we call it the *length* of λ. For more details about integer partitions, we refer to [1].

Our purpose for this section is to introduce an ECO operator and give the recursive structure of the corresponding generating tree $T_{d\text{-}P}$ which generates all *d-strict partitions* of positive integers.

3.1. d-strict Partitions and ECO Operator

For the set of all d-strict partitions, we define the operator $\vartheta : d\text{-}\mathcal{P}(n) \rightarrow 2^{d\text{-}\mathcal{P}(n+1)}$ as follows: For each $a = (a_1, a_2, \ldots, a_l) \in d\text{-}\mathcal{P}(n)$, $a^{\downarrow 1} = (a_1 + 1, a_2, \ldots, a_l)$ is an element of $\vartheta(a)$, called *the left child* of a. Furthermore, if a begins with a staircase and ending at a leap: $p, p-d, \ldots, p-id, p-(i+1)d-1,$

then a has another element $a^{\downarrow i+2}$ (called *the right child* of a) which begins with $p, p-d, \ldots, p-id, p-(i+1)d$.

The corresponding generating tree $T_{2\text{-}P}$ for operator ϑ coincides with the spanning tree of the lattice $2\text{-}\mathcal{P}(\infty)$ from the point of view discrete dynamical system in Sec. 2 and is illustrated in Fig. 3.

Lemma 3.1. *ϑ is an ECO operator, i.e. ϑ satisfies the two following conditions:*
(i) *For each $b \in d\text{-}\mathcal{P}(n+1)$ there is $a \in d\text{-}\mathcal{P}(n)$ such that $b \in \vartheta(a)$,*
(ii) *For every $a, a' \in d\text{-}\mathcal{P}(n)$ with $a \neq a'$, $\vartheta(a) \cap \vartheta(a') = \emptyset$.*

Proof. (i) Suppose that $b = (b_1, b_2, \ldots, b_l) \in d\text{-}\mathcal{P}(n+1)$. If $b_1 - b_2 \geq d+1$ then let $a = (b_1 - 1, b_2, \ldots, b_l)$. It is clear that $a \in d\text{-}\mathcal{P}(n)$ and $a^{\downarrow 1} = b$, thus $b \in \vartheta(a)$.

If $b_1 = b_2 + d$, we can write b as

$$b = (b_1, b_1 - d, b_1 - 2d, \ldots, b_1 - id, b_1 - (i+1)d, b_{i+3}, \ldots, b_l)$$

with $b_1 - (i+2)d - 1 \geq b_{i+3} \geq 0$. Let

$$a = (b_1, b_1 - d, b_1 - 2d, \ldots, b_1 - id, b_1 - (i+1)d - 1, b_{i+3}, \ldots, b_l).$$

It is clear that $a \in d\text{-}\mathcal{P}(n)$ and $b = a^{\downarrow i+2}$, thus $b \in \vartheta(a)$.

(ii) For $a, a' \in d\text{-}\mathcal{P}(n)$, when $a \neq a'$, it is obvious that $a^{\downarrow 1} \neq a'^{\downarrow 1}$. On the other hand, if a has a *right child*, then it is no doubt that it differs from $a'^{\downarrow 1}$. Similarly, if a' has a *right child*, then it differs from $a^{\downarrow 1}$. If a and a' have *right child* b and b' respectively. If $b = b'$ then $a = a'$, which gives an obvious contradiction. ■

3.2. Recursive Structure of $T_{d\text{-}P}$ and Generating Function
Denote by $d\text{-}p(n) = |d\text{-}\mathcal{P}(n)|$, the number of all *d-strict partition* of n. Denote by $d\text{-}\mathcal{P} = \bigcup_n d\text{-}\mathcal{P}(n)$ the set of all d-strict partitions. Define

$$d\text{-}\mathcal{P}(n,k) = \{\lambda \in d\text{-}\mathcal{P}(n) : a(\lambda) \leq k\},\ d\text{-}p_k(n) = |d\text{-}\mathcal{P}(n,k)|.$$

We first define a certain kind of subtrees of $T_{d\text{-}P}$ which gives the recursive structure of $T_{d\text{-}P}$.

Definition 3.2. [20] We will call X_k *subtree* any subtree T of $T_{d\text{-}P}$ which is rooted at an element $a = (m, m-d, m-2d, \ldots, m-(k-1)d, a_{k+1}, \ldots)$ with $a_{k+1} \leq m - kd - 1$ and which is either the whole subtree of $T_{d\text{-}P}$ rooted at a if a has only one son, or a and its left subtree if a has two sons. Moreover, we define X_0 as a simple node.

By using the same technique in [20], we can show that all the X_k subtrees are isomorphic (see Fig. 4).

Proposition 3.3. *A X_k subtree, with $k \geq 1$, is composed by a chain of $k+1$ nodes (the rightmost chain) whose edges are labeled with $1, 2, \ldots, k$ and whose i-th node is the root of a X_{i-1} subtree for all i between 1 and $k+1$.*

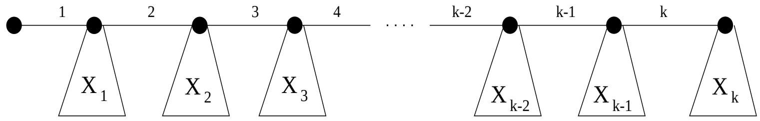

Fig. 4 Self-referencing structure of X_k subtrees

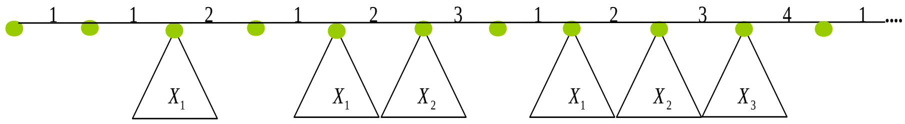

Fig. 5 Representation of $T_{d\text{-}P}$ as a chain.

This recursive structure allows us to give a compact representation of the tree $T_{d\text{-}P}$ by a chain. In the case $d = 0$, we receive the result of the tree T_P in [20] (see Fig. 6) and in the case $d = 1$, it is exactly the representation of the tree T_{SP} as a chain [17] (see Fig. 8):

Theorem 3.4. [20] *The tree T_P can be represented by the infinite chain defined as follows: the i-th node of this chain, $b = (\underbrace{1, 1, \ldots, 1}_{i-1})$, is linked to the following node in the chain by an edge labeled with i and is the root of a X_{i-1} subtree.*

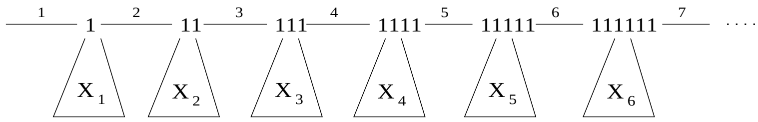

Fig. 6 Representation of T_P as a chain

Moreover, we can prove a stronger property of each subtree in this chain:

Corollary 3.5. *The X_k subtree of $T_{d\text{-}P}$ with root $(1+(k-1)d, 1+(k-2)d, \ldots, 1+2d, 1+d, 1)$ contains exactly the partitions of length k.*

Note that, in the general case the tree $T_{d\text{-}P}$ is not of the form $X_\infty := lim_{k\to\infty} X_k$. However, this is true in the case $d = 0$.

Corollary 3.6. [20] *The tree T_P is of the form X_∞.*

3.3. Sylvester's Bijection

To give a new point of view about the bijection between the set of strict partitions (i.e. partitions with distinct parts) of a given number n and the set of partitions of n with odd parts, especially the Silvester's bijection, we present generating trees for strict partitions and for odd partitions, then we prove the bijection between them.

Recall that $\mathcal{SP}(n)$ is the set of strict partitions of n. Denote by $\mathcal{SP} = \bigcup_n \mathcal{SP}(n)$, the set of all strict partitions. And denote by $\mathcal{ODD}(n)$ the set of partitions of n with odd parts and by $\mathcal{ODD} = \bigcup_n \mathcal{ODD}(n)$ the set of all odd partitions.

By Lemma 3.1, we have the ECO operator for strict partitions as follows:

$\vartheta_1 : \mathcal{SP}(n) \rightarrow 2^{\mathcal{SP}(n+1)}$ such that: for each $a = (a_1, a_2, \dots, a_l) \in \mathcal{SP}(n)$, $a^{\downarrow 1} = (a_1+1, a_2, \dots, a_l)$ is an element of $\vartheta_1(a)$, (*the left child*) of a. Furthermore, if a is of the form $(m, m-1, \dots, m-i, m-i-2, a_{i+3}, \dots, a_l)$ (and $a_{i+3} \leq m-i-3$) then $\vartheta_1(a)$ has another element (*the right child* of a): $a^{\downarrow i+2} = (m, m-1, \dots, m-i, m-i-1, a_{i+3}, \dots, a_l)$.

This operator gives the corresponding labeled tree of strict partitions T_{SP} which is illustrated in Fig. 7.

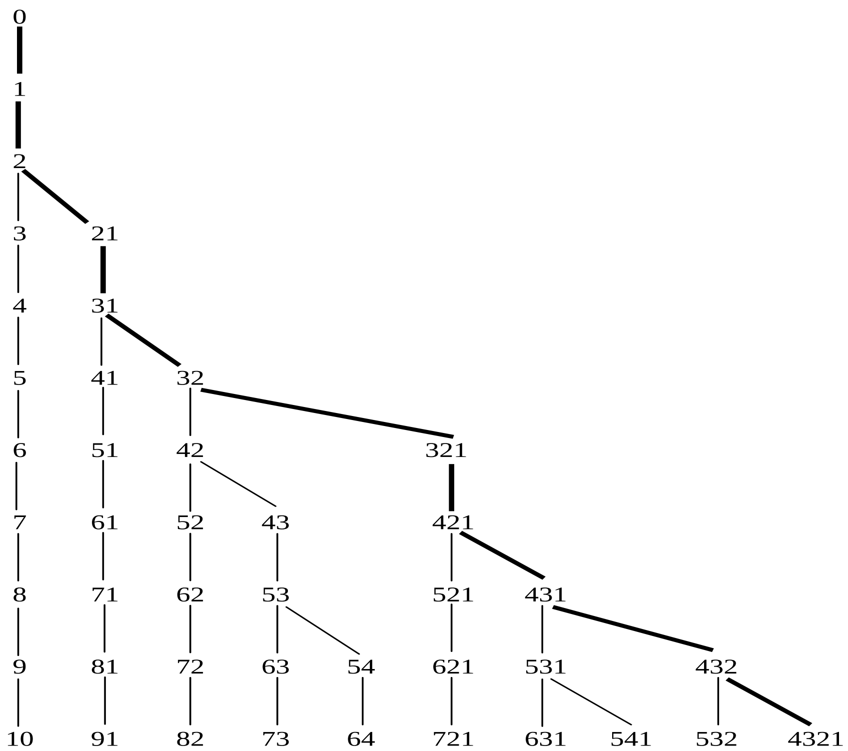

Fig. 7 The tree of strict partitions

Keep in mind the notation of X_k in Section 3.2, we can now give a result of the recursive structure of the tree T_{SP} as a special case:

Lemma 3.7. *Let* $a = (m, m-1, \ldots, m-k, a_{i+2}, \ldots)$ *(with* $a_{i+2} \leq m-k-2$*), if* a *has only one child then the subtree rooted at* a *is of the form* X_{k+1}*, and if* a *has two children then the tree consisting of* a *and the left subtree of* a *is of the form* X_{k+1}*.*

This recursive structure allows us to give a compact representation of the tree T_{SP} by a chain (see Fig. 8).

Theorem 3.8. *The tree* T_{SP} *can be represented by the infinite chain* $(), 1, 2, 21, 31, 32, 321, \ldots, (n-1, n-2, \ldots, 1), (n, n-2, \ldots, 2, 1), \ldots, (n, n-1, \ldots, 3, 2), (n, n-1, \ldots, 3, 2, 1), \ldots$ *with corresponding edges* $1, 1, 2, 1, 2, 3, \ldots, 1, 2, \ldots, n, \ldots$*; each node before an edge* $k \geq 1$ *is the root of a* X_{k-1} *subtree.*

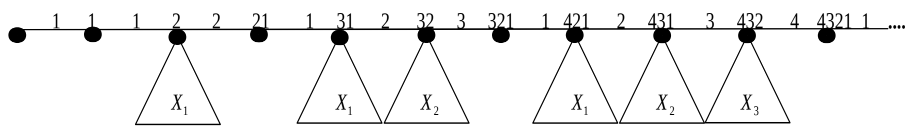

Fig. 8 Representation of T_{SP} as a chain.

Moreover, we can prove a stronger property of each subtree in this chain:

Proposition 3.9. *The* X_k *subtree of* T_{SP} *with root* $(k, k-1, \ldots, 2, 1)$ *contains exactly the strict partitions of length* k*.*

We now define the ECO operator for odd partition $\vartheta_2 : \mathcal{ODD}(n) \to 2^{\mathcal{ODD}(n+1)}$ and its corresponding labeled tree T_{ODD} (see Fig. 9) as follows (each labeled arrow represents a father-child relation in T_{ODD}):

(i) For each odd partition $u = (u_1, u_2, \ldots, u_k), u \xrightarrow{1} (u_1, \ldots, u_k, 1)$,

(ii) $(2i-1)^{i+1} \xrightarrow{2i} (2i+1)^i$,

(iii) $((2k+1)^i, \ldots, 2l+1, 2i-1) \xrightarrow{2i} ((2k+3)^i, \ldots, 2l+1)$,

(iv) $(2i+3)^i \xrightarrow{2i+1} (2i+1)^{i+1}$,

(v) $((2k+3)^i, 2k+1, \ldots, 2l+1) \xrightarrow{2i+1} ((2k+1)^{i+1}, \ldots, 2l+1, 2i+1)$ for $k \geq l \geq i \geq 1$,

where a^k is the tuple $\underbrace{(a, \ldots, a)}_{k}$.

Lemma 3.10. *The operator* ϑ_2 *is an* ECO operator*.*

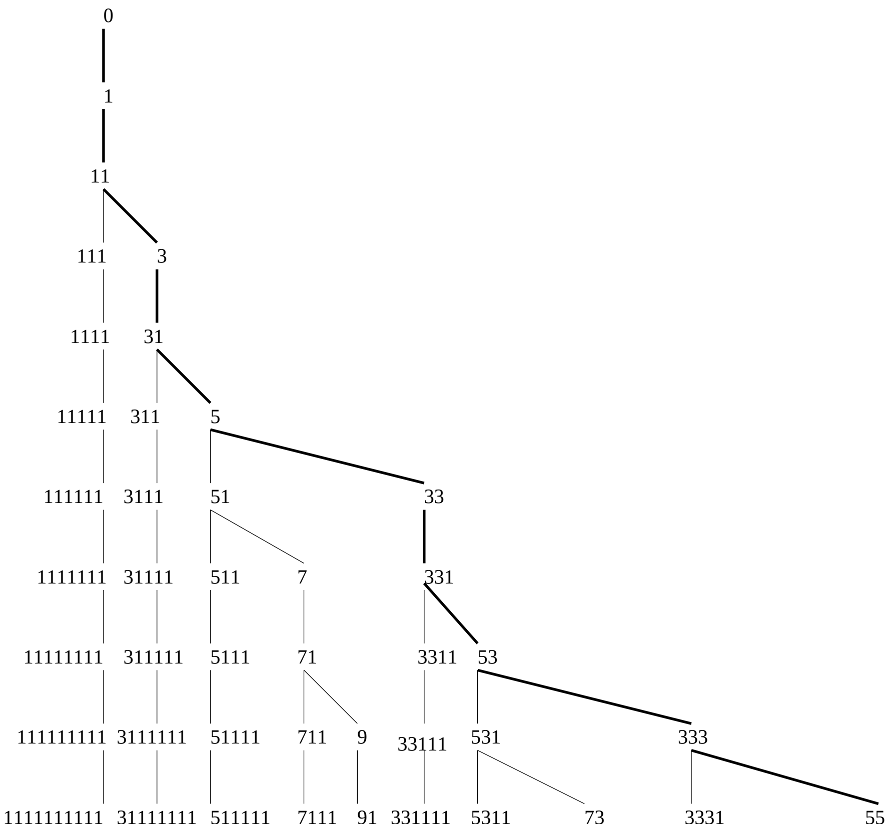

Fig. 9 The tree of odd partitions

Proof. Firstly, for $b \in \mathcal{ODD}(n+1)$, we will take $a \in \mathcal{ODD}(n)$ such that $b \in \vartheta_2(a)$. We consider all possible cases of b :

- If $b = (b_1, \ldots, b_k, 1)$ then we take $a = (b_1, \ldots, b_k)$ and hence $a \xrightarrow{1} b \in \vartheta_2(a)$.
- If all parts of b are equal, (i.e. $b = (2k+1)^i$), there are four different cases:
  *) If $1 \leq i < k$ then $a = ((2k-1)^i, 2i-1)$ and we have $a \xrightarrow{2i} b$.
  *) If $1 \leq i = k$ then $a = (2i-1)^{i+1}$ and we have $a \xrightarrow{2i} b$.
  *) If $1 \leq i = k+1$ then $a = (2i+1)^i$ and we have $a \xrightarrow{2i-1} b$.
  *) If $i-1 > k \geq 1$ then $a = ((2k+3)^k, (2k+1)^{i-k-1}, \ldots)$ and we have $a \xrightarrow{2k} b$.
- The last possibility is that $b = ((2k+1)^i, \ldots, 2\ell+1)$ with $k > \ell$.
  *) If $1 \leq \ell < i$ then $a = ((2k+3)^l, (2k+1)^{i-l}, \ldots)$ and we have $a \xrightarrow{2\ell+1} b$.
  *) If $\ell \geq i$ then $a = ((2k-1)^i, \ldots, 2\ell+1, 2i-1)$ and we have $a \xrightarrow{2i} b$.

So, for $b \in \mathcal{ODD}(n+1)$ there exists $a \in \mathcal{ODD}(n)$ such that $b \in \vartheta_2(a)$. On the other hand, one can check that in each case, b cannot be obtained by any other relation, this implies that for all $a, a' \in \mathcal{ODD}(n)$, if $a \neq a'$ then $\vartheta_2(a) \cap \vartheta_2(a') = \emptyset$. The proof is complete. ■

Corollary 3.11. *The infinite tree T_{ODD} is a binary tree which generates all odd partitions. Moreover each level n of this tree contains exactly all odd partitions of n.*

To show that T_{SP} and T_{ODD} are isomorphic, we prove that T_{ODD} can be represented by certain chain of subtrees Y_k and Y_k are isomorphic to X_k for all k.

Keep in mind the notation in the definition of ϑ_2 and T_{ODD}. Let Y_{2i} *subtree* of T_{ODD} be any left subtree of an odd partition of the form (iv) or (v), and Y_{2i-1} *subtree* be any left subtree of an odd partition of the form (ii) or (iii). Moreover, let Y_0 be a simple node. Remark that if a node has the right relation $\xrightarrow{k+1}$, then its left subtree is of the form Y_k.

Proposition 3.12. *A Y_k subtree, with $k \geq 1$, is composed by a chain of $k+1$ nodes (the rightmost chain) whose edges are labeled by $1, 2, \ldots, k$; and whose i^{th} node is the root of a Y_i subtree for all i between 1 and k.*

Proof. Let us consider an odd partition of the form iii), the other cases will be proved in the same way. So let $v = ((2k+1)^i, \ldots, 2l+1, 2i-1)$ with $k \geq l \geq i \geq 1$. Let us denote by v^{2j} the odd partition $((2k+1)^j, (2k+1)^{i-j}, \ldots, 2l+1, 2j-1)$, (for $i > j \geq 0$) and by v^{2j+1} the odd partition $((2k+3)^j, \ldots, 2l+1)$ (for $i \geq j \geq 1$). It is easy to check that v^0 is v and that $v^{j-1} \xrightarrow{j} v^j$ for all $i \geq j \geq 1$. So the left subtree of each v^j is a tree of the form Y_j, which implies the proposition. ■

Comparing this result and Proposition 3.3, we see that Y_k and X_k have the same structure for all $k \geq 1$.

On the other hand, it is straightforward to see that the rightmost chain in the tree T_{ODD} is

$(), 1, 1^2, 3, 31, 5, 3^2, \ldots, (2n+1)^n, (2n+1)^n 1,$
$(2n+3)(2n+1)^{n-1}, \ldots, (2n+1)^{n+1}, \ldots, (2n+3)^{n+1}, \ldots.$

By using the above remark, we can prove that

Theorem 3.13. *The tree T_{ODD} can be represented by the infinite chain*

$(), 1, 1^2, 3, 31, 5, 3^2, \ldots, (2n+1)^n, (2n+1)^n 1,$
$(2n+3)(2n+1)^{n-1}, \ldots, (2n+1)^{n+1}, \ldots, (2n+3)^{n+1}, \ldots$

with corresponding edges

$$1, 1, 2, 1, 2, 3, \ldots, 1, 2, \ldots, 2n+1, 1, 2, \ldots, 2n+2, \ldots;$$

where each node before an edge k is the root of a Y_{k-1} subtree.

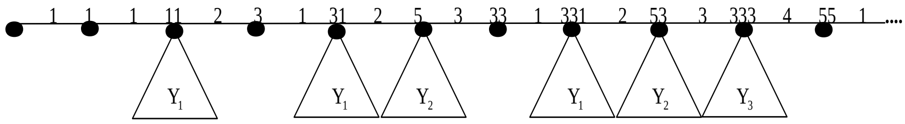

Fig. 10 Representation of T_{ODD} as a chain.

This theorem implies that T_{ODD} has the same structure as T_{SP}, so they are in a bijection. We denote by $\mathcal{LP}$ this bijection. Therefore, some attributes can be received such as $|\mathcal{SP}(n)| = |\mathcal{ODD}(n)|$.

Corollary 3.14. *The set of strict partitions of length* $2k-1$ *(with* $k \geq 1$*) is in bijection to the set of odd partitions such that the* k^{th} *part is* $2k-1$.

Proof. All strict partitions of length $2k-1$ belong to a subtree rooted at $(2k-1, 2k-2, \ldots, 2, 1)$ of T_{SP} (see Fig. 7). All transitions in this subtree are labeled with an integer smaller than or equal to $2k-1$. The corresponding subtree in T_{ODD} is a subtree with root $(2k-1)^k$ (see Fig. 9). Therefore a transition labeled with i ($i \leq 2k-1$) does not change the k^{th} part of an odd partition in T_{ODD}; the proposition is then proved. ■

The following corollary is straightforward:

Corollary 3.15. *The set of strict partitions of length* $2k$ (*with* $k \geq 1$) *is in bijection to the set of odd partitions such that the* k^{th} *part is greater than or equal to* $2k+1$ *and the* $(k+1)^{th}$ *part is smaller than or equal to* $2k-1$.

4. Some Computation on Infinite Tree

By using recursive structure of generating tree T_P we present the calculus of generating functions for integer partitions and prove some identities on partitions.

First of all, let us define the generating functions for infinite trees. Let T be an infinite tree in which each arrow is assigned by a parameter t, and some arrows are assigned by another parameter s. For a node a of T, let $n_t(a)$ (resp. $n_s(a)$) be the number of t (resp. s) in the path from the root of T to a. The *generating function* (*of one parameter*) of T is

$$f_T(t) = \sum_{a \in T} t^{n_t(a)},$$

and the *generating function of two parameters* of T is

$$f_T(t, s) = \sum_{a \in T} t^{n_t(a)} s^{n_s(a)}.$$

It is easy to check that $f_T(t)$ is equal to $\sum_{n=0}^{\infty} p(n)t^n$, with $p(n)$ being the number of nodes in the level n.

From now on, we assign always each arrow of any tree by parameter t. We state the first result on X_k and T_P.

Lemma 4.1. *The generating function for each X_k is given by*

$$f_{X_k}(t) = \frac{1}{(1-t)(1-t^2)\dots(1-t^k)}.$$

Moreover, the generating function of the tree T_P is given by

$$f_{T_P}(t) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)}, \quad \textit{or by } f_{T_P}(t) = 1 + \sum_{n=1}^{\infty} t^n \prod_{k=1}^{n} \frac{1}{(1-t^k)}.$$

Proof. First, it is obvious that $f_{X_1}(t) = \frac{1}{1-t}$. Then, from Proposition 2.4, we have $f_{X_k}(t) = f_{X_{k-1}}(t) + t^k f_{X_k}(t)$. So, by induction we obtain the formula for $f_{X_k}(t)$. Moreover, from Corollary 3.6, T_P is of the form of X_∞, so we have the first formula for f_{T_P}. On the other hand, from Theorem 3.4, we obtain the second formula for f_{T_P}. ■

Because T_P is the tree of all integer partitions, so the generating function $P(t)$ of integer partitions is equal to the generating function $f_{T_P}(t)$, that means

$$P(t) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)}.$$

We then use generating function on two parameters of T_P to prove the following results.

Theorem 4.2. (The first Euler Identity) *The following identity is due to Euler:*

$$1 + \sum_{k=1}^{\infty} \frac{s^k t^k}{(1-t)\dots(1-t^k)} = \prod_{i=1}^{\infty} \frac{1}{1-st^i}.$$

Proof. We assign the tree T_P by another parameter s in two different ways and then we prove that the two assignments give the same generating function. The first assignment consists of giving a parameter s on each arrow $a \to a^{\downarrow 1}$. So, for each node λ in the tree T_P, the number of s in the path from the root to λ is equal to the largest part $a(\lambda)$ of λ. By this assignment, we see that T_P is always of the form X_∞. Let denote by $f^1(t,s)$ the corresponding generating function of this assignment.

It is easy to see that $f^1_{X_1}(t,s) = \frac{1}{1-st}$ and that $f^1_{X_k}(t,s) = f^1_{X_{k-1}}(t,s) + st^k f^1_{X_k}(t,s)$, which give the following results

$$\sum_{l(\lambda)\le k} t^{|\lambda|}s^{a(\lambda)} = f^1_{X_k}(t,s) = \prod_{i=1}^{k}\frac{1}{1-st^i},$$

and

$$\sum_{\lambda\in P} t^{|\lambda|}s^{a(\lambda)} = f^1_{T_P}(t,s) = \prod_{i=1}^{\infty}\frac{1}{1-st^i}.$$

On the other hand, the second assignment consists of giving one parameter s on each arrow in the rightmost path of T_P. Let us denote by $f^2(t,s)$ the corresponding generating function of this assignment. For each node λ in T_P, the number of s in the path from the root of T_P to λ is equal to the length $l(\lambda)$ of λ. One can see that there is no parameter s inside each X_k, so $f^2_{X_k}$ is equal to f_{X_k}. However, by this assignment, T_P is not of the form X_∞ but it is a sum of X_k and we have

$$\sum_{\lambda\in P} t^{|\lambda|}s^{l(\lambda)} = f^2_{T_P}(t,s) = 1+\sum_{k=1}^{\infty} s^k t^k f_{X_k}(t) = 1+\sum_{k=1}^{\infty}\frac{s^k t^k}{(1-t)\dots(1-t^k)}.$$

At the end, because the number of partitions of n of largest part k is equal to the number of partitions of n of length k. So the two generating functions $f^1_{T_P}(t,s)$ and $f^2_{T_P}(t,s)$ are equal, and we have the Euler's identity:

$$1+\sum_{k=1}^{\infty}\frac{s^k t^k}{(1-t)\dots(1-t^k)} = \prod_{i=1}^{\infty}\frac{1}{1-st^i}.$$

■

Now we calculate the generating function for strict partitions by using the same arguments as in Theorem 4.2 and obtain another *Euler identity.*

Theorem 4.3. (The second Euler Identity) *The following identity is due to Euler:*

$$1+\sum_{k=1}^{\infty}\frac{s^k t^{k(k+1)/2}}{\prod_{i=1}^{k}(1-t^i)} = \prod_{i=1}^{\infty}(1+st^i).$$

Proof. By using the assignment t to each arrow of the tree T_{SP}, we obtain the following formula for the generating function of T_{SP}:

$$f_{T_{SP}}(t) = \sum_{k=0}^{\infty} sp(k)t^k = 1+\sum_{k=1}^{\infty} t^{k(k+1)/2} f_{X_k}(t) = 1+\sum_{k=1}^{\infty}\frac{t^{k(k+1)/2}}{\prod_{i=1}^{k}(1-t^i)}.$$

On the other hand, the generating function of strict partitions is $SP(t) = \prod_{k=0}^{\infty}(1+t^k)$, so we have

$$1+\sum_{k=1}^{\infty}\frac{t^{k(k+1)/2}}{\prod\limits_{i=1}^{k}(1-t^i)}=\prod_{k=0}^{\infty}(1+t^k).$$

Similarly, we calculate generating function of two parameters. We define

$$SP(t,s)=\sum_n\left(\sum_{\lambda\vdash n}s^{l(\lambda)}\right)t^n$$

generating functions for the set of strict partitions of integer, where the powers of s represent the length of strict partitions. Firstly, it is easy to see that $SP(t,s) = \prod\limits_{i=1}^{\infty}(1+st^i)$.

Moreover, in T_{SP}, if we assign the arrow $() \to 1$ and each arrow $(m, m-1, \ldots, 3, 2) \to (m, m-1, \ldots, 3, 2, 1), m \geq 2$ in the rightmost path by s, then the powers of s represent the lenght of strict partitions, therefore the generating function of two parameters for the tree T_{SP} is equal to $SP(t,s)$. By using the recursive structure of T_{SP} (see Theorem 3.8), we have

$$f_{T_{SP}}(t,s)=\sum_n(\sum_{\lambda\vdash n}s^{l(\lambda)})t^n=1+\sum_{k=1}^{\infty}\frac{s^k t^{k(k+1)/2}}{\prod\limits_{i=1}^{k}(1-t^i)},$$

which implies the second Euler identity. ∎

Now, we investigate a different way of assignment of s to calculate the generating function of strict partitions on weight and on the largest part.

Proposition 4.4.

$$\sum_{n\geq 0}\sum_{\lambda\in SP(n)}t^{|\lambda|}s^{a(\lambda)}=\frac{1-t}{1-st}\prod_{k\geq 1}(1+t^k).$$

Proof. Firstly, in the tree T_{ODD}, if we assign each arrow $a \to a^{\downarrow 1}$ by s, then the power of s represents the number of parts equal to 1 in a partition and we have another two parameters generating function

$$f^2_{T_{ODD}}(t,s)=\sum_{n\geq 0}\sum_{\lambda\in ODD(n)}t^{|\lambda|}s^{|\{\lambda_i=1\}|}=\frac{1}{1-st}\prod_{k\geq 1}\frac{1}{1-t^{2k+1}}.$$

On the other hand, in the tree T_{SP}, if we assign each arrow $a \to a^{\downarrow 1}$ by s, then the power of s represents the largest part of partitions, and we have the corresponding generating function

$$f^2_{T_{SP}}(s,t)=\sum_{n\geq 0}\sum_{\lambda\in SP(n)}t^{|\lambda|}s^{a(\lambda)}.$$

Because the two trees T_{ODD} and T_{SP} are in a bijection, so we can calculate the two parameters generating function of T_{SP}, which completes our proof:

$$f^2_{T_{SP}}(t,s) = f^2_{ODD}(t,s) = \frac{1}{1-st}\prod_{k\geq 1}\frac{1}{1-t^{2k+1}}$$
$$= \frac{1-t}{1-st}\prod_{k\geq 0}\frac{1}{1-t^{2k+1}}, = \frac{1-t}{1-st}\prod_{k\geq 1}(1+t^k).$$

■

We end this section by giving the formula of the generating functions of the tree $T_{d\text{-}P}$. Denote by $d\text{-}p(n) = |d\text{-}\mathcal{P}(n)|$, the number of all *d-strict partition* of n. We define generating functions of one and two parameters respectively for the tree $T_{d\text{-}P}$ as follows:

$$f_{T_{d\text{-}P}}(t) = \sum_{n=0}^{\infty} d\text{-}p(n)t^n \text{ and } f_{d\text{-}P}(t,s) = \sum_{n=0}^{\infty}\left(\sum_{\lambda\in d\text{-}\mathcal{P}(n)} s^{l(\lambda)}\right)t^n.$$

By using the same assigning technique, we can deduce the generating functions of one and two parameters for this tree as below:

$$f_{T_{d\text{-}P}}(t) = \sum_{n=0}^{\infty} d\text{-}P(n)t^n = 1 + \sum_{n\geq 1}\frac{t^{\frac{n(nd+2-d)}{2}}}{(1-t)(1-t^2)\dots(1-t^n)}.$$
$$f_{T_{d\text{-}P}}(t,s) = \sum_{n=0}^{\infty}(\sum_{\lambda\in d\text{-}\mathcal{P}(n)} s^{l(\lambda)})t^n = 1 + \sum_{n\geq 1}\frac{s^n t^{\frac{n(nd+2-d)}{2}}}{(1-t)(1-t^2)\dots(1-t^n)}.$$

In conclusion, on the case $d = 2$, we get the formula

$$f_{T_{2\text{-}P}}(t) = 1 + \sum_{n\geq 1}\frac{t^{n^2}}{(1-t)(1-t^2)\dots(1-t^n)},$$

which is the left side of the following very famous *Ramanujan's identity:*

$$1 + \sum_{n\geq 1}\frac{t^{n^2}}{(1-t)(1-t^2)\dots(1-t^n)} = \prod_{i=0}^{\infty}\frac{1}{(1-t^{5i+1})(1-t^{5i+4})}.$$

An idea to prove this identity is therefore to construct a generating tree for its right side (integer partitions with parts of the form $5d+1$ or $5d+4$). Moreover, we hope that our method using generating tree with different ways of assignment can be applied to prove other identities and formulas on partitions.

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