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Applications of Fixed Point Theorems for Acyclic Maps – A Review

Sehie Park

The National Academy of Sciences, Republic of Korea, Seoul 137–044; and
Department of Mathematical Sciences,
Seoul National University, Seoul 151–747, Korea

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Abstract. We review applications of our fixed point theorems on compact compositions of acyclic maps. Our applications are mainly on acyclic polyhedra, locally convex topological vector spaces, admissible (in the sense of Klee) convex sets, and almost convex or Klee approximable sets in topological vector spaces. Those applications are concerned with general equilibrium problems like as (collective) fixed point theorems, the von Neumann type intersection theorems, the von Neumann type minimax theorems, the Nash type equilibrium theorems, cyclic coincidence theorems, best approximation theorems, (quasi-) variational inequalities, and the Gale-Nikaido-Debreu theorem. Finally, we briefly introduce some related results mainly appeared in other author's works.

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1. Introduction

John von Neumann's 1928 minimax theorem [61] and 1937 intersection lemma [62] have numerous generalizations and applications. Kakutani's 1941 fixed point theorem [19] was to give simple proofs of the above-mentioned results. Nash [27, 28] obtained his 1950 equilibrium theorem based on the Brouwer or Kakutani fixed point theorem. In 1952, Fan [11] and Glicksberg [15] extended the

Kakutani theorem to locally convex topological vector spaces. Their result was applied by them to generalize the von Neumann intersection lemma and the Nash equilibrium theorem. Further generalizations were followed by Ma [26] and others. For the literature, see [41] and references therein.

An upper semicontinuous (u.s.c.) multimap with nonempty compact convex values is called a *Kakutani map*. The Fan-Glicksberg theorem was extended by Himmelberg [18] in 1972 for compact Kakutani maps instead of assuming compactness of domains. The Himmelberg theorem has numerous applications and generalizations; see [48]. Especially, in 1990, Lassonde [23] extended the Himmelberg theorem to multimaps factorizable by Kakutani maps through convex sets in topological vector spaces. Moreover, Lassonde [23] applied the theorem to game theory and obtained a von Neumann type intersection theorem for finite number of sets and a Nash type equilibrium theorem.

On the other hand, in 1946, the Kakutani fixed point theorem was extended for acyclic maps by Eilenberg and Montgomery [10]. This result was applied by Park [40] to give acyclic versions of the social equilibrium existence theorem due to Debreu [8], saddle point theorems, minimax theorems, and the Nash equilibrium theorem. Moreover, Park [38, 43] obtained a fixed point theorem for compact compositions of acyclic maps defined on admissible (in the sense of Klee) convex subsets of topological vector spaces. This new theorem was applied in [43] to deduce acyclic versions of the von Neumann intersection lemma, the minimax theorem, the Nash equilibrium theorem, and others. Further, in [45], Park obtained a cyclic coincidence theorem for acyclic maps, generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, and the von Neumann type minimax theorems. More generalizations of our fixed point theorems and applications followed in [46]-[50]. Consequently, many applications of the Kakutani maps to equilibrium problems are extended to acyclic maps in a number of works of the present author; see the references in the end of this paper.

In this paper, we review applications of our fixed point theorems on compact compositions of acyclic maps on acyclic polyhedra (in Sec. 3), locally convex topological vector spaces (in Sec. 4), admissible (in the sense of Klee) convex sets (in Sec. 5), and almost convex or Klee approximable sets in topological vector spaces (in Sec. 6). Those applications are mainly concerned with general equilibrium problems like as (collective) fixed point theorems, the von Neumann type intersection theorems, the von Neumann type minimax theorems, the Nash type equilibrium theorems, cyclic coincidence theorems, best approximation theorems, (quasi-) variational inequalities, and the Gale-Nikaido-Debreu theorem. Finally, in Sec. 7, we briefly introduce our new fixed point theorem and, in Sec. 8, some related results mainly appeared in other author's works.

2. Preliminaries

Let $\{X_i\}_{i\in I}$ be a family of nonempty sets, and let $i \in I$ be fixed. Let

$$X := \prod_{j\in I} X_j \quad \text{and} \quad X^i := \prod_{j\in I\setminus\{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I \setminus \{i\}$, let x^i_j denote the j-th coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its i-th coordinate is x_i and, for $j \neq i$, the j-th coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x onto X^i.

For $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A(x^i) := \{y_i \in X_i \mid [x^i, y_i] \in A\} \quad \text{and} \quad A(x_i) := \{y^i \in X^i \mid [y^i, x_i] \in A\}.$$

A *multimap* or *map* $F : X \multimap Y$ is a function from a set X into the set 2^Y of nonempty subsets of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \mid x \in A\}$; and for any $B \subset Y$, the (*lower*) *inverse* of B under F is defined by $F^-(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}$.

For topological spaces X and Y, a map $F : X \multimap Y$ is said to be *closed* if its graph

$$\mathrm{Gr}(F) := \{(x, y) \mid y \in F(x),\ x \in X\}$$

is closed in $X \times Y$, and *compact* if $F(X)$ is contained in a compact subset of Y.

A map $F : X \multimap Y$ is said to be *upper semicontinuous* (*u.s.c.*) if, for each closed set $B \subset Y$, $F^-(B)$ is closed in X; *lower semicontinuous* (*l.s.c.*) if, for each open set $B \subset Y$, $F^-(B)$ is open in X; and *continuous* if it is u.s.c. and l.s.c.

If F is u.s.c. with closed values, then F is closed whenever Y is regular. The converse is true whenever Y is compact.

Recall that an extended real-valued function $f : X \to \overline{\mathbf{R}}$ on a topological space is *lower* [resp., *upper*] *semicontinuous* (*l.s.c.*) [resp., *u.s.c.*] if $\{x \in X \mid f(x) > r\}$ [resp. $\{x \in X \mid f(x) < r\}$ is open for each $r \in \mathbf{R}$. If X is a convex set in a vector space, then f is *quasiconcave* [resp., *quasiconvex*] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is convex for each $r \in \mathbf{R}$.

For a subset X of a topological vector space E and $x \in X$, the *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, resp., are defined by Halpern as follows:

$$I_X(x) := \{x + r(u - x) \in E \mid u \in X,\ r > 0\},$$
$$O_X(x) := \{x - r(u - x) \in E \mid u \in X,\ r > 0\}.$$

The closures of $I_X(x)$ and $O_X(x)$ are denoted by $\overline{I_X(x)}$ and $\overline{O_X(x)}$, resp.

All spaces are assumed to be Hausdorff, a t.v.s. means a topological vector space, and a locally convex space means a locally convex t.v.s.

3. For Acyclic Polyhedra

In 1998 [40], an acyclic version of the social equilibrium existence theorem of Debreu [8] is obtained as follows:

A *polyhedron* is a set in $\mathbf{R}^n$ homeomorphic to a union of a finite number of compact convex sets in $\mathbf{R}^n$. The product of two polyhedra is a polyhedron [8].

A nonempty topological space is said to be *acyclic* whenever its reduced homology groups over a field of coefficients vanish. The product of two acyclic spaces is acyclic by the Künneth theorem.

The following is due to Eilenberg and Montgomery [10] or, more generally, to Begle [5]:

Lemma 3.1. *Let Z be an acyclic polyhedron and $T : Z \multimap Z$ an acyclic map (that is, u.s.c. with acyclic values). Then T has a fixed point $\hat{x} \in Z$; that is, $\hat{x} \in T(\hat{x})$.*

In this section, we assume $I = \{1, 2, \ldots, n\}$.

The following is a *collective fixed point theorem* equivalent to Lemma 3.1:

Theorem 3.2. [40] *Let $\{X_i\}_{i\in I}$ be a finite family of acyclic polyhedra, and $T_i : X \multimap X_i$ an acyclic map for each $i \in I$. Then there exists an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for each $i \in I$.*

From this, we have the following acyclic version of the *social equilibrium existence theorem* of Debreu [8]:

Theorem 3.3. [40] *Let $\{X_i\}_{i\in I}$ be a finite family of acyclic polyhedra, $A_i : X^i \multimap X_i$ closed maps, and $f_i, g_i : \mathrm{Gr}(A_i) \to \overline{\mathbf{R}}$ u.s.c. functions for each $i \in I$ such that*

(1) *$g_i(x) \leq f_i(x)$ for all $x \in \mathrm{Gr}(A_i)$;*
(2) *$\varphi_i(x^i) := \max_{y\in A_i(x^i)} g_i[x^i, y]$ is an l.s.c. function of $x^i \in X^i$; and*
(3) *for each $i \in I$ and $x^i \in X^i$, the set*

$$M(x^i) := \{x_i \in A_i(x^i) \mid f_i[x^i, x_i] \geq \varphi_i(x^i)\}$$

is acyclic.

Then there exists an equilibrium point $\hat{a} \in \mathrm{Gr}(A_i)$ for all $i \in I$; that is,

$$\hat{a}_i \in A_i(\hat{a}^i) \quad \textit{and} \quad f_i(\hat{a}) = \max_{a_i \in A(\hat{a}^i)} g_i[\hat{a}^i, a_i] \quad \textit{for all} \quad i \in I.$$

In [40], this is applied to deduce acyclic versions of theorems on saddle points, minimax theorems, and the following *Nash equilibrium theorem*:

Corollary 3.4. [40] *Let $\{X_i\}_{i\in I}$ be a finite family of acyclic polyhedra, $X = \prod_{i=1}^n X_i$, and for each i, $f_i : X \to \overline{\mathbf{R}}$ a continuous function such that*

(0) *for each $x^i \in X^i$ and each $\alpha \in \overline{\mathbf{R}}$, the set*

$$\{x_i \in X_i \mid f_i[x^i, x_i] \geq \alpha\}$$

is empty or acyclic.

Then there exists a point $\hat{a} \in X$ such that

$$f_i(\hat{a}) = \max_{y_i \in X_i} f_i[\hat{a}^i, y_i] \quad \text{for all} \quad i \in I.$$

4. For Locally Convex Spaces

From now on, a topological space is said to be *acyclic* if all of its reduced Čech homology groups over rationals vanish. For nonempty subsets in a t.v.s., convex $\Longrightarrow$ star-shaped $\Longrightarrow$ contractible $\Longrightarrow$ ω-connected $\Longrightarrow$ acyclic $\Longrightarrow$ connected, and not conversely in each step.

For topological spaces X and Y, a map $F : X \multimap Y$ is called a *Kakutani map* whenever Y is a convex subset of a t.v.s. and F is u.s.c. with compact convex values; and an *acyclic map* whenever F is u.s.c. with compact acyclic values.

Let $\mathbb{V}(X,Y)$ be the class of all acyclic maps $F : X \multimap Y$, and $\mathbb{V}_c(X,Y)$ all finite compositions of acyclic maps, where the intermediate spaces are arbitrary topological spaces. Instead of $\mathbb{V}$, $\mathbb{K}$ can be used for Kakutani maps, where the intermediate spaces are Lassonde type convex spaces [22].

The following is given in [32, Theorem 7]:

Theorem 4.1. [32] *Let X and C be nonempty convex subsets of a locally convex space E. Let $F : X \multimap X + C$ be a compact acyclic maps. Suppose that one of the following conditions holds:*

(i) *X is closed and C is compact.*

(ii) *X is compact and C is closed.*

(iii) *$C = \{0\}$.*

Then there is an $\hat{x} \in X$ such that $F\hat{x} \cap (\hat{x} + C) \neq \emptyset$.

Lassonde [22] first obtained Theorem 4.1 for the particular class $\mathbb{K}$. Since then it is extended by the present author to various map classes $\mathbb{V}$, $\mathbb{V}_c$, $\mathfrak{A}_c$, and $\mathfrak{B}$ by step by step; see [34, 38, 48, 57].

The following case (iii) reduces to Himmelberg [18, Theorem 2] when F is a Kakutani map; see also [48]:

Theorem 4.2. [32] *Let X be a nonempty convex subset of a locally convex space E. Then any compact map $F \in \mathbb{V}(X,X)$ has a fixed point.*

This was further generalized for broader classes of multimaps; see [46, 47]. Especially, in 2007 [46], we showed that Theorem 4.2 holds for almost convex subsets instead of convex subsets; see Corollary 6.4.

In 1992, we obtained the following *cyclic coincidence theorem* for acyclic maps, where $\mathbf{Z}_k := \{0, 1, \ldots, k-1\}$ with $(k-1)+1$ interpreted as 0:

Theorem 4.3. [33] *Let $k \geq 1$ and, for each $h \in \mathbf{Z}_k$, let Y_h be a nonempty compact convex subset of a locally convex space E_h, and $V_h \in \mathbb{V}(Y_h, Y_{h+1})$. Then there exists $(y_0, y_1, \ldots, y_{k-1}) \in Y_0 \times Y_1 \times \cdots \times Y_{k-1}$ such that $y_{h+1} \in \mathbb{V}_h y_h$ for all $h \in \mathbf{Z}_k$.*

In [57], Theorem 4.2 is extended to more general $\mathbb{V}_c$ than $\mathbb{V}$ as follows:

Theorem 4.4. [57] *Let X be a nonempty convex subset of a locally convex space E and $T \in \mathbb{V}_c(X, X)$. If T is compact, then T has a fixed point $x_0 \in X$.*

From this we obtained the following *best approximation result* in [57]:

Theorem 4.5. [57] *Let C be a nonempty approximatively compact, convex subset of a locally convex space E, and suppose that $\mathbb{V}_c(C, E)$ is a compact map. Then for each continuous seminorm p on E there exists an $(x_0, y_0) \in \mathrm{Gr}(F)$ such that*

$$p(x_0 - y_0) \leq p(x - y_0) \quad \textit{for all} \;\; x \in \overline{I_C(x_0)}.$$

In 1995 [35], we showed that Theorem 4.2 has an equivalent formulation of a form of quasi-variational inequality; see Theorem 5.2.

For the definition of an *lc* space, see [5, 31]. Note that an ANR (metric) is an *lc* space and a finite union of compact convex subsets of a locally convex space is an *lc* space.

In [35], we had the following theorem on *quasi-variational inequalities* (QVI):

Theorem 4.6. [35] *Let X be a compact acyclic lc space, $f : X \times X \to \mathbf{R}$ a u.s.c. function, and $S : X \to 2^X$ a u.s.c. map with compact values. Suppose that*

(1) *the function M on X defined by*

$$M(x) := \sup_{y \in S(x)} f(x, y) \quad \textit{for} \quad x \in X$$

is l.s.c.; and

(2) *for each $x \in X$, the set*

$$\{y \in S(x) \mid f(x, y) = M(x)\}$$

is acyclic.

Then there exists an $\hat{x} \in Y$ such that

$$\hat{x} \in S(\hat{x}) \quad \textit{and} \quad f(\hat{x}, \hat{x}) = M(\hat{x}).$$

In 1995 [36], we obtained the following *variational inequality* (VI):

Theorem 4.7. [36] *Let X be a compact convex space, Y a topological space, and $T \in \mathbb{V}(X,Y)$. Let $g : X \times Y \to \mathbf{R}$ be a continuous function such that for each $y \in Y$, $x \mapsto g(x,y)$ is quasi-convex on X. Then there exists an $(x_0, y_0) \in \mathrm{Gr}(T)$ such that*

$$g(x_0, y_0) \le g(x, y_0) \quad \textit{for all} \quad x \in X.$$

In the sequel, $W(x)$ denotes either $\overline{I_X(x)}$ or $\overline{O_X(x)}$.

Let $\mathcal{P}$ denote the family of all continuous seminorms on a locally convex space E. As an application of Theorem 4.7, we obtain the following Fan type *fixed point or best approximation theorem* for acyclic maps:

Theorem 4.8. [36] *Let X be a compact convex subset of a locally convex space E and $T \in \mathbb{V}(X,E)$. Then either T has a fixed point or there exist an $(x_0, y_0) \in \mathrm{Gr}(T)$ and a $p \in \mathcal{P}$ such that*

$$0 < p(x_0 - y_0) \le p(x - y_0) \quad \textit{for all} \quad x \in W(x_0).$$

This is applied in [36] to several fixed point theorems for acyclic maps $T \in \mathbb{V}(X,E)$ having various boundary conditions.

A direct consequence of Theorem 4.7 is the following *best approximation theorem*:

Theorem 4.9. [37] *Let E be a metric t.v.s. where the metric d on E has been chosen so that balls are convex, X a compact convex subset of E, and $T \in \mathbb{V}(X,E)$. Then there exists an $(x_0, y_0) \in \mathrm{Gr}(T)$ such that*

$$d(x_0, y_0) \le d(x, y_0) \quad \textit{for all} \quad x \in X.$$

Further, if $T \in \mathbb{V}(X,X)$, then T has a fixed point.

The following particular form of [7, Theorem 2] is a consequence of Theorem 4.2:

Theorem 4.10. [7] *Let X be a convex subset of a locally convex space E, $S : X \multimap X$ a compact closed l.s.c. map (hence, continuous), F a locally convex t.v.s., $T \in \mathbb{V}(X,F)$ a compact map, and C a convex subset of F containing $\overline{T(X)}$, and $\phi : X \times C \times X \to \overline{\mathbf{R}}$ a continuous function. Suppose that for each $(x,y) \in X \times C$,*

(1) $\phi(x,y,x) \le 0$; *and*

(2) $M(x,y) = \{u \in S(x) \mid \phi(x,y,u) = \max_{s \in S(x)} \phi(x,y,s)\}$ *is acyclic.*

Then there exist an $\overline{x} \in S(\overline{x})$ and a $\overline{y} \in T(\overline{x})$ such that

$$\phi(\overline{x}, \overline{y}, x) \le 0 \quad \textit{for all} \quad x \in S(\overline{x}).$$

Further, if $\phi(x,y,x) = 0$ *for all* $(x,y) \in X \times C$, $S(\overline{x})$ *is convex, and* $x \mapsto \phi(\overline{x},\overline{y},x)$ *is concave, then*

$$\phi(\overline{x},\overline{y},x) \le 0 \quad \text{for all} \quad x \in \overline{I}_{S(\overline{x})}(\overline{x}).$$

In 1997 [7], this is applied to obtain known variational or quasi-variational inequalities due to the following authors in the chronological order.

Hartman-Stampacchia, Browder, Lions-Stampacchia, Mosco, Saigal, Chan-Pang, Shih-Tan, Kim, Chang, Parida-Sen, Yang-Chen, Guo-Kung, Yao, Parida-Sahoo-Kumar, Behera-Panda, Siddiqi-Khaliq-Ansari, Ben-El-Mechaiekh and Isac.

5. For Admissible Convex Sets

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

It is well-known that every nonempty convex subset of a locally convex space is admissible. Other examples of admissible t.v.s. are ℓ^p, $L^p(0,1)$, H^p for $0 < p < 1$, and many others; see [39, 41] and references therein.

The following is a particular form of a fixed point theorem due to the author [39]:

Theorem 5.1. *Let* E *be a t.v.s. and* X *an admissible convex subset of* E. *Then any compact map* $T \in \mathbb{V}_c(X,X)$ *has a fixed point.*

In 2000 [56], this is applied to generalize Smol'yakov's saddle point theorem replacing the convexity of the involved sets by the acyclicity, and the continuity of the involved functions by lower or upper semicontinuity.

Theorem 5.1 is the basis of our arguments in 2000 [42], where we had a *quasi-equilibrium theorem* equivalent to Theorem 5.1:

Theorem 5.2. [42] *Let* X *be an admissible convex subset of a t.v.s.* E, $f : X \times X \to \mathbf{R}$ *a u.s.c. function, and* $S : X \multimap X$ *a compact closed map. Suppose that*

(1) *the function* M *defined on* X *by*

$$M(x) := \max_{y \in S(x)} f(x,y) \quad \text{for} \quad x \in X$$

is l.s.c.; and

(2) *for each* $x \in X$, *the set*

$$\{y \in S(x) \mid f(x,y) = M(x)\}$$

is acyclic.

Then there exists an $\hat{x} \in X$ *such that*

$$\hat{x} \in S(\hat{x}) \quad \text{and} \quad f(\hat{x}, \hat{x}) = M(\hat{x}).$$

For a convex subset of a locally convex space, Theorem 5.2 reduces to [35, Theorem 2].

In 2000 [43] and 2002 [45], we applied Theorem 5.1 to a cyclic coincidence theorem for acyclic maps, generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, and the von Neumann minimax theorem.

The following *collective fixed point theorem* is equivalent to Theorem 5.1 for $\mathbb{V}(X, X)$:

Theorem 5.3. [43] *Let* $\{X_i\}_{i=1}^n$ *be a family of convex sets, each in a t.v.s.* E_i, K_i *a nonempty compact subset of* X_i, *and* $T_i : X = \prod_{j=1}^n X_j \multimap K_i$ *an acyclic map for each* i, $1 \leq i \leq n$. *If* X *is admissible in the t.v.s.* $E = \prod_{j=1}^n E_j$, *there exists an* $\hat{x} \in K = \prod_{j=1}^n K_j$ *such that* $\hat{x}_i \in T_i(\hat{x})$ *for each* i.

From this, we have the following *von Neumann type intersection theorem*:

Theorem 5.4. [43] *Let* $\{X_i\}_{i=1}^n$ *be a family of convex sets, each in a t.v.s.* E_i, K_i *a nonempty compact subset of* X_i, *and* A_i *a closed subset of* $X = \prod_{j=1}^n X_j$ *such that* $A_i(x^i)$ *is an acyclic subset of* K_i *for each* $x^i \in X^i$, *where* $1 \leq i \leq n$. *If* X *is admissible in* $E = \prod_{j=1}^n E_j$, *then* $\bigcap_{j=1}^n A_j \neq \emptyset$.

From Theorem 5.1, we have the following *intersection theorem*:

Theorem 5.5. [43] *Let* X *be a compact space and* Y *an admissible compact convex subset of a t.v.s.* E. *Let* A *and* B *be two closed subsets of* $X \times Y$ *such that*

(1) *for each* $x \in X$, $A(x) := \{y \in Y \mid (x, y) \in A\}$ *is acyclic; and*

(2) *for each* $y \in Y$, $B(y) := \{x \in X \mid (x, y) \in B\}$ *is acyclic.*

Then $A \cap B \neq \emptyset$.

From Theorem 5.5, we have the following *von Neumann type minimax theorem*:

Theorem 5.6. [47] *Let* X *be a compact space and* Y *an admissible compact convex subset of a t.v.s., and* $f : X \times Y \to \mathbf{R}$ *a continuous real function. Suppose that for each* $x_0 \in X$ *and* $y_0 \in Y$, *the sets*

$$\{x \in X \mid f(x, y_0) = \max_{\zeta \in X} f(\zeta, y_0)\}$$

and

$$\{y \in Y \mid f(x_0, y) = \min_{\eta \in Y} f(x_0, \eta)\}$$

are acyclic. Then

(1) *f has a saddle point $(x_0, y_0) \in X \times Y$; that is,*

$$\min_{\eta \in Y} f(x_0, y) = f(x_0, y_0) = \max_{\zeta \in X} f(\zeta, y_0).$$

(2) *We have the minimax equality*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Theorem 5.6 includes [43, Theorem 4] and [45, Corollary 6.2] as particular forms. For Euclidean spaces or locally convex spaces, if acyclicity is replaced by convexity, then Theorem 5.6 reduces to the von Neumann minimax theorem [61] or Fan [11, Theorem 3], resp.

The following generalization of the von Neumann minimax theorem [61] is a simple consequence of Theorems 5.5 and 5.6.

Theorem 5.7. [43, 47] *Let X, Y, and f be the same as in Theorem 5.6. Suppose that*

(1) *for every $x \in X$ and $\alpha \in \mathbf{R}$, $\{y \in Y \mid f(x, y) \le \alpha\}$ is acyclic; and*

(2) *for every $y \in Y$ and $\beta \in \mathbf{R}$, $\{x \in X \mid f(x, y) \ge \beta\}$ is acyclic.*

Then we have

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

The following examples are generalized forms of *quasi-equilibrium theorems* or *social equilibrium existence theorems* which directly imply generalizations of *the Nash-Ma type equilibrium existence theorem* [26]. Theorem 5.3 has the following equivalent form of quasi-equilibrium theorem:

Theorem 5.8. [43] *Let $\{X_i\}_{i=1}^n$ be a family of convex sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, $S_i : X \multimap K_i$ a closed map, and $f_i, g_i : X = X^i \times X_i \to \mathbf{R}$ u.s.c. functions for each i.*

Suppose that for each i,

(i) *$g_i(x) \le f_i(x)$ for each $x \in X$;*

(ii) *the function M_i defined on X by*

$$M_i(x) := \max_{y \in S_i(x)} g_i(x^i, y) \quad \textit{for} \quad x \in X$$

is l.s.c.; and

(iii) *for each $x \in X$, the set*

$$\{y \in S_i(x) \mid f_i(x^i, y) \ge M_i(x)\}$$

is acyclic.

If X is admissible in $E = \prod_{j=1}^n E_j$, then there exists an $\hat{x} \in K$ such that for each i,

$$\hat{x}_i \in S_i(\hat{x}) \quad \text{and} \quad f_i(\hat{x}^i, \hat{x}_i) \geq g_i(\hat{x}^i, y) \quad \text{for all} \quad y \in S_i(\hat{x}).$$

From Theorem 5.8, we have the following generalization of *the Nash equilibrium theorem*:

Theorem 5.9. [43] *Let $\{X_i\}_{i=1}^n$ be a family of nonempty compact convex subsets, each in a t.v.s. E_i and for each i, let $f_i : X \to \mathbf{R}$ be a continuous function such that*

(0) *for each $x^i \in X^i$ and each $\alpha \in \mathbf{R}$, the set*

$$\{x_i \in X_i \mid f_i(x^i, x_i) \geq \alpha\}$$

is empty or acyclic.

If X is admissible in $E = \prod_{j=1}^n E_j$, there exists a point $\hat{x} \in X$ such that

$$f_i(\hat{x}) = \max_{y_i \in X_i} f_i(\hat{x}^i, y_i) \quad \text{for all} \quad i,\ 1 \leq i \leq n.$$

In 2001 [44], Theorem 5.1 for $\mathbb{V}$ is applied to an existence theorem for a *quasi-equilibrium problem* (simply, QEP) (in the sense of Noor and Oettli [30]) and the following *symmetric quasi-equilibrium theorem*:

Theorem 5.10. [44] *Let X and Y be admissible convex subsets of t.v.s. E and F, resp., $S : X \times Y \multimap X$ and $T : X \times Y \multimap Y$ compact acyclic maps, and $f, g : X \times Y \to \overline{\mathbf{R}}$ l.s.c. functions such that*

(i) *the functions*

$$F(x,y) := \min\{f(\xi, y) \mid \xi \in S(x,y)\},$$
$$G(x,y) := \min\{g(x, \eta) \mid \eta \in T(x,y)\}$$

are u.s.c. on $X \times Y$; and

(ii) *for each $(x,y) \in X \times Y$, the sets*

$$A(x,y) := \{\xi \in S(x,y) \mid f(\xi, y) = F(x,y)\},$$
$$B(x,y) := \{\eta \in T(x,y) \mid g(x, \eta) = G(x,y)\}$$

are acyclic.

Then there exists an $(\overline{x}, \overline{y}) \in X \times Y$ such that

$$\overline{x} \in S(\overline{x}, \overline{y}),\ f(x, \overline{y}) \geq f(\overline{x}, \overline{y}) \text{ for all } x \in S(\overline{x}, \overline{y}),$$
$$\overline{y} \in T(\overline{x}, \overline{y}),\ g(\overline{x}, y) \geq g(\overline{x}, \overline{y}) \text{ for all } y \in T(\overline{x}, \overline{y}).$$

Some other applications of Theorem 5.1 for $\mathbb{V}$ to generalized QEPs were given in Lin and Park [25].

In 2003 [6], the following existence theorem for solutions of QEPs with respect to a multimap $\Phi : X \times X \multimap \overline{\mathbf{R}}$ was given:

Theorem 5.11. [6] *Let X be an admissible convex subset of a t.v.s. E, $S : X \multimap X$ a compact closed map, and $\Phi : X \times X \multimap \overline{\mathbf{R}}$ a u.s.c. function with compact values. Suppose that*

(1) *the function m defined on X by*

$$m(x) := \sup_{u \in S(x)} \Phi(x, u) \quad \text{for all} \quad x \in X$$

is l.s.c.; and

(2) *for each $x \in X$, the set*

$$M(x) = \{y \in S(x) \mid m(x) \in \Phi(x, y)\}$$

is acyclic.

Then there exists an $\hat{x} \in X$ such that $\hat{x} \in S(\hat{x})$ and $m(\hat{x}) \in \Phi(\hat{x}, \hat{x})$.

Further, if $\sup \Phi(x, x) \leq 0$ *for all $x \in X$, then there exists an $\hat{x} \in X$ such that*

$$\hat{x} \in S(\hat{x}) \quad \text{and} \quad \Phi(\hat{x}, y) \leq 0 \quad \text{for all} \quad y \in S(\hat{x}).$$

Several applications on QEPs follow in [6].

In 2009 [50], from Theorem 5.1, we deduced the following generalization of Gwinner's extension [17] of the *Walras theorem*:

Theorem 5.12. [50] *Let K and L be compact convex subsets of t.v.s. E and F, resp., such that K is admissible. Let $c \in \mathbf{R}$, $f : K \times L \to \mathbf{R}$ a continuous function, and $T : K \multimap L$ a multimap. Suppose*

(1) *for each $y \in L$, $f(\cdot, y)$ is quasiconvex;*

(2) *T is an acyclic map; and*

(3) *(Walras law) for each $x \in K$ and $y \in T(x)$, we have $f(x, y) \geq c$.*

Then there exists a Walras equilibrium, that is, there exist $\overline{x} \in K$, $\overline{y} \in L$ such that

$$\overline{y} \in T(\overline{x}) \quad \text{and} \quad f(x, \overline{y}) \geq c \quad \text{for all} \quad x \in K.$$

Since every convex subset of a locally convex space is admissible, Theorem 5.12 is also valid when E is a locally convex space.

To specialize Theorem 5.12 towards the *Gale-Nikaido-Debreu theorem*, as in [17], we boil down the function f to a bilinear form $\langle \cdot, \cdot \rangle$ for a dual system $(E, F, \langle \cdot, \cdot \rangle)$ of t.v.s. E and F.

For a convex cone P of E, the *dual cone* is defined by

$$P^+ := \{y \in F \mid \langle p, y\rangle \geq 0,\ p \in P\}.$$

Theorem 5.13. [50] *Let $(E, F, \langle\cdot,\cdot\rangle)$ be a dual system of t.v.s. E and F such that the bilinear form $\langle\cdot,\cdot\rangle$ is continuous on compact subsets of $E \times F$. Let K and L be compact convex subsets of E and F, resp., such that K is admissible; and P the convex cone $\bigcup\{rK \mid r \geq 0\}$. Let $T : K \multimap L$ be an acyclic map such that $\langle x, y\rangle \geq 0$ for all $x \in K$ and $y \in T(x)$. Then there exists $\overline{x} \in K$ such that $T(\overline{x}) \cap P^+ \neq \emptyset$.*

Theorem 5.13 is also valid if E is a locally convex space instead of the admissibility of K, and for a Kakutani map T instead of an acyclic map.

With the choice $P := \{x \in \mathbf{R}^n \mid x_i \geq 0;\ i = 1, 2, \ldots, n\}$ and $K = L := \{x \in P \mid x_1 + x_2 + \cdots + x_n = 1\}$ (the standard simplex), the Gale-Nikaido-Debreu theorem ([14, Principal Lemma], [29, Theorem 16.6], [9, 5.6(1)]) can be immediately obtained.

6. For Almost Convex or Klee Approximable Subsets

A subset X of a t.v.s. E is said to be *almost convex* [18] if for any neighborhood V of the origin 0 of E and for any finite subset $A := \{x_1, x_2, \ldots, x_n\}$ of X, there exists a subset $B := \{y_1, y_2, \ldots, y_n\}$ of X such that $y_i - x_i \in V$ for each $i = 1, 2, \ldots, n$ and $\operatorname{co} B \subset X$.

Recently it is known that any u.s.c. map with compact values having *trivial shape* (that is, contractible in each neighborhood) belongs to our well-known 'better' admissible class $\mathfrak{B}(X, Y)$; see [46].

A *polytope* P in a subset X of a t.v.s. E is a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

A nonempty subset K of a t.v.s. E is said to be *Klee approximable* if for every neighborhood V of the origin 0 of E, there exists a continuous function $h : K \to E$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope of E. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K)$ is contained in a polytope in X.

Examples of Klee approximable sets can be seen in [46]. Here we give an example:

Lemma 6.1. [46] *Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Then every compact subset K of Y is Klee approximable into X.*

In this section, we begin with applications of the following particular case of [46, Theorem 3.4]:

Theorem 6.2. [46] *Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Let $G : Y \multimap Y$ be a compact closed map such that $G(x)$ is acyclic (resp., has trivial shape) for all $x \in X$. Then G has a fixed point.*

Corollary 6.3. [46] *Let X be an almost convex dense subset of a closed subset Y of a locally convex t.v.s. E. Let $G : Y \multimap Y$ be a compact u.s.c. multimap with closed values such that $G(x)$ is acyclic for all $x \in X$. Then G has a fixed point.*

Corollary 6.4. [46] *Let X be an almost convex admissible subset of a t.v.s. E. Then any compact closed map $G : X \multimap X$ such that $G(x)$ is acyclic* (*resp., has trivial shape*) *for all $x \in X$ has a fixed point.*

When X is a subset of a locally convex space and $G(x)$ is convex, Corollary 6.4 appeared in Park and Tan [58, 59].

If X is a convex subset of a locally convex space E, then Corollary 6.4 reduces to Theorem 4.2. Further if G has convex values, then Corollary 6.4 reduces to Himmelberg [18, Theorem 2], which has numerous applications.

As in Himmelberg [18], we conclude from Theorem 6.2 the following generalization of *the Fan intersection theorem* [11]:

Theorem 6.5. [46] *Let $\{E_i\}_{i=1}^n$ be a family of t.v.s. For each i, let X_i be an almost convex dense subset of an admissible subset Y_i of E_i, and K_i a compact subset of Y_i. Let $\{A_i\}_{i=1}^n$ be a family of closed subsets of $Y := \prod_{i=1}^n Y_i$ such that, for each i, the section $A_i(y^i)$ is acyclic* (*resp., has trivial shape*) *for all $y^i \in X^i := \prod_{j\neq i} X_j$ and nonempty for all $y^i \in Y^i := \prod_{j\neq i} Y_j$. Then $\bigcap_{i=1}^n A_i \neq \emptyset$.*

If each $A_i(y^i)$ is assumed to be convex and each E_i is locally convex, then Theorem 6.5 holds for a family not-necessarily finite as in Himmelberg [18, Theorem 3].

We deduce from Theorem 6.5 the following:

Theorem 6.6. [46] *Let Y_1, Y_2 be compact admissible subsets of t.v.s. E_1, E_2, resp., and X_1, X_2 be almost convex dense subsets of Y_1, Y_2, resp. Let f be a continuous real function defined on $Y_1 \times Y_2$ such that for any $x_1 \in X_1,\ y_2 \in X_2$, the sets*

$$\{x \in Y_1 \mid f(x, y_2) = \max_{\xi \in Y_1} f(\xi, y_2)\}$$

and

$$\{y \in Y_2 \mid f(x_1, y) = \min_{\eta \in Y_2} f(x_1, \eta)\}$$

are acyclic (*resp., has trivial shape*), *then*

$$\max_{x\in Y_1} \min_{y\in Y_2} f(x, y) = \min_{y\in Y_2} \max_{x\in Y_1} f(x, y).$$

When $Y_1,\ Y_2$ are subsets of locally convex t.v.s. E_1, E_2, resp., and if we assume the convexity instead of the acyclicity, Theorem 6.6 reduces to [18, Theorem 4].

In the continuation [47] of [46], we deduce some *collective fixed point theorems* for families of maps and, then, various von Neumann type intersection theorems.

Theorem 6.7. [47] *Let $\{E_i\}_{i=1}^n$ be a family of t.v.s. For each i, let X_i be a subset of E_i, $X := \prod_{i=1}^n X_i$, K_i a nonempty compact subset of X_i, and $F_i : X \multimap K_i$ a closed map with acyclic values (resp., values of trivial shape). If $K := \prod_{i=1}^n K_i$ is Klee approximable into X, then there exists an $\overline{x} = (\overline{x_i})_{i=1}^n \in X$ such that $\overline{x_i} \in F_i(\overline{x})$ for each i.*

From Theorem 6.7, we obtain the following *von Neumann type intersection theorem*:

Theorem 6.8. [47] *Let $\{X_i\}_{i=1}^n$ be a family of sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and A_i a closed subset of X such that $A_i(x^i)$ is an acyclic subset of K_i for each $x^i \in X^i$, where $1 \leq i \leq n$. If X is an almost convex admissible subset of E, then $\bigcap_{j=1}^n A_j \neq \emptyset$.*

Similarly, we can obtain a more general result than Theorem 6.8 as follows:

Theorem 6.9. [47] *Let I be any index set, $\{X_i\}_{i\in I}$ a family of sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and A_i a closed subset of X for each $i \in I$. Suppose that for each $x^i \in X^i$, $A_i(x^i)$ is a convex subset of K_i except a finite number of i's for which $A_i(x^i)$ is an acyclic subset of K_i. If X is an almost convex admissible subset of E, then $\bigcap_{j\in I} A_j \neq \emptyset$.*

If $I = \{1, 2\}$, E_i are Euclidean, $X_i = K_i$, and $A_i(x^i)$ are nonempty and convex, then Theorem 6.8 or 6.9 reduces to the intersection lemma of von Neumann [62].

We have another intersection theorem:

Theorem 6.10. [47] *Let X_0 be a topological space and $\{X_i\}_{i=1}^n$ a family of sets, each in a t.v.s. E_i. For each $i = 0, 1, 2, \ldots, n$, let K_i be a nonempty subset of X_i which is compact except possibly K_n, and $F_i \in \mathbb{V}_c(X^i, X_i)$. If K^0 is Klee approximable into X^0, then $\bigcap_{i=0}^n \mathrm{Gr}(F_i) \neq \emptyset$.*

In case when each X_i is convex for $i \geq 1$ and X^0 is admissible in E^0, Theorem 6.10 reduces to [45, Theorem 4]. Particular forms of Theorem 6.10 were given by von Neumann, Fan, Lassonde, Chang, and Park; see [45]. The following is one of them:

Corollary 6.11. [47] *Let X be a topological space, Y a subset in a t.v.s. E, and $F \in \mathbb{V}_c(X, Y)$ and $G \in \mathbb{V}_c(Y, X)$. If F is compact and $F(X)$ is Klee approximable into Y, then $\mathrm{Gr}(F) \cap \mathrm{Gr}(G) \neq \emptyset$.*

From Corollary 6.11, we have the following:

Corollary 6.12. [47] *Let X be a topological space and Y a compact subset of a t.v.s. E. Let A and B be two closed subsets of $X \times Y$ such that*

(1) *for each $x \in X$, $A(x) := \{y \in Y \mid (x, y) \in A\}$ is acyclic; and*

(2) *for each $y \in Y$, $B(y) := \{x \in X \mid (x, y) \in B\}$ is acyclic.*

If $A(X) := \bigcup\{A(x) \mid x \in X\}$ is Klee approximable into Y, then $A \cap B \neq \emptyset$.

If Y is an admissible, compact, and almost convex subset of E, then $A(X)$ is Klee approximable into Y. Especially, for the particular case when X is compact and Y is convex, Corollary 6.12 and some consequences were obtained in [43].

From Theorem 6.10, we deduce the following generalized form of *the quasi-equilibrium theorem or the social equilibrium existence theorem* in the sense of Debreu [8]:

Theorem 6.13. [47] *Let X_0 be a topological space, and $\{X_i\}_{i=1}^n$ a family of sets, each in a t.v.s. E_i. For $i = 0, 1, \ldots, n$, let K_i be a nonempty subset of X_i which is compact except possibly K_n, $S_i : X^i \multimap K_i$ a closed map with compact values, and $f_i, g_i : X = X^i \times X_i \to \mathbf{R}$ u.s.c. real functions.*

Suppose that for each $i = 0, 1, \ldots, n$,

(i) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*

(ii) *the real function $M_i : X^i \to \mathbf{R}$ defined by*

$$M_i(x^i) := \max_{y_i \in S_i(x^i)} g_i[x^i, y_i] \quad for \quad x^i \in X^i$$

is l.s.c.; and

(iii) *for each $x^i \in X^i$, the set*

$$\{y_i \in S_i(x^i) \mid f_i[x^i, y_i] \geq M_i(x^i)\}$$

is acyclic.

If K^0 is Klee approximable into X^0 and if S_n is u.s.c., then there exists an equilibrium point $\hat{x} \in X$; that is,

$$\hat{x}_i \in S_i(\hat{x}^i) \quad and \quad f_i(\hat{x}^i, \hat{x}_i) \geq \max_{y_i \in S_i(x^i)} g_i[x^i, y_i] \quad for\ each \ \ i \in \mathbb{Z}_{n+1}.$$

If X^0 is admissible, Theorem 6.13 reduces to [45, Theorem 5].

Moreover, from Theorem 6.13, we have the following particular form:

Theorem 6.14. [47] *Let X_0 be a topological space, and $\{X_i\}_{i=1}^n$ a family of sets, each in a t.v.s. E_i. For $i = 0, 1, \ldots, n$, let K_i be a nonempty subset of X_i which is compact except possibly K_n, $S_i : X^i \multimap K_i$ a continuous multimap with compact values, and $f_i : X = X^i \times X_i \to \mathbf{R}$ a continuous real function.*

Suppose that for each $i = 0, 1, \ldots, n$, the following holds:

(0) *for each $x^i \in X^i$ and each $\alpha \in \mathbf{R}$, the set*

$$\{x_i \in S_i(x^i) \mid f_i[x^i, x_i] \geq \alpha\}$$

is empty or acyclic.

If K^0 is Klee approximable into X^0, there exists an equilibrium point $\hat{x} \in X$.

If each X_i is convex and if X is admissible in E, then Theorem 6.14 reduces to [45, Theorem 6]. For other particular forms of Theorems 6.13 and 6.14, see [43, 45].

The following generalizes the Nash theorem:

Corollary 6.15. [47] *Let X_0 be a compact topological space, and $\{X_i\}_{i=1}^n$ a family of convex sets, each in a t.v.s. E_i, such that each X_i is compact except X_n. For $i = 0, 1, \dots, n$, let $f_i : X = X^i \times X_i \to \mathbf{R}$ be a continuous real function such that*

(1) *for each $x^i \in X^i$ and each $\alpha \in \mathbf{R}$, the set*

$$\{x_i \in X_i \mid f_i[x^i, x_i] \geq \alpha\}$$

is empty or acyclic.

If X^0 is admissible, then there exists an equilibrium point $\hat{x} \in X$; that is,

$$f_i(\hat{x}) = \max_{y_i \in X_i} f_i[\hat{x}^i, y_i] \quad \textit{for all} \quad i \in \mathbf{Z}_{n+1}.$$

This slightly extends [43, Theorem 7].

If all X_i are compact convex subsets of Euclidean spaces and if $x_i \mapsto f_i[x^i, x_i]$ is quasiconcave for each $x^i \in X^i$, then Corollary 6.15 reduces to Nash [28, Theorem].

It is noted that our fixed point theorems can be applied to various types of QVIs on *almost convex sets*. For example, from Theorem 6.2, we have the following:

Theorem 6.16. [47] *Let X be an almost convex admissible subset of a t.v.s. E, $S : X \multimap X$ a compact map, F a t.v.s., $T \in \mathbb{V}(X, F)$ a compact map, and C an almost convex admissible subset of F containing $\overline{T(X)}$, and $\phi : X \times C \times X \to \overline{\mathbf{R}}$ a u.s.c. function. Suppose that*

(1) *$\phi(x, y, x) \leq 0$ for all $(x, y) \in X \times C$; and*

(2) *the map $M : X \times C \to X$ defined by*

$$M(x, y) := \{u \in S(x) \mid \phi(x, y, u) = \max_{s \in S(x)} \phi(x, y, s)\} \quad \textit{for } (x, y) \in X \times C$$

belongs to $\mathbb{V}(X \times C, X)$.

Then there exist an $\overline{x} \in S(\overline{x})$ and a $\overline{y} \in T(\overline{x})$ such that

$$\phi(\overline{x}, \overline{y}, x) \leq 0 \quad \textit{for all} \quad x \in S(\overline{x}).$$

Further, if $\phi(x, y, x) = 0$ for all $(x, y) \in X \times C$, $S(\overline{x})$ is convex, and $x \mapsto \phi(\overline{x}, \overline{y}, x)$ is concave and l.s.c., then

$$\phi(\overline{x}, \overline{y}, x) \leq 0 \quad \textit{for all} \quad x \in \overline{I_{S(\overline{x})}(\overline{x})}.$$

This implies the following:

Theorem 6.17. [47] *Let X, S, F, T and C be the same as in Theorem 6.16. Suppose that $S \in \mathbb{V}(X, X)$ is l.s.c. (hence continuous). Let $\phi : X \times C \times X \to \overline{\mathbf{R}}$ be continuous such that, for each $(x, y) \in X \times C$,*

(1) $\phi(x, y, x) \leq 0$; *and*

(2) $M(x, y) := \{u \in S(x) \,|\, \phi(x, y, u) = \max_{s \in S(x)} \phi(x, y, s)\}$ *is acyclic.*

Then there exist an $\overline{x} \in S(\overline{x})$ and a $\overline{y} \in T(\overline{x})$ such that

$$\phi(\overline{x}, \overline{y}, x) \leq 0 \quad for\ all \quad x \in S(\overline{x}).$$

Further, if $\phi(x, y, x) = 0$ for all $(x, y) \in X \times C$, $S(\overline{x})$ is convex, and $x \mapsto \phi(\overline{x}, \overline{y}, x)$ is concave, then

$$\phi(\overline{x}, \overline{y}, x) \leq 0 \quad for\ all \quad x \in \overline{I_{S(\overline{x})}(\overline{x})}.$$

When X is convex, Theorems 6.16 and 6.17 were given in [44].

In our previous work [7], particular forms of Theorems 6.16 and 6.17 are applied to more than fifteen known variational or quasi-variational inequalities due to Hartman-Stampacchia, Browder, Lions-Stampacchia, Mosco, Saigal, and many others. Now all of them can be stated for almost convex sets instead of convex sets.

Moreover, when X is a convex subset of a locally convex space E and $F := E^*$, its topological dual, Theorem 6.17 reduces to the main result of Kum [21], which improves earlier works of Kim, Shih and Tan.

7. A General Fixed Point Theorem

In this paper, we are mainly concerned with acyclic maps on t.v.s. This class of maps can be extended to a large class of maps on abstract convex spaces without having any linear structure; see our previous works [34, 38, 39, 41], [46]-[49], [51]. In this section, we introduce a recent result which subsumes most of fixed point theorems in this paper. For details, see [51] and references therein.

In order to improve the KKM theory, in 2006-09, we proposed new concepts of abstract convex spaces and new fixed point theory on them. For details, see [51] and references therein.

Definition 7.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ*-convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a *Γ-convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 7.2. An *abstract convex uniform space* $(E, D; \Gamma; \mathcal{U})$ is the one with a basis $\mathcal{U}$ of a uniform structure of E.

Definition 7.3. Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space. A subset K of E is said to be *Klee approximable* if, for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to E$ satisfying conditions

(1) $(x, h(x)) \in U$ for all $x \in K$;

(2) $h(K) \subset \Gamma_N$ for some $N \in \langle D \rangle$; and

(3) there exist continuous functions $p : K \to \Delta_n$ and $\phi_N : \Delta_n \to \Gamma_N$ with $|N| = n + 1$ such that $h = \phi_N \circ p$.

Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K) \subset \Gamma_N \subset X$ for some $N \in \langle D \rangle$ in condition (2).

A subset X of E is *admissible* (in the sense of Klee) if and only if every compact subset K of X is Klee approximable into E.

Definition 7.4. Let $(E, D; \Gamma)$ be an abstract convex space, X a nonempty subset of E, and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of maps from X into Y as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, and for any continuous function $p : F(\Gamma_N) \to \Delta_n$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n) \subset X$.

This definition works for G-convex spaces or ϕ_A-spaces and there are lots of examples including acyclic maps.

We have the following main result in [51]:

Theorem 7.5. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space, $X \subset Y$ subsets of E, and $F : Y \multimap Y$ a map such that $F|_X \in \mathfrak{B}(X, Y)$ and $F(X)$ is Klee approximable into X. Then F has the almost fixed point property (that is, for any $V \in \mathcal{U}$, $F|_X$ has a V-fixed point $x_V \in X$ satisfying $F(x_V) \cap (x_V + V) \neq \emptyset$)*

Further if $(E, \mathcal{U})$ is Hausdorff, F is closed, and $\overline{F(X)}$ is compact in Y, then F has a fixed point $x_0 \in Y$ (that is, $x_0 \in F(x_0)$).

In [34, 38, 39, 41], [46]-[49], [51] and others, we gave some of our previous results which are direct consequences of Theorem 7.5. Note that Theorem 7.5 subsumes nearly one hundred particular cases appeared in the history of generalizations of the Brouwer fixed point theorem.

8. Other Related Results

In this supplementary section, we briefly introduce some related results mainly appeared in other author's works.

1) For early developments of the KKM theory and the analytical fixed point theory, see Granas [16] and Park [51].

2) In 1994, Park et al. [52] obtained a general Fan type geometric property of convex sets and applied it to existence of maximizable quasiconcave functions, new minimax inequalities, and fixed point theorems for upper hemicontinuous multimaps. These results generalize works of Fan, Himmelberg, Ha, Jiang, and many others.

3) In 1998, Wu and Li [64] applied Lemma 2.1 [52] to a well-known Lefshetz type fixed point theorem for compositions of acyclic maps. In the same year, Wu and Xu [65] applied Theorem 2.2 [52] to a general geometric property.

4) In 2002, Lin and Cheng [24] applied Park's 1992 fixed point theorem (Theorem 4.2) to the existence results of two types of equilibrium problems - the constrained or the competitive Nash type equilibrium problems with multivalued payoff functions. In these two problems, the authors found a strategy combination such that each player wishes to find a minimal lose from his multivalued payoff function.

5) In 2003, motivated by Park's works, Balaj [3] obtained a fixed point theorem for the composition of an acyclic map defined on a generalized convex space with a Φ-map, and a matching theorem which can be restated as a KKM theorem involving acyclic maps. We note that most of the results have already known variants for multimaps in the more general class than that of acyclic maps.

6) Summary of Balaj [4] in 2005: "Applying a KKM-type theorem due to S. Park and H. Kim, we obtain an acyclic version of a minimax inequality established by Fan [12]. The result is applied to formulate acyclic versions of other minimax results and a theorem of Fan concerning compatibility of some systems of inequalities."

Note that, in MR2199111 (2006j:49019), the paper of Park and Kim is misquoted as Y. S. Park and S. H. Rim [*J. Korean Math. Soc.* **31** (3) (1994), 439–456; MR1297429 (95i:16030)] by the editor.

7) Amini et al. [1] claimed that Fakhar and Zafarani had shown in 2005 that those multifunctions defined on G-convex spaces which are closed, compact and acyclic-valued have the KKM property. However, this was already known by Park and Kim in 1997.

8) In 2005, Farajzadeh [13] applied Theorem 5.10 to the symmetric vector quasi-equilibrium problems (SVQEP) in a t.v.s. A coincidence theorem and a solution of vector optimization problem for a pair of vector-valued maps are obtained.

9) The following is a particular case of [54, Theorem 4]:

Theorem 8.1. *Let X be a nonempty compact admissible subset of a hyperconvex metric space (H, d) and $F \in \mathbb{V}(X, X)$. Then F has a fixed point.*

In 2008, Amini-Harandi et al. [2] defined a best proximity pair for a map $F : A \multimap B$ with respect to a map $g : A \to B$ for subsets A, B of metric spaces and, by applying the above theorem, proved new existence theorems on best proximity pairs for upper semicontinuous multimaps with respect to a homeomorphism in hyperconvex metric spaces.

10) Abstract of Kim et al. [20] in 2008: "In this paper, using Lassonde's fixed point theorem for Kakutani factorizable multifunctions and Park's fixed point theorem for acyclic factorizable multifunctions, we prove new existence theorems for general proximity pairs and equilibrium pairs for free abstract economies, which generalize the previous best proximity theorems due to Srinivasan and Veeramani, and Kim and Lee in several aspects."

Here, Park's theorem is Theorem 4.4.

11) In 2009, Sach and Tuan [60] applied Theorem 4.2 to generalizations of vector quasivariational inclusion problems with set-valued maps.

12) In 2009, Wang and Fu [63] obtained some existence theorems for solutions of the Stampacchia type of generalized vector quasi-equilibrium problem (in short, GVQEP) on set-valued mapping without any monotonicity assumption. They are based on a continuous selection theorem and a particular form of Theorem 4.2 for a compact convex subset X.

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