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Local T-pluripolarity and T-pluripolarity of a Subset and some Cegrell's Pluricomplex Energy Classes Associated to a Positive Closed Current

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Abstract. In this paper we establish the equivalence between the locally T-pluripolarity and the T-pluripolarity of a subset E in a domain Ω in $\mathbb{C}^n$. At the same time, in the same way as Cegrell, we introduce notions of Cegrell's pluricomplex energy classes associated to a positive closed current T of dimension $q \geq 1$: $\mathcal{E}_0^T(\Omega), \mathcal{F}^T(\Omega), \mathcal{E}^T(\Omega)$. We give several results on the convergence, some inequalities in classes of pluricomplex energy in this new setting.

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1. Introduction

In the paper published on Documenta Math in 2006 Dabbek and Elkhadhra gave the notion of the relative capacity of a compact subset in an open set $\Omega \subset \mathbb{C}^n$ associated to a closed positive current T of dimension $q \geq 1$ and the notions of the local T-pluripolarity and the T-pluripolarity of a subset E in a domain Ω in $\mathbb{C}^n$ (see details in Sec. 2) and put the following open question: Is E T-pluripolar if E is locally T-pluripolar in Ω? The first aim in the paper is to answer the above question. We prove in Sec. 3 that if Ω is a domain in $\mathbb{C}^n$ and $E \subset \Omega$ is locally T-pluripolar then E is T- pluripolar. Also here we prove Xing type inequality for currents. We should remark that our inequality is stronger than Xing's inequality stated and proved at Lemma 1 in [13]. Here we replace the factor $\frac{1}{(n!)^2}$ by $\frac{1}{n!}$. In next sections, we combine Cegrell's pluricomplex energy classes and current theory in order to investigate Cegrell's pluricomplex energy class on positive closed currents. For details of these notions, see Sec. 4, 5. We show that the new energy classes inherit many properties of Cegrell's pluricomplex energy classes.

The paper is organized as follows. In Sec. 3 we prove the equivalence between the local T- pluripolarity and T-pluripolarity of a subset E in a domain Ω in $\mathbb{C}^n$ and Xing type inequality for currents. In Sec. 4 and Sec. 5 we introduce the classes of Cegrell's pluricomplex energy on positive closed currents: $\mathcal{E}_0^T(\Omega), \mathcal{F}^T(\Omega), \mathcal{E}^T(\Omega)$ and study some their properties.

2. Preliminaries

First we recall the definition of the relative capacity of a compact subset associated to a positive closed current T of dimension $q \geq 1$ presented in [6] and list some its properties. Let Ω be an open subset in $\mathbb{C}^n$ and K be a compact subset in Ω and T be a positive closed current of dimension $q \geq 1$ in Ω. We denote the set of plurisubharmonic functions in Ω by $\mathrm{psh}(\Omega)$ and the set of negative plurisubharmonic functions in Ω by $\mathrm{psh}^-(\Omega)$. By $L^\infty_{loc}(\Omega)$ we denote the set of locally bounded functions on Ω. In [6] the authors gave the notion of the relative capacity of K associated to T which is defined by

$$C_T(K,\Omega) = C_T(K) = \sup\left\{\int\limits_K (dd^c v)^q \wedge T, v \in \mathrm{psh}(\Omega), 0 < v < 1\right\}.$$

If $E \subset \Omega$ is a Borel subset, we define

$$C_T(E,\Omega) = \sup\{C_T(K), K \text{ compact}, K \subset E\}.$$

Then $C_T(E)$ is a Choquet capacity. The C_T has the following properties:

Proposition 2.1. [6]
1. *If E is a Borel set, then we have*

$$C_T(E,\Omega) = C_T(E) = \sup\left\{\int\limits_E (dd^c v)^q \wedge T, v \in psh(\Omega), 0 < v < 1\right\}.$$

2. *If $E_1 \subset E_2$, then $C_T(E_1,\Omega) \leq C_T(E_2,\Omega)$.*
3. *If $E \subset \Omega_1 \subset \Omega_2$, then $C_T(E,\Omega_1) \geq C_T(E,\Omega_2)$.*
4. *If $\{E_j\}_{j\geq 1} \subset \Omega$ is a sequence of Borel sets, then*

$$C_T\left(\bigcup_{j\geq 1} E_j, \Omega\right) \leq \sum_{j=1}^{\infty} C_T(E_j, \Omega).$$

5. *If* $E_1 \subset E_2 \subset \ldots$ *are Borel sets in* Ω*, then*

$$C_T\left(\bigcup_{j\geq 1} E_j, \Omega\right) = \lim_{j\to\infty} C_T(E_j, \Omega).$$

6. *Let* $f : \Omega_1 \to \Omega_2$ *be a proper holomorphic map on* $\operatorname{Supp} T$ *and suppose that* $\mathcal{O}$ *is an open subset in* Ω_2*. Then we have*

$$C_{f_*T}(\mathcal{O}, \Omega_2) \leq C_T(f^{-1}(\mathcal{O}), \Omega_1),$$

the equality holds if f *is a biholomorphic map.*

Locally bounded plurisubharmonic functions are quasi-continuous with respect to T by the following theorem

Theorem 2.2. [6] *Let* Ω *be an open bounded subset in* $\mathbb{C}^n$, $u \in psh(\Omega) \cap L^\infty_{loc}(\Omega)$ *and let* T *be a positive closed current in* Ω *of dimension* $q \geq 1$*. Then for all* $\epsilon > 0$*, there exists an open subset* O *in* Ω *such that* $C_T(O, \Omega) < \epsilon$ *and* u *is continuous on* $\Omega \setminus O$*.*

Let T be a positive closed current of bidimension $q \geq 1$. Then we can write T in terms of a differential form with coefficients $T_{I,J}$ which is complex measures as follows

$$T = \sum_{|I|=n-q, |J|=n-q} T_{I,J} dz^I \wedge d\bar{z}^J.$$

The quantity $\|T\| = \sum_I |T_{I,I}|$ is the total variation of T, where $|T_{I,J}|$ is the variation of the measures $T_{I,J}$. By [12], $\|T\|$ can be defined by

$$\|T\|(U) = \sup\{|T(\varphi)| : \varphi \in \mathcal{D}^{(q,q)}(U), |\varphi(x)| \leq 1, \forall x \in U\},$$

where $\mathcal{D}^{(q,q)}(U)$ is the space of differential forms which have coefficients of class $\mathcal{C}^\infty$ with compact supports in U and have (q, q) bidegree.

Recall that if $\{u_\alpha\}$ is a family of plurisubharmonic functions on Ω which is locally bounded from above then the function

$$u(z) = \sup_\alpha u_\alpha(z)$$

generally, is not plurisubharmonic, but its upper semicontinuous regularization

$$u^*(z) = \overline{\lim_{\zeta\to z}}\, u(\zeta) \geq u(z)$$

is plurisubharmonic. Now we recall the definition of a T-pluripolar subset $E \subset \Omega \subset \mathbb{C}^n$ in [6]. A subset $E \subset \Omega$ is called to be T-pluripolar if $C_T(E) = 0$. In [6], the authors do not give the notion about $T-$negligible subsets. They used this notion in the Corollary 2.7 (see [6]). We give the notion as follows. A subset $E \subset \Omega \subset \mathbb{C}^n$ is said to be T-negligible if $\|T\|(E) = 0$ where T is a positive closed current of dimension $q \geq 1$ on Ω. By using the notion of the relative capacity of a subset $E \subset \Omega$ associated to a positive closed current T defined as above and applying the Chern-Levine-Nirenberg inequality in [3] it follows that if E is a Borel subset of Ω and E is T-negligible then E is T-pluripolar. The converse is true if Ω is a bounded domain in $\mathbb{C}^n$. Indeed, let Ω be a bounded domain in $\mathbb{C}^n$ and E be T-pluripolar, where T is a positive closed current of dimension $q \geq 1$. Then

$$\|T\|(E) \leq \mathrm{C}_1 \int\limits_E T \wedge \beta^q = \int\limits_E T \wedge (\frac{1}{4} dd^c \|z\|^2)^q \leq \mathrm{C}_2 C_T(E),$$

where $\beta = \frac{1}{4} dd^c \|z\|^2 = \frac{i}{2} \sum\limits_{j=1}^{n} dz_j \wedge d\overline{z}_j$ is the canonical Kähler form of $\mathbb{C}^n$ and C_1, C_2 are some constants, the second inequality follows from the boundedness of Ω. Hence, from the above proof we can state the following

Proposition 2.3. *Let Ω be a bounded domain in $\mathbb{C}^n$ and T be a positive closed current of dimension $q \geq 1$ on Ω. Then a subset $E \subset \Omega$ is T-negligible if and only if it is T-pluripolar.*

3. The Local *T*-pluripolarity and *T*-pluripolarity of a Subset and Xing's Inequality for Currents

In this section we establish the equivalence between the local T-pluripolarity and the T-pluripolarity of a subset which answer the fourth open problem put in [6]. First we recall the following notion in [6]. Let Ω be an open subset in $\mathbb{C}^n$ and T a positive closed current of dimension $q \geq 1$ on Ω. A subset E of Ω is said to be locally $T-$pluripolar in Ω if for each $a \in E$ there exists an open neighborhood V of a in Ω such that $E \cap V$ is T-pluripolar in V. Dabbek and Elkhadhra in [6] put the following problem. Is E T-pluripolar in Ω if E is locally T-pluripolar in Ω. The answer is in the following

Proposition 3.1. *Let Ω be an open subset in $\mathbb{C}^n$ and T be a positive closed current of dimension $q \geq 1$ in Ω. If $E \subset \Omega$ is locally T-pluripolar then it is T-pluripolar in Ω.*

Proof. From the hypothesis we can find a sequence of open subsets $\{V_j\}, V_j \subset \Omega$ and $\{E_j\}, E_j = E \cap V_j \subset V_j$ such that $E \subset \bigcup_j E_j$ and E_j is T-pluripolar in V_j. Then $C_T(E_j, V_j) = 0$. Proposition 2.1 implies that $C_T(E_j, V_j) \geq C_T(E_j, \Omega)$ and, consequently, $C_T(E_j, \Omega) = 0$ for all $j \geq 1$. The desired conclusion follows from the following

$$C_T(E, \Omega) \le C_T(\bigcup_j E_j, \Omega) \le \sum_j C_T(E_j, \Omega).$$

■

Next we prove the following Xing's type inequality for currents.

Theorem 3.2. *Let Ω be a bounded open subset in $\mathbb{C}^n$ and $u, v \in psh(\Omega) \cap L^\infty(\Omega)$ satisfing $\liminf_{z \to \partial\Omega}(u(z) - v(z)) \ge 0$ and let T be a positive closed current of dimension $q \ge 1$ on Ω. Then for any constant $r \ge 1$ and all $w_j \in psh(\Omega)$ with $0 \le w_j \le 1, j = 1, \dots, q$ we have*

$$\frac{1}{(q!)} \int_{\{u<v\}} (v-u)^q dd^c w_1 \wedge \ldots \wedge dd^c w_q \wedge T + \int_{\{u<v\}} (r - w_1)(dd^c v)^q \wedge T$$
$$\le \int_{\{u<v\}} (r - w_1)(dd^c u)^q \wedge T.$$

Proof. We first prove that Theorem 3.2 holds for continuous functions u, v in Ω. In this case, without loss of generality, we can assume that $\Omega = \{u < v\}$. For each constant $\varepsilon > 0$ we define a function $v_\epsilon = \max(u, v - \varepsilon) \uparrow v$ in Ω as $\varepsilon \downarrow 0$ and $v_\varepsilon = u$ near the boundary $\partial\Omega$. Stokes' formula implies that

$$\int_\Omega (v_\varepsilon - u)^q dd^c w_1 \wedge \ldots \wedge dd^c w_q \wedge T$$
$$= \int_\Omega (w_q - 1)(dd^c(v_\varepsilon - u)^q) \wedge dd^c w_1 \wedge \ldots \wedge dd^c w_{q-1} \wedge T$$
$$= q(q-1) \int_\Omega (w_q - 1)(v_\varepsilon - u)^{q-2} d(v_\varepsilon - u) \wedge d^c(v_\varepsilon - u) \wedge dd^c w_1$$
$$\wedge \ldots \wedge dd^c w_{q-1} \wedge T + q \int_\Omega (w_q - 1)(v_\varepsilon - u)^{q-1} dd^c(v_\varepsilon - u) \wedge dd^c w_1$$
$$\wedge \ldots \wedge dd^c w_{q-1} \wedge T$$
$$\le q \int_\Omega (1 - w_q)(v_\varepsilon - u)^{q-1} dd^c u \wedge dd^c w_1 \wedge \ldots \wedge dd^c w_{q-1} \wedge T$$
$$\le q \int_\Omega (v_\varepsilon - u)^{q-1} dd^c u \wedge dd^c w_1 \wedge \ldots \wedge dd^c w_{q-1} \wedge T$$
$$\le q(q-1) \int_\Omega (v_\varepsilon - u)^{q-2} (dd^c u)^2 \wedge dd^c w_1 \wedge \ldots \wedge dd^c w_{q-2} \wedge T$$
$$\le \cdots \le q(q-1)\cdots 2 \int_\Omega (v_\varepsilon - u)(dd^c u)^{q-1} \wedge dd^c w_1 \wedge T$$

$$\leq q!\int_{\Omega}(v_\varepsilon-u)\Big(\sum_{s=0}^{q-1}(dd^cu)^{q-1-s}\wedge(dd^cv_\varepsilon)^s\Big)\wedge dd^cw_1\wedge T$$
$$=q!\int_{\Omega}(w_1-r)(dd^c(v_\varepsilon-u))\Big(\sum_{s=0}^{q-1}(dd^cu)^{q-1-s}\wedge(dd^cv_\varepsilon)^s\Big)\wedge T$$
$$=q!\int_{\Omega}(w_1-r)\Big((dd^cv_\varepsilon)^q-(dd^cu)^q\Big)\wedge T$$
$$=q!\int_{\Omega}(r-w_1)(dd^cu)^q\wedge T-q!\int_{\Omega}(r-w_1)(dd^cv_\varepsilon)^q\wedge T.$$

But Theorem 1.7 in [7] implies that

$$(r-w_1)(dd^cv_\varepsilon)^q\wedge T\to(r-w_1)(dd^cv)^q\wedge T$$

as currents when $\varepsilon\downarrow 0$ then

$$\int_{\Omega}(r-w_1)(dd^cv)^q\wedge T\leq\liminf_{\varepsilon\to 0}\int_{\Omega}(r-w_1)(dd^cv_\varepsilon)^q\wedge T$$

and hence we obtain the required inequality for continuous functions u and v.

The general case will then follow by an approximation argument. As in the proof of Theorem 4.1 in [1], we may assume that for each $\delta>0$ there exists an open $E\Subset\Omega$ such that $u(z)-v(z)\geq\delta>0$ for all $z\in\Omega\setminus E$. Otherwise, we can replace u by $u+2\delta$ and then let $\delta\downarrow 0$. We can choose two decreasing sequences of smooth psh functions u_k and v_j in a neighborhood Ω' of $\bar{E}$ such that $\lim\limits_{k\to\infty}u_k=u$, $\lim\limits_{j\to\infty}v_j=v$ in Ω' and $u_k\geq v_j$ near the boundary $\partial\Omega'$. For smooth functions u_k and u_j in Ω', we have proved the following inequality

$$\frac{1}{q!}\int_{\{u_k<v_j\}}(v_j-u_k)^q dd^cw_1\wedge\ldots\wedge dd^cw_q\wedge T+\int_{\{u_k<v_j\}}(r-w_1)(dd^cv_j)^q\wedge T$$
$$\leq\int_{\{u_k<v_j\}}(r-w_1)(dd^cu_k)^q\wedge T.$$

Letting $j\to\infty$ and then $k\to\infty$ and using Fatou Lemma, we get that the limit inferior of the first term in the left hand side exceeds

$$\frac{1}{q!}\int_{\{u<v\}}(v-u)^q dd^cw_1\wedge\ldots\wedge dd^cw_q\wedge T.$$

To handle the other terms in the same inequality when $j\to\infty$ and $k\to\infty$, we first observe that $(r-w_1)(dd^cu_k)^q\wedge T\to(r-w_1)(dd^cu)^q\wedge T$ and $(r-w_1)(dd^cv_j)^q\wedge T\to(r-w_1)(dd^cv)^q\wedge T$ as currents, see [7]. Completely repeating the proof of

Theorem 4.1 in [1] and notice that the quasi-continuity of psh functions in that proof will be replaced by the T-quasicontinuity as the Theorem 2.2, we get

$$\frac{1}{q!}\int\limits_{\{u<v\}}(v-u)^q dd^c w_1\wedge\ldots\wedge dd^c w_q\wedge T+\int\limits_{\{u<v\}}(r-w_1)(dd^c v)^q\wedge T$$
$$\le\int\limits_{\{u\le v\}}(r-w_1)(dd^c u)^q\wedge T.$$

Finally, applying the last inequality for functions $u+\sigma$ instead of u and letting $\sigma\downarrow 0$ we obtain the required inequality and hence the proof is complete. ■

In particular, consequently, we obtain the following result in [6].

Corollary 3.3. [6] [Comparison principle] *Let Ω be a bounded open subset in $\mathbb{C}^n$ and T be a positive closed current of dimension $q\ge 1$ on $\Omega, u, v\in \mathrm{psh}(\Omega)\cap L^\infty(\Omega)$ satisfying $\liminf\limits_{\xi\to\partial\Omega}(u(\xi)-v(\xi))\ge 0$ then we have*

$$\int\limits_{\{u<v\}}(dd^c v)^q\wedge T\le\int\limits_{\{u<v\}}(dd^c u)^q\wedge T.$$

4. Cegrell's Pluricomplex Energy Class $\mathcal{E}_0^T(\Omega)$

In this section we introduce the Cegrell pluricomplex energy class $\mathcal{E}_0^T(\Omega)$ associated to a positive closed current T of dimension $q\ge 1$. Let $\Omega\subset\mathbb{C}^n$ be a bounded domain and T a positive closed current of dimension $q\ge 1$. We denote by $\mathcal{E}_0^T(\Omega)$ the class of negative and bounded plurisubharmonic functions φ on Ω such that $\lim\limits_{z\to\xi}\varphi(z)=0,\ \forall\xi\in\partial\Omega$ and $\int(dd^c\varphi)^q\wedge T<+\infty$. Then we have

(i) $\mathcal{E}_0^T(\Omega)$ is a convex cone, and
(ii) If $\varphi\in\mathcal{E}_0^T(\Omega)$ and $\psi\in \mathrm{psh}^-(\Omega)$ then $\max(\varphi,\psi)\in\mathcal{E}_0^T(\Omega)$.

Indeed, it is easy to see that if $\varphi\in\mathcal{E}_0^T(\Omega)$ then $\alpha\varphi\in\mathcal{E}_0^T(\Omega), \forall\ \alpha\in\mathbb{R}^+$. Now, if $\varphi,\psi\in\mathcal{E}_0^T(\Omega)$, then

$$\int\limits_{\Omega}(dd^c(\varphi+\psi))^q\wedge T=\int\limits_{\varphi<\psi}(dd^c(\varphi+\psi))^q\wedge T+\int\limits_{\psi\le\varphi}(dd^c(\varphi+\psi))^q\wedge T.$$

However, $\{\varphi<\psi\}\subset\{2\varphi<\varphi+\psi\}$, by Corollary 3.3, we have

$$\int\limits_{2\varphi<\varphi+\psi}(dd^c(\varphi+\psi))^q\wedge T \le \int\limits_{2\varphi<\varphi+\psi}(dd^c(2\varphi)^q\wedge T$$
$$= 2^q\int\limits_{2\varphi<\varphi+\psi}(dd^c\varphi)^q\wedge T$$
$$\le 2^q\int\limits_{\Omega}(dd^c\varphi)^q\wedge T < +\infty.$$

Hence,

$$\int\limits_{\varphi<\psi}(dd^c(\varphi+\psi))^q\wedge T < +\infty.$$

Similarly, take $\lambda > 1$. Then $\lambda\psi < \psi$ and $\{\psi \le \varphi\} \subset \{\lambda\psi < \varphi\} \subset \{(\lambda+1)\psi < \varphi+\psi\}$. Hence, Corollary 3.3 implies that

$$\int\limits_{\{\psi\le\varphi\}}(dd^c(\varphi+\psi))^q\wedge T \le (\lambda+1)^q\int\limits_{\Omega}(dd^c\psi)^q\wedge T < +\infty.$$

Thus $\int\limits_{\Omega}(dd^c(\varphi+\psi))^q\wedge T < +\infty$ and, hence $\varphi+\psi \in \mathcal{E}_0^T(\Omega)$. Finally, for all $k > 1$, by Corollary 3.3 we have

$$\int\limits_{\Omega}(dd^c\max(\varphi,\psi))^q\wedge T = \int\limits_{k\varphi<\max(\varphi,\psi)}(dd^c\max(\varphi,\psi))^q\wedge T$$
$$\le k^p\int\limits_{k\varphi<\max(\varphi,\psi)}(dd^c\varphi)^q\wedge T = k^p\int\limits_{\Omega}(dd^c\varphi)^q\wedge T. \quad (1)$$

From the boundedness of $\int\limits_{\Omega}(dd^c\varphi)^q\wedge T$, the desired conclusion follows.

We also get the following Blocki type inequality which is true in the setting of class $\mathcal{E}_0^T(\Omega)$.

Theorem 4.1. *If* $v,\varphi \in \mathcal{E}_0^T(\Omega)$ *then*

$$\int(-\varphi)^{q+1}(dd^cv)^q\wedge T \le (q+1)!(\sup(-v))^q\int(-\varphi)(dd^c\varphi)^q\wedge T.$$

Proof. Integrating by parts we have

$$\begin{aligned}\int(-\varphi)^{q+1}(dd^cv)^q\wedge T=&\int vdd^c(-\varphi)^{q+1}\wedge(dd^cv)^{q-1}\wedge T\\ &=(q+1)q\int v(-\varphi)^{q-1}d\varphi\wedge d^c\varphi\wedge(dd^cv)^{q-1}\wedge T\\ &\quad+(q+1)\int(-v)(-\varphi)^q dd^c\varphi\wedge(dd^cv)^{q-1}\wedge T\\ &\le(q+1)(\sup(-v))\int(-\varphi)(dd^c\varphi)\wedge(dd^cv)^{q-1}\wedge T.\end{aligned}$$

Repeating this process $q-1$ times more, we have the delivered inequality. The proof is complete. ■

Theorem 4.2. *Suppose $u, v \in \mathcal{E}_0^T(\Omega)$. If $p \geq 1$ then*

$$\begin{aligned}&\int_\Omega(-u)^p(dd^cu)^j\wedge(dd^cv)^{q-j}\wedge T\\ &\le D_{j,q}\left(\int_\Omega(-u)^p(dd^cu)^q\wedge T\right)^{\frac{p+j}{p+q}}\left(\int_\Omega(-v)^p(dd^cv)^q\wedge T\right)^{\frac{q-j}{p+q}},\end{aligned}$$

where $D_{j,p}=p^{\frac{(p+j)(q-j)}{p-1}}$ if $s>1$ and $D_{j,1}=e^{(1+j)(q-j)}$.

Proof. Cf. [6]. ■

5. Cegrell's Pluricomplex Energy Classes $\mathcal{E}^T(\Omega)$ and $\mathcal{F}^T(\Omega)$

In this section we will deal with Cegrell's pluricomplex energy classes $\mathcal{E}^T(\Omega)$ and $\mathcal{F}^T(\Omega)$, where Ω is a hyperconvex domain in $\mathbb{C}^n$. We recall that a set $\Omega\subset\mathbb{C}^n$ is said to be a hyperconvex domain if it is open, bounded, connected and if there exists $\varphi\in\mathrm{psh}^-(\Omega)$ such that $\{z\in\Omega:\varphi(z)<-c\}\Subset\Omega,\forall\ c>0$. Such function is called an exhaustion function for Ω.

In hyperconvex domains, we have the following useful approximation theorem.

Theorem 5.1. *Suppose Ω is a hyperconvex domain and assume that $u\in\mathrm{psh}^-(\Omega)$, $\lim\limits_{z\to\partial\Omega}u(z)=0$ and T is a positive closed current of dimension q such that $\int_\Omega(dd^cu)^q\wedge T<+\infty$. Then there exists a decreasing sequence of functions $u_j\in\mathcal{E}_0^T(\Omega)$ with $u_j\mid_{\partial\Omega}=0,\forall j\in\mathbb{N}$, $\lim\limits_{j\to\infty}u_j(z)=u(z),\ \forall z\in\Omega$.*

Proof. By Theorem 2.1 in [5] it follows that there exists a decreasing sequence $\{u_j\}\subset\mathrm{psh}^-(\Omega)\cap\mathrm{C}(\overline{\Omega})$ such that $u_j\mid_{\partial\Omega}=0$, $\forall j\in\mathbb{N}$, $\lim\limits_{j\to\infty}u_j(z)=u(z)$, $\forall z\in\Omega$. The hypothesis implies that

$$\liminf_{z\to\partial\Omega}(u(z)-u_j(z))\geq 0.$$

Hence, Corollary 3.3 implies that

$$\int_{\Omega}(dd^c u_j)^q\wedge T=\int_{\{u<u_j\}}(dd^c u_j)^q\wedge T\leq\int_{\Omega}(dd^c u)^q\wedge T<\infty$$

and the desired conclusion follows. ■

Now we state the following result which is similar to Theorem 2.1 in [5].

Lemma 5.2. *If Ω is a hyperconvex domain, then*

$$\mathcal{C}_0^\infty(\Omega)\subset\mathcal{E}_0^T(\Omega)\cap\mathcal{C}(\bar{\Omega})-\mathcal{E}_0^T(\Omega)\cap\mathcal{C}(\bar{\Omega}).$$

Notice that, in [5], Cegrell proved the above statement for the Cegrell class $\mathcal{E}_0$, but in our setting, the proof of Cegrell is still valid, for $\mathcal{E}_0^T(\Omega)$ is a convex cone. For details, see [5] and the previous section.

The following result is similar as in [5].

Theorem 5.3. *Suppose Ω is a hyperconvex domain, $u,v\in\mathrm{psh}^-(\Omega), u\neq 0, \lim_{z\to\xi}u(z)=0$, $\forall\xi\in\partial\Omega$, and T a positive closed current of dimension q, $0\leq q\leq n$. Then $(dd^c u)^q\wedge T$ is a well-defined positive measure on Ω. Furthermore, if*

$$\int v(dd^c u)^q\wedge T>-\infty$$

then $(dd^c v)^q\wedge T$ is also a well-defined positive measure on Ω and

$$\int v(dd^c u)^q\wedge T\leq\int u(dd^c v)^q\wedge T.$$

Proof. Using Sibony's arguments as in [12] we can assume that T is a positive and closed current of dimension 1. The proof now follows from [5]. ■

Corollary 5.4. *Suppose $u,v\in\mathrm{psh}(\Omega)$, $\lim_{z\to\xi}u(z)=\lim_{z\to\xi}v(z)=0$, $\forall\xi\in\partial\Omega$ and that T is a closed positive current of dimension $q\geq 1$. If*

$$\int v(dd^c u)^q\wedge T>-\infty$$

then

$$\int u(dd^c v)^q\wedge T>-\infty$$

and

$$\int v(dd^c u)^q\wedge T=\int u(dd^c v)^q\wedge T.$$

Now we give the definition of the classes of Cegrell's T-pluricomplex energy.

Definition 5.5. Let $u \in \mathrm{psh}^-(\Omega)$. u is said to belong to $\mathcal{E}^T(\Omega)$ if for each $z_0 \in \Omega$ there exist a neighborhood ω of z_0 in Ω and a decreasing sequence $\{h_j\} \in \mathcal{E}_0^T(\Omega)$ such that $h_j \downarrow u$ on ω and $\sup_j \int_\Omega (dd^c h_j)^q \wedge T < \infty$.

In order to give the definition of measures for the class of Cegrell's T-pluricomplex energy $\mathcal{E}^T(\Omega)$, we establish the following convergence theorem.

Theorem 5.6. *Suppose $u^p \in \mathcal{E}^T(\Omega), 1 \le p \le q$. If $g_j^p \in \mathcal{E}_0^T(\Omega)$ decreases to u^p as $j \to \infty$, then $dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^q \wedge T$ is weak*-convergent and the limit measure does not depend on the particular sequence $\{g_j^p\}_{j=1}^\infty$.*

Proof. Suppose first that $\sup_j \int (dd^c g_j^p)^q \wedge T < +\infty$. Then, for $h \in \mathcal{E}_0^T(\Omega)$ we have that

$$\int h dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^q \wedge T$$

is a decreasing sequence by Corollary 5.4 and since

$$\int h (dd^c g_j^p)^q \wedge T \ge \left(\inf_\Omega h\right) \sup_j \int (dd^c g_j^p)^q \wedge T > -\infty,$$

we conclude that $\lim_{j\to\infty} \int h dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^q \wedge T$ exists for all $h \in \mathcal{E}_0^T(\Omega)$. By Lemma 5.2, $dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^q \wedge T$ is weak*-convergent.

If v_j^p is another sequence decreasing to u^p, we obtain

$$\begin{aligned}
&\int h dd^c v_j^1 \wedge dd^c v_j^2 \wedge \ldots \wedge dd^c v_j^q \wedge T \\
&= \int v_j^1 dd^c h \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^q \wedge T \\
&\ge \int u^1 dd^c h \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^q \wedge T \\
&= \lim_{s_1\to\infty} \int g_{s_1}^1 dd^c h \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^q \wedge T \\
&= \lim_{s_1\to\infty} \int v_j^2 dd^c h \wedge dd^c g_{s_1}^1 \wedge \ldots \wedge dd^c g_j^q \wedge T \\
&\ge \lim_{s_1\to\infty} \lim_{s_2\to\infty} \cdots \lim_{s_q\to\infty} \int h dd^c g_{s_1}^1 \wedge dd^c g_{s_2}^2 \wedge \ldots \wedge dd^c g_{s_q}^q \wedge T \\
&\ge \lim_{s_1,s_2,\cdots,s_q\to\infty} \int h dd^c g_{s_1}^1 \wedge dd^c g_{s_2}^2 \wedge \ldots \wedge dd^c g_{s_q}^q \wedge T.
\end{aligned}$$

Therefore, $\lim_{j\to\infty} \int h dd^c v_j^1 \wedge dd^c v_j^2 \wedge \ldots \wedge dd^c v_j^q \wedge T$ exists and is minorized by

$$\lim_{s_1, s_2 \cdots, s_q \to \infty} \int h dd^c g^1_{s_1} \wedge dd^c g^2_{s_2} \wedge \ldots \wedge dd^c g^q_{s_q} \wedge T.$$

But this is a symmetric situation so we conclude that the limits are equal.
To complete the proof it remains to remove the restriction

$$\sup_j \int (dd^c g^p_j)^q \wedge T < +\infty.$$

Let K be a given compact subset of Ω. Cover K with finitely many $W_s, s = 1 \ldots, N$ as in the definition of $\mathcal{E}^T(\Omega)$. Let $\{h^{ps}_j\}_{j=1}^{\infty}, 1 \leq s \leq N, 1 \leq p \leq q$ be the corresponding u^p and put $w^p_j = \sum_{s=1}^{N} h^{ps}_j$. Then $w^p_j \in \mathcal{E}^T_0(\Omega), w^p_j \leq g^p_j$ on $\bigcup_s W_s$ and

$$\sup_j \int_{\Omega} (dd^c w^p_j)^q \wedge T < +\infty.$$

Thus, if we define $v^p_j = \max(g^p_j, w^p_j) \in \mathcal{E}^T_0(\Omega)$ and we have $\sup_j \int_{\Omega} (dd^c v^p_j)^q \wedge T < +\infty$ and $v^p_j = g^p_j$ near K. ■

Definition 5.7. For $u^p \in \mathcal{E}^T(\Omega), 1 \leq p \leq q$, we define $dd^c u^1 \wedge dd^c u^2 \wedge \ldots dd^c u^q \wedge T$ to be the limit measure obtained in Theorem 5.6.

Definition 5.8. We denote by $\mathcal{F}^T(\Omega)$ the subclass of functions u in $\mathcal{E}^T(\Omega)$ such that there exists a decreasing sequence $u_j \in \mathcal{E}^T_0(\Omega)$ such that $u_j \downarrow u$ on Ω and $\sup_j \int_{\Omega} (dd^c u_j)^q \wedge T < +\infty$.

Remark 5.9. It is easy to see that $\mathcal{E}^T_0(\Omega) \subset \mathcal{F}^T(\Omega) \subset \mathcal{E}^T(\Omega)$. Moreover, to study Cegrell's pluricomplex energy classes $\mathcal{E}^T_0(\Omega)$, $\mathcal{F}^T(\Omega)$ and $\mathcal{E}^T(\Omega)$ associated to a positive closed current T of bidimension $q \geq 1$, it will be useful to understand deeply Cegrell pluricomplex energy classes $\mathcal{E}_0(\Omega), \mathcal{F}(\Omega)$ and $\mathcal{E}(\Omega)$ introduced and investigated in [4] and [5].

Remark 5.10. From Definition 5.8 it follows that if $u \in \mathcal{F}^T(\Omega)$ then

$$\int_{\Omega} (dd^c u)^q \wedge T < \infty.$$

Remark 5.11. It follows from Corollary 5.4 and Theorem 5.6 that integration by parts is allowed in $\mathcal{F}^T(\Omega)$.

Remark 5.12. We also can show that every $u \in \mathcal{E}^T(\Omega)$ is locally in $\mathcal{F}^T(\Omega)$. This means for every $u \in \mathcal{E}^T(\Omega)$ and every ω open and relatively compact in Ω, there is an $u_\omega \in \mathcal{F}^T(\Omega)$ with $u = u_\omega$ in ω.

Remark 5.13. Similarly, from the convexity of $\mathcal{E}_0^T(\Omega)$ the convexity of $\mathcal{E}^T(\Omega)$ and $\mathcal{F}^T(\Omega)$ follows.

Lemma 5.14. *Let $u, v \in \mathcal{F}^T(\Omega)$ be such that $u \leq v$ on Ω then*

$$\int_\Omega \varphi(dd^c u)^q \wedge T \leq \int_\Omega \varphi(dd^c v)^q \wedge T,$$

where $\varphi \in \mathrm{psh}^-(\Omega)$.

Proof. First, assume that $\varphi \in \mathcal{E}_0^T(\Omega)$. Then Corollary 5.4 implies that

$$\int_\Omega (-\varphi)(dd^c u)^q \wedge T = \int_\Omega (-u)(dd^c \varphi) \wedge (dd^c u)^{q-1} \wedge T.$$

By hypothesis $(-v) \leq (-u)$ it follows that

$$\int_\Omega (-\varphi)(dd^c u)^q \wedge T \geq \int_\Omega (-v)(dd^c \varphi) \wedge (dd^c u)^{q-1} \wedge T.$$

Again by Corolarry 5.4 we obtain

$$\int_\Omega (-\varphi)(dd^c u)^q \wedge T \geq \int_\Omega (-\varphi)(dd^c v) \wedge (dd^c u)^{q-1} \wedge T.$$

Repeating this process $n-1$ times more, we conclude

$$\int_\Omega (-\varphi)(dd^c u)^q \wedge T \geq \int_\Omega (-\varphi)(dd^c v)^q \wedge T.$$

Hence

$$\int_\Omega \varphi(dd^c u)^q \wedge T \leq \int_\Omega \varphi(dd^c v)^q \wedge T.$$

Now if $\varphi \in \mathrm{psh}(\Omega), \varphi \leq 0$ is arbitrary, then by Theorem 5.1, we can find a decreasing sequence $\{\varphi_j\} \subset \mathcal{E}_0^T(\Omega)$ such that $\{\varphi_j\}$ converges pointwise to φ as $j \to \infty$. The above inequality yields

$$\int_\Omega \varphi_j(dd^c u)^q \wedge T \leq \int_\Omega \varphi_j(dd^c v)^q \wedge T.$$

Now the lemma follows from the monotone convergence theorem. The proof is complete. ∎

Corollary 5.15. *Let $u, v \in \mathcal{F}^T(\Omega)$ be given such that $u \leq v$ on Ω then*

$$\int\limits_{\Omega}(dd^c v)^q \wedge T \leq \int\limits_{\Omega}(dd^c u)^q \wedge T.$$

Proof. The proof follows from Lemma 5.14, by tending φ to -1. ■

Proposition 5.16. *Suppose $u^p \in \mathcal{F}^T(\Omega), 1 \leq p \leq q$ and $h \in \mathrm{psh}^-(\Omega)$, $\lim\limits_{z\to\partial\Omega} h(z) = 0$. If $g_j^p \in \mathcal{E}_0^T(\Omega)$ decreases to u^p as $j \to \infty$ then*

$$\lim_{j\to\infty}\int h dd^c g_{s_1}^1 \wedge dd^c g_{s_2}^2 \wedge \ldots \wedge dd^c g_{s_q}^q \wedge T = \int h dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^q \wedge T.$$

Proof. If h is in $\mathcal{E}_0^T(\Omega)\cap\mathcal{C}(\Omega)$ then from the proof of Theorem 5.6 it follows that

$$\lim_{j\to\infty}\int h dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^q \wedge T \int h dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^q \wedge T.$$

Suppose now $h \in \mathrm{psh}^-(\Omega)$, $\lim\limits_{z\to\partial\Omega} h(z) = 0$ and that $\int h dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^q \wedge T$ is finite. For each j, by Theorem 5.1 we can choose $h_j \in \mathcal{E}_0^T(\Omega)\cap\mathcal{C}(\Omega)$ decreasing to h, q_j and s_j such that

$$\begin{aligned}
&\int -h dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^q \wedge T \\
&\leq \int -h_j dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^q \wedge T + \frac{1}{j} \\
&\leq \int -h_j dd^c g_{q_j}^1 \wedge dd^c g_{q_j}^2 \wedge \ldots \wedge dd^c g_{q_j}^q \wedge T + \frac{2}{j} \\
&\leq \int -h dd^c g_{q_j}^1 \wedge dd^c g_{q_j}^2 \wedge \ldots \wedge dd^c g_{q_j}^q \wedge T + \frac{2}{j} \\
&\leq \int -h_{s_j} dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^q \wedge T + \frac{4}{j} \\
&\leq \int -h dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^q \wedge T + \frac{4}{j}.
\end{aligned}$$

Letting $j \to \infty$ we get

$$\lim_{j\to\infty}\int h dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^q \wedge T = \int h dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^q \wedge T.$$

Also, if

$$\int h dd^c u^1 \wedge dd^c u^2 \wedge \ldots \wedge dd^c u^q \wedge T = -\infty$$

then by the standard arguments we infer that

$$\lim_{j\to\infty}\int h dd^c g_j^1 \wedge dd^c g_j^2 \wedge \ldots \wedge dd^c g_j^q \wedge T = -\infty.$$

The proof is complete. ■

Lemma 5.17. *Suppose $u, v \in \mathcal{F}^T(\Omega), h \in \mathcal{E}_0^T(\Omega)$ and that p, s are positive natural numbers such that $p + s = q$. Then*

$$\int_\Omega -h(dd^c u)^p \wedge (dd^c v)^s \wedge T \leq \left(\int_\Omega -h(dd^c u)^q \wedge T\right)^{\frac{p}{q}} \left(\int_\Omega -h(dd^c v)^q \wedge T\right)^{\frac{s}{q}}. \tag{2}$$

Proof. By Proposition 5.15 it suffices to prove (2) for the case $u, v \in \mathcal{E}_0^T(\Omega)$. First, let $\tilde{T} = (dd^c u)^r \wedge (dd^c v)^t \wedge T$, where r, t are arbitrary natural numbers such that $r + t = q - 2$. By using the arguments as in [5] we have

$$\int_\Omega -h(dd^c u) \wedge (dd^c v) \wedge \tilde{T} \leq \left(\int_\Omega -h(dd^c u)^2 \wedge \tilde{T}\right)^{1/2} \left(\int_\Omega -h(dd^c v)^2 \wedge \tilde{T}\right)^{1/2}.$$

Now assume that (5.1) holds for $p + s + r + t = q, p + s = m < q$, $m \geq 2$ and $\tilde{T} = (dd^c u)^r \wedge (dd^c v)^t \wedge T$. This means that the following

$$\int_\Omega -h(dd^c u)^p \wedge (dd^c v)^s \wedge \tilde{T}$$
$$\leq \left(\int_\Omega -h(dd^c u)^{p+s} \wedge \tilde{T}\right)^{\frac{p}{p+s}} \left(\int_\Omega -h(dd^c v)^{p+s} \wedge \tilde{T}\right)^{\frac{s}{p+s}} \tag{3}$$

holds. Now, we have to prove (2) for $p + s = m + 1, p + s + r + t = q$ and $\tilde{T} = (dd^c u)^r \wedge (dd^c v)^t \wedge T$. By the same arguments in [5], we first prove if $p' + s' = m$ then

$$\int_\Omega -h(dd^c u)^{p'+s'} \wedge (dd^c v) \wedge \tilde{T}$$
$$\leq \left(\int_\Omega -h(dd^c u)^{p'+s'+1} \wedge \tilde{T}\right)^{\frac{p'+s'}{p'+s'+1}} \left(\int_\Omega -h(dd^c v)^{p'+s'+1} \wedge \tilde{T}\right)^{\frac{1}{p'+s'+1}}.$$

Indeed, by applying (3) it follows that

$$\int_\Omega -h(dd^c u)^{p'+s'} \wedge (dd^c v) \wedge \tilde{T}$$
$$= \int_\Omega -h(dd^c u)^{p'+s'-1} \wedge (dd^c v) \wedge (dd^c u) \wedge \tilde{T}$$

$$
\begin{aligned}
&\leq \left(\int\limits_{\Omega} -h(dd^c u)^{p'+s'} \wedge (dd^c u) \wedge \tilde{T}\right)^{\frac{p'+s'-1}{p'+s'}} \times \\
&\quad \times \left(\int\limits_{\Omega} -h(dd^c v)^{p'+s'} \wedge (dd^c u) \wedge \tilde{T}\right)^{\frac{1}{p'+s'}} \\
&= \left(\int\limits_{\Omega} -h(dd^c u)^{p'+s'+1} \wedge \tilde{T}\right)^{\frac{p'+s'-1}{p'+s'}} \left(\int\limits_{\Omega} -h(dd^c v)^{p'+s'} \wedge (dd^c u) \wedge \tilde{T}\right)^{\frac{1}{p'+s'}} \\
&\leq \left(\int\limits_{\Omega} -h(dd^c u)^{p'+s'+1} \wedge \tilde{T}\right)^{\frac{p'+s'-1}{p'+s'}} \times \\
&\quad \times \Bigg[\left(\int\limits_{\Omega} -h(dd^c v)^{p'+s'+1} \wedge \tilde{T}\right)^{\frac{p'+s'}{p'+s'+1}} \times \\
&\quad \times \left(\int\limits_{\Omega} -h(dd^c u)^{p'+s'+1} \wedge (dd^c v) \wedge \tilde{T}\right)^{\frac{1}{p'+s'+1}} \Bigg]^{\frac{1}{p'+s'}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int\limits_{\Omega} -h(dd^c u)^{p'+s'} \wedge (dd^c v) \wedge \tilde{T} \\
&\quad \leq \left(\int\limits_{\Omega} -h(dd^c u)^{p'+s'+1} \wedge \tilde{T}\right)^{\left(\frac{p'+s'-1}{p'+s'} + \frac{1}{(p'+s)(p'+s'+1)}\right)} \times \\
&\qquad \times \left(\int\limits_{\Omega} -h(dd^c v)^{p'+s'+1} \wedge \tilde{T}\right)^{\frac{1}{p'+s'+1}}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\int\limits_{\Omega} -h(dd^c u)^{p'+s'} \wedge (dd^c v) \wedge \tilde{T} \\
&\leq \left(\int\limits_{\Omega} -h(dd^c u)^{p'+s'+1} \wedge \tilde{T}\right)^{\frac{p'+s'}{p'+s'+1}} \left(\int\limits_{\Omega} -h(dd^c v)^{p'+s'+1} \wedge \tilde{T}\right)^{\frac{1}{p'+s'+1}}.
\end{aligned}
$$

Using this, we have

$$\int_{\Omega} -h(dd^c v)^{p'+1} \wedge (dd^c u)^{s'} \wedge \tilde{T}$$

$$= \int_{\Omega} -h(dd^c v)^{p'} \wedge (dd^c u)^{s'} \wedge (dd^c v) \wedge \tilde{T}$$

$$\leq \left(\int_{\Omega} -h(dd^c v)^{p'+s'} \wedge (dd^c v) \wedge \tilde{T} \right)^{\frac{p'}{p'+s'}} \left(\int_{\Omega} -h(dd^c u)^{p'+s'} \wedge (dd^c v) \wedge \tilde{T} \right)^{\frac{s'}{p'+s'}}$$

$$= \left(\int_{\Omega} -h(dd^c u)^{p'+s'} \wedge (dd^c v) \wedge \tilde{T} \right)^{\frac{s'}{p'+s'}} \left(\int_{\Omega} -h(dd^c v)^{p'+s'+1} \wedge \tilde{T} \right)^{\frac{p'}{p'+s'}}$$

$$\leq \left[\left(\int_{\Omega} -h(dd^c u)^{p'+s'+1} \wedge \tilde{T} \right)^{\frac{p'+s'}{p'+s'+1}} \left(\int_{\Omega} -h(dd^c v)^{p'+s'+1} \wedge \tilde{T} \right)^{\frac{1}{p'+s'+1}} \right]^{\frac{s'}{p'+s'}} \times$$

$$\times \left(\int_{\Omega} -h(dd^c v)^{p'+s'+1} \wedge \tilde{T} \right)^{\frac{p'}{p'+s'}}$$

$$= \left(\int_{\Omega} -h(dd^c u)^{p'+s'+1} \wedge \tilde{T} \right)^{\frac{s'}{p'+s'+1}} \left(\int_{\Omega} -h(dd^c v)^{p'+s'+1} \wedge \tilde{T} \right)^{\frac{p'+1}{p'+s'+1}}.$$

■

Theorem 5.18. *Suppose $u_1, u_2, \ldots, u_q \in \mathcal{F}^T(\Omega)$ and $h \in \mathcal{E}_0^T(\Omega)$. Then*

$$\int -hdd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$$
$$\leq \left(\int -h(dd^c u_1)^q \wedge T \right)^{\frac{1}{q}} \ldots \left(\int -h(dd^c u_q)^q \wedge T \right)^{\frac{1}{q}}.$$

Proof. Using the definition of $\mathcal{F}^T(\Omega)$ and Proposition 5.15, we see that it is enough to consider the case when $u_1, u_2, \ldots, u_q \in \mathcal{E}_0^T(\Omega)$. Lemma 5.16 implies that

$$\int -hdd^c u_1 \wedge (dd^c u_2)^{q-1} \wedge T \leq \left(\int -h(dd^c u_1)^q \wedge T \right)^{\frac{1}{q}} \left(\int -h(dd^c u_2)^q \wedge T \right)^{\frac{q-1}{q}}.$$

Assume the theorem is true for $u_{s+1} = \ldots = u_q = u$. Suppose that $u_{s+2} = \ldots = u_q = u$. Then

$$\begin{aligned}
&\int -hdd^c u_1 \wedge \ldots \wedge dd^c u_{s+1} \wedge (dd^c u)^{q-s-1} \wedge T \\
&\leq \left(\int -h(dd^c u_{s+1})^{q-s} \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_s \wedge T \right)^{\frac{1}{q-s}} \times \\
&\quad \times \left(\int -h(dd^c u)^{q-s} \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_s \wedge T \right)^{\frac{q-s-1}{q-s}} \\
&\leq \left[\left(\int -h(dd^c u_1)^q \wedge T \right)^{\frac{1}{q}} \ldots \left(\int -h(dd^c u_s)^q \wedge T \right)^{\frac{1}{q}} \times \right. \\
&\quad \left. \times \left(\int -h(dd^c u_{s+1})^q \wedge T \right)^{\frac{q-s}{q}} \right]^{\frac{1}{q-s}} \left[\left(\int -h(dd^c u_1)^q \wedge T \right)^{\frac{1}{q}} \ldots \times \right. \\
&\quad \left. \times \left(\int -h(dd^c u_s)^q \wedge T \right)^{\frac{1}{q}} \left(\int -h(dd^c u)^q \wedge T \right)^{\frac{q-s}{q}} \right]^{\frac{q-s-1}{q-s}} \\
&\leq \left(\int -h(dd^c u_1)^q \wedge T \right)^{\frac{1}{q}} \ldots \left(\int -h(dd^c u_s)^q \wedge T \right)^{\frac{1}{q}} \times \\
&\quad \times \left(\int -h(dd^c u_{s+1})^q \wedge T \right)^{\frac{1}{q}} \left(\int -h(dd^c u)^q \wedge T \right)^{\frac{q-s-1}{q}} .
\end{aligned}$$

The proof is complete. ■

Corollary 5.19. *Suppose* $u_1, u_2, \ldots, u_q \in \mathcal{F}^T(\Omega)$. *Then*

$$\int dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T \leq \left(\int (dd^c u_1)^q \wedge T \right)^{\frac{1}{q}} \ldots \left(\int (dd^c u_q)^q \wedge T \right)^{\frac{1}{q}} .$$

Proof. Tending h to -1 in Theorem 5.17 we are done. ■

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