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Oscillation for a Nonlinear Difference Equation

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Abstract. In this paper, some oscillatory results for nonlinear difference equations

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}), \quad n = 0, 1, \dots$$

where $m_i, \forall i = 1, \dots, r$ are fixed positive integers, $\{\lambda_n\}_n$ is a sequence of positive real numbers and the function F is defined on the set of the real numbers are obtained.

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1. Introduction

Recently there has been a considerable interest in the oscillation of solutions of difference equations of the form

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}), \quad n = 0, 1, 2, \dots \tag{1}$$

where $r, m_1, m_2, \dots, m_r$ are fixed positive integers, the functions $\alpha_i(n)$ are defined on the set of positive integers, and the function F is defined on the set of the real numbers, (see for example the work in [1, 2, 3] and the references cited therein). In [1, 2], the authors investigated the oscillation of (1) in case $\lambda_n = 1$ or $\lambda_n = \lambda \geqslant 1$ and $\alpha_i(n) \geqslant 0, \forall i = 1, 2, \dots, r$, $\forall n = 0, 1, \dots$. In [3], the authors studied the oscillation of difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{r} \alpha_i(n) x_{n-m_i} = 0, \quad n = 0, 1, 2, \ldots$$

in case the positivity of $\alpha_i(n)$ is not required.

Motivated by the above work, in the present paper, we aim to study the oscillation of (1) where $\{\lambda_n\}_n$ is an arbitrary sequence of positive real numbers and the positivity of $\alpha_i(n)$ is not required.

2. The Results

In this section, by $\mathbb{N}$ we denote the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and by $\mathbb{R}$ the set of the real numbers. For all $\ell \in \mathbb{N}_0$ we use $\mathbb{N}_\ell = \{n \in \mathbb{N} : n \geqslant \ell\}$.

Theorem 2.1. *Let F be a nonincreasing function, $xF(x) < 0,\ x \neq 0, -F(x) \geqslant x,\ F(ax) = aF(x),\ \forall a, x \in \mathbb{R}$, and*

$$\liminf_{n\to\infty} \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} > 0,\ \limsup_{n\to\infty} \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} > 1 - \liminf_{n\to\infty} \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}},$$

where $\alpha_i(n) \geqslant 0, n \in \mathbb{N}_0, i = 1, \ldots, r$. Then, (1) is oscillatory.

Proof. Consider the following inequalities

$$\frac{x_{n+1}}{\lambda_n^{n+1}} - \frac{x_n}{\lambda_n^n} - \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} F\left(\frac{x_{n-m_i}}{\lambda_n^{n-m_i}}\right) \leqslant 0, \quad n \in \mathbb{N}_0 \tag{2}$$

and

$$\frac{v_{n+1}}{\lambda_n^{n+1}} - \frac{v_n}{\lambda_n^n} - \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} F\left(\frac{v_{n-m_i}}{\lambda_n^{n-m_i}}\right) \geqslant 0, \quad n \in \mathbb{N}_0. \tag{3}$$

We will prove that (2) has no eventually positive solution and (3) has no eventually negative solution. Indeed, let $\{x_n\}_n$ be a solution of (2), $x_n > 0$ for $n \in \mathbb{N}_{n_1}, n_1 \in \mathbb{N}_0$. By the hypothesis

$$\liminf_{n\to\infty} \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} = \beta > 0,$$

so there exists $0 < \epsilon < \beta$ and $n_2 \in \mathbb{N}, n_2 \geqslant n_1$ such that

$$\sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} \geqslant \beta - \epsilon > 0$$

for all $n \in \mathbb{N}_{n_2}$. Put

$$n_3 = \max\{n_1 + m^*, n_2\}, \quad m^* = \max_{1 \leqslant i \leqslant r} m_i.$$

We have $\left\{\frac{x_n}{\lambda_n^n}\right\}_n$ being eventually nonincreasing for all $n \in \mathbb{N}_{n_3}$. Since F is a nonincreasing function on $\mathbb{R}$, we get

$$\frac{x_n}{\lambda_n^n} \geqslant -\sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} F\left(\frac{x_{n-m_i}}{\lambda_n^{n-m_i}}\right) \geqslant -\sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} F\left(\frac{x_{n-1}}{\lambda_n^{n-1}}\right) \geqslant (\beta-\epsilon)\frac{x_{n-1}}{\lambda_n^{n-1}},$$

for all $n \in \mathbb{N}_{n_3}$. On the other hand,

$$\begin{aligned}
0 &\geqslant \frac{x_{n+1}}{\lambda_n^{n+1}} - \frac{x_n}{\lambda_n^n} - \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} F\left(\frac{x_{n-m_i}}{\lambda_n^{n-m_i}}\right) \\
&\geqslant \frac{x_{n+1}}{\lambda_n^{n+1}} - \frac{x_n}{\lambda_n^n} - \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} F\left(\frac{x_n}{\lambda_n^n}\right) \\
&\geqslant (\beta-\epsilon)\frac{x_n}{\lambda_n^n} - \frac{x_n}{\lambda_n^n} + \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} \frac{x_n}{\lambda_n^n}.
\end{aligned}$$

Hence

$$\frac{x_n}{\lambda_n^n}\left[\beta - \epsilon - 1 + \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}}\right] \leqslant 0.$$

It implies

$$\sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} \leqslant 1 - \beta + \epsilon$$

for all $n \in \mathbb{N}_{n_3}$. Therefore,

$$\limsup_{n\to\infty} \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} \leqslant 1 - \beta + \epsilon.$$

Since $\epsilon > 0$ is arbitrary we have

$$\limsup_{n\to\infty} \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} \leqslant 1 - \beta = 1 - \liminf_{n\to\infty} \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}},$$

which contradicts the hypothesis. Hence, (2) has no eventually positive solution.

Next, we assume that $\{v_n\}_n$ is a solution of (3) such that $v_n < 0$ for all $n \in \mathbb{N}_{n_1}, n_1 \in \mathbb{N}_0$. Putting $x_n = -v_n,\ n \in \mathbb{N}_0$, we obtain a contradiction. The proof is complete. ■

Theorem 2.2. *Let F be a nonincreasing and continuous function; $F(0) = 0$; $xF(x) < 0, -F(x) \geqslant x, \quad x \neq 0$; $F(ax) = aF(x),\ \forall a, x \in \mathbb{R}$; and*

$$\liminf_{x\to 0} \frac{F(x)}{x} = M < 0.$$

Suppose further that

$$\limsup_{n\to\infty}\sum_{i=1}^{r}\sum_{\ell=n}^{n+m^*}\frac{\alpha_i(\ell)}{\lambda_\ell^{m_i+1}} > -\frac{1}{M},$$

where $m^* = \max\limits_{1\leqslant i\leqslant r} m_i,\ \alpha_i(n) \geqslant 0, n \in \mathbb{N}_0, i = 1,\dots,r$. *Then,* (1) *is oscillatory.*

Proof. We write the equation (1) in the following form

$$\frac{x_{n+1}}{\lambda_n^{n+1}} - \frac{x_n}{\lambda_n^n} = \sum_{i=1}^{r}\frac{\alpha_i(n)}{\lambda_n^{m_i+1}}F\left(\frac{x_{n-m_i}}{\lambda_n^{n-m_i}}\right), \quad n \in \mathbb{N}_0. \tag{4}$$

Let $\{x_n\}_n$ be a solution of (4), $x_n > 0$ for $n \in \mathbb{N}_{n_1}, n_1 \in \mathbb{N}_0$. Since $\alpha_i(n) \geqslant 0$ for all $i = 1,\dots,r$ and $n \in \mathbb{N}_{n_1+m^*}$, from (4) we have $\left\{\frac{x_n}{\lambda_n^n}\right\}_n$ that is nonincreasing for all $n \in \mathbb{N}_{n_1+m^*}$. Thus, there exists $\lim\limits_{n\to\infty}\frac{x_n}{\lambda_n^n}$. Put $\lim\limits_{n\to\infty}\frac{x_n}{\lambda_n^n} = \beta$. We have $\beta \geqslant 0$. Taking the limit in (4), we obtain $F(\beta) = 0$. This implies $\beta = 0$. Summing the equation (4) from n_2 to $n_2 + m^*$ (where $n_2 \in \mathbb{N}_{n_1}$) we obtain

$$\frac{x_{n_2+m^*+1}}{\lambda_n^{n_2+m^*+1}} - \frac{x_{n_2}}{\lambda_n^{n_2}} = \sum_{i=1}^{r}\sum_{\ell=n_2}^{n_2+m^*}\frac{\alpha_i(\ell)}{\lambda_\ell^{m_i+1}}F\left(\frac{x_{\ell-m_i}}{\lambda_\ell^{\ell-m_i}}\right).$$

It implies

$$\frac{x_{n_2+m^*+1}}{\lambda_n^{n_2+m^*+1}} - \frac{x_{n_2}}{\lambda_n^{n_2}} - \sum_{i=1}^{r}\sum_{\ell=n_2}^{n_2+m^*}\frac{\alpha_i(\ell)}{\lambda_\ell^{m_i+1}}F\left(\frac{x_{\ell-m_i}}{\lambda_\ell^{\ell-m_i}}\right) \leqslant 0,$$

$$\frac{x_{n_2+m^*+1}}{\lambda_n^{n_2+m^*+1}} - \frac{x_{n_2}}{\lambda_n^{n_2}}\left[1 + \sum_{i=1}^{r}\sum_{\ell=n_2}^{n_2+m^*}\frac{\alpha_i(\ell)}{\lambda_\ell^{m_i+1}}\frac{F\left(\frac{x_{n_2}}{\lambda_n^{n_2}}\right)}{\frac{x_{n_2}}{\lambda_n^{n_2}}}\right] \leqslant 0,$$

$$\frac{x_{n_2+m^*+1}}{\lambda_n^{n_2+m^*+1}} - \frac{x_{n_2}}{\lambda_n^{n_2}}\left[1 + \sum_{i=1}^{r}\sum_{\ell=n_2}^{n_2+m^*}\frac{\alpha_i(\ell)}{\lambda_\ell^{m_i+1}}M\right] \leqslant 0.$$

Therefore

$$1 + \sum_{i=1}^{r}\sum_{\ell=n_2}^{n_2+m^*}\frac{\alpha_i(\ell)}{\lambda_\ell^{m_i+1}}M \geqslant 0,$$

and

$$\limsup_{n_2\to\infty}\sum_{i=1}^{r}\sum_{\ell=n_2}^{n_2+m^*}\frac{\alpha_i(\ell)}{\lambda_\ell^{m_i+1}} \leqslant -\frac{1}{M}.$$

This contradicts the hypothesis. The proof is complete. ■

Theorem 2.3. *Assume that*

$$xF(x) < 0,\ x \neq 0 \text{ and } \liminf_{x\to 0}\frac{F(x)}{x} = M < 0.$$

Then, (1) *is oscillatory if the following inequality holds*

$$(-M)\frac{(\tilde{m}+1)^{\tilde{m}+1}}{\tilde{m}^{\tilde{m}}}\sum_{i=1}^{r}\liminf_{n\to\infty}\frac{\alpha_i(n)}{\lambda_n}>1, \tag{5}$$

where $\alpha_i(n)\geqslant 0, n\in\mathbb{N}, 1\leqslant i\leqslant r$, $\tilde{m}=\min_{1\leqslant i\leqslant r} m_i$.

Proof. We first prove that the inequality

$$\frac{x_{n+1}}{\lambda_n^{n+1}}-\frac{x_n}{\lambda_n^{n}}-\sum_{i=1}^{r}\frac{\alpha_i(n)}{\lambda_n^{m_i+1}}F\left(\frac{x_{n-m_i}}{\lambda_n^{n-m_i}}\right)\leqslant 0,\quad n\in\mathbb{N}_0 \tag{6}$$

has no eventually positive solution. Assume, for the sake of contradiction, that (6) has a solution $\{x_n\}_n$ with $x_n>0$ for all $n\geqslant n_1, n_1\in\mathbb{N}$. Setting $v_n=\frac{x_n\lambda_n}{x_{n+1}}$ and dividing this inequality by $\frac{x_n}{\lambda_n^n}$, we obtain

$$\frac{1}{v_n}\leqslant 1+\left[\sum_{i=1}^{r}\frac{\alpha_i(n)}{\lambda_n}\frac{F\left(\frac{x_{n-m_i}}{\lambda_n^{n-m_i}}\right)}{\frac{x_{n-m_i}}{\lambda_n^{n-m_i}}}\prod_{\ell=1}^{m_i}v_{n-\ell}\right], \tag{7}$$

where $n\geqslant n_1+m,\quad m=\max_{1\leqslant i\leqslant r} m_i$.

Clearly, $\left\{\frac{x_n}{\lambda_n^n}\right\}_n$ is nonincreasing with $n\geqslant n_1+m$, and so $v_n\geqslant 1$ for all $n\geqslant n_1+m$. From (5) and (7) we see that $\{v_n\}_n$ is an above bounded sequence. Putting $\liminf_{n\to\infty} v_n=\beta$, we get

$$\limsup_{n\to\infty}\frac{1}{v_n}=\frac{1}{\beta}\leqslant 1+\liminf_{n\to\infty}\left\{\sum_{i=1}^{r}\frac{\alpha_i(n)}{\lambda_n}\frac{F\left(\frac{x_{n-m_i}}{\lambda_n^{n-m_i}}\right)}{\frac{x_{n-m_i}}{\lambda_n^{n-m_i}}}v_{n-m_i}\cdots v_{n-1}\right\},$$

or

$$\frac{1}{\beta}\leqslant 1+\sum_{i=1}^{r}\liminf_{n\to\infty}\frac{\alpha_i(n)}{\lambda_n}\cdot M\cdot\beta^{m_i}. \tag{8}$$

Since

$$\beta^{m_i}\geqslant\beta^{\tilde{m}},\quad \forall i=1,\dots,r,$$

we have

$$-\liminf_{n\to\infty}\sum_{i=1}^{r}\frac{\alpha_i(n)}{\lambda_n}M\leqslant\frac{\beta-1}{\beta^{\tilde{m}+1}}.$$

We shall prove that

$$\frac{\beta-1}{\beta^{\tilde{m}+1}}\leqslant\frac{\tilde{m}^{\tilde{m}}}{(\tilde{m}+1)^{\tilde{m}+1}}.$$

Consider the function $f(\beta) = \frac{\beta-1}{\beta^{\tilde{m}+1}}$, $\beta \geqslant 1$. It is easy to check that $f'(\beta) = \frac{\beta^{\tilde{m}}(-\tilde{m}\beta+\tilde{m}+1)}{(\beta^{\tilde{m}+1})^2}$, $f'(\beta) = 0 \Leftrightarrow \beta = \frac{\tilde{m}+1}{\tilde{m}}$, $f'(\beta) < 0$ for $\beta > \frac{\tilde{m}+1}{\tilde{m}}$, $f'(\beta) > 0$ for $\beta < \frac{\tilde{m}+1}{\tilde{m}}$. Thus,

$$\max_{\beta \geqslant 1} f(\beta) = f\Big(\frac{\tilde{m}+1}{\tilde{m}}\Big) = \frac{\tilde{m}^{\tilde{m}}}{(\tilde{m}+1)^{\tilde{m}+1}}.$$

Therefore

$$(-M)\frac{(\tilde{m}+1)^{\tilde{m}+1}}{\tilde{m}^{\tilde{m}}}\sum_{i=1}^{r}\liminf_{n\to\infty}\frac{\alpha_i(n)}{\lambda_n} \leqslant 1,$$

which contradicts condition (5). Hence, (6) has no eventually positive solution.

Similarly, we can prove that the inequality

$$\frac{x_{n+1}}{\lambda_n^{n+1}} - \frac{x_n}{\lambda_n^n} - \sum_{i=1}^{r}\frac{\alpha_i(n)}{\lambda_n^{m_i+1}}F\left(\frac{x_{n-m_i}}{\lambda_n^{n-m_i}}\right) \geqslant 0, \quad n \in \mathbb{N}$$

has no eventually negative solution. So, the proof is complete. ∎

Corollary 2.4. *Assume that the condition* (5) *in Theorem 2.3 is replaced by*

$$r\left(\prod_{i=1}^{r}\liminf_{n\to\infty}\frac{\alpha_i(n)}{\lambda_n}\cdot(-M)\right)^{\frac{1}{r}} > \frac{\hat{m}^{\hat{m}}}{(\hat{m}+1)^{\hat{m}+1}}, \tag{9}$$

where $\alpha_i(n) \geqslant 0, n \in \mathbb{N}, 1 \leqslant i \leqslant r$ *and* $\hat{m} = \frac{1}{r}\sum_{i=1}^{r} m_i$. *Then,* (1) *is oscillatory.*

Proof. We will prove that the inequality (6) has no eventually positive solution. Assume, for the sake of contradiction, that (6) has a solution $\{x_n\}_n$ with $x_n > 0$ for all $n \geqslant n_1, n_1 \in \mathbb{N}$. Using arithmetic and geometric mean inequality, we obtain

$$\sum_{i=1}^{r}\liminf_{n\to\infty}\frac{\alpha_i(n)}{\lambda_n}\cdot(-M)\cdot\beta^{m_i} \geqslant r\left(\prod_{i=1}^{r}\liminf_{n\to\infty}\frac{\alpha_i(n)}{\lambda_n}\cdot(-M)\cdot\beta^{m_i}\right)^{\frac{1}{r}},$$

which is the same as

$$\sum_{i=1}^{r}\liminf_{n\to\infty}\frac{\alpha_i(n)}{\lambda_n}\cdot(-M)\cdot\beta^{m_i} \geqslant r\left(\prod_{i=1}^{r}\liminf_{n\to\infty}\frac{\alpha_i(n)}{\lambda_n}\cdot(-M)\right)^{\frac{1}{r}}\beta^{\hat{m}}.$$

This yields

$$\frac{1}{\beta} \leqslant 1 - r\left(\prod_{i=1}^{r}\liminf_{n\to\infty}\frac{\alpha_i(n)}{\lambda_n}\cdot(-M)\right)^{\frac{1}{r}}\beta^{\hat{m}}.$$

It implies

$$r\left(\prod_{i=1}^{r}\liminf_{n\to\infty}\frac{\alpha_i(n)}{\lambda_n}\cdot(-M)\right)^{\frac{1}{r}} \leqslant \frac{\hat{m}^{\hat{m}}}{(\hat{m}+1)^{\hat{m}+1}}.$$

which contradicts condition (9). Hence, (6) has no eventually positive solution. ■

Example 2.5. Consider the difference equation

$$x_{n+1} = \sqrt[n]{1+2^{n(-1)^n}}x_n + \sum_{i=1}^{r}(1+2^{n(-1)^n})^{\frac{2(m_i+1)}{n}}(-x_{n-m_i}),\ n=0,1,2,\ldots, \tag{10}$$

where $r, m_1, m_2, \ldots, m_r$ are fixed positive integers. It is clear that this equation is a particular case of (1), where $\lambda_n = \sqrt[n]{1+2^{n(-1)^n}}$, $\alpha_i(n) = (1+2^{n(-1)^n})^{\frac{2(m_i+1)}{n}}, \forall n \in \mathbb{N}, 1 \leqslant i \leqslant r$, and $F(x) \equiv -x$.

After some computations, we obtain

$$\liminf_{n\to\infty}\sum_{i=1}^{r}\frac{\alpha_i(n)}{\lambda_n^{m_i+1}} = \liminf_{n\to\infty}\sum_{i=1}^{r}\left[(1+2^{n(-1)^n})^{\frac{1}{n}}\right]^{m_i+1} = r > 0$$

and

$$\limsup_{n\to\infty}\sum_{i=1}^{r}\frac{\alpha_i(n)}{\lambda_n^{m_i+1}} = 2\sum_{i=1}^{r}2^{m_i} > 1 - \liminf_{n\to\infty}\sum_{i=1}^{r}\frac{\alpha_i(n)}{\lambda_n^{m_i+1}};$$

$$\limsup_{n\to\infty}\sum_{i=1}^{r}\sum_{\ell=n}^{n+m^*}\frac{\alpha_i(\ell)}{\lambda_\ell^{m_i+1}} = 2m^*\sum_{i=1}^{r}2^{m_i} > 1 = -\frac{1}{M}, m^* = \max_{1\leqslant i\leqslant r} m_i;$$

$$\begin{aligned}&(-M)\frac{(\tilde{m}+1)^{\tilde{m}+1}}{\tilde{m}^{\tilde{m}}}\sum_{i=1}^{r}\liminf_{n\to\infty}\frac{\alpha_i(n)}{\lambda_n}\\ &= 1\cdot\frac{(\tilde{m}+1)^{\tilde{m}+1}}{\tilde{m}^{\tilde{m}}}\cdot\sum_{i=1}^{r}\liminf_{n\to\infty}\left[(1+2^{n(-1)^n})^{\frac{1}{n}}\right]^{2m_i+1}\\ &= 1\cdot\frac{(\tilde{m}+1)^{\tilde{m}+1}}{\tilde{m}^{\tilde{m}}}\cdot r > 1, \tilde{m} = \min_{1\leqslant i\leqslant r} m_i.\end{aligned}$$

It is easy to verify that all conditions of Theorems 2.1, 2.2, 2.3 hold. Hence (10) is oscillatory.

Proposition 2.6. *Let F be a nonincreasing function, $xF(x) < 0, x \neq 0, -F(x) \geqslant x$, $F(ax) = aF(x), \forall a, x \in \mathbb{R}$, $m_1 > m_2 > \ldots > m_r > 0$ and suppose there exists a sufficiently large integer N such that*

$$\alpha_1(n) \geqslant 0, \alpha_1(n)\lambda_n^{n-m_1} + \alpha_2(n)\lambda_n^{n-m_2} \geqslant 0, \cdots, \sum_{i=1}^{r}\alpha_i(n)\lambda_n^{n-m_i} \geqslant 0 \ \textit{for}\ n \geqslant N.$$

Assume further that for any given positive integer n_1 there exists an integer $n_2 \geqslant n_1$ such that $\alpha_i(n) \geqslant 0, \quad i = 1, \ldots, r$ for $n \in [n_2, n_2 + m_1]$. Let $\{x_n\}_n$ be a solution of (1) *such that $\{x_n\}_n$ is eventually positive. Then $\left\{\frac{x_n}{\lambda_n^n}\right\}_n$ is eventually nonincreasing and*

$$\sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) \leqslant -\frac{x_{n-m_r}}{\lambda_n^{n-m_r}} \sum_{i=1}^{r} \alpha_i(n) \lambda_n^{n-m_i} \tag{11}$$

holds eventually.

Proof. Let $x_{n-m_1} > 0$ for $n \geqslant N$. Then there exists $n_2 \geqslant N$ such that $\alpha_i(n) \geqslant 0, \quad i = 1, \ldots, r, n \in [n_2, n_2 + m_1]$. This implies that

$$\frac{x_{n+1}}{\lambda_n^{n+1}} - \frac{x_n}{\lambda_n^n} = \frac{1}{\lambda_n^{n+1}} \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) \leqslant 0 \text{ for } n \in [n_2, n_2 + m_1].$$

We shall show that $\left\{\frac{x_n}{\lambda_n^n}\right\}_n$ is nonincreasing for $n \in [n_2 + m_1, n_2 + m_1 + m_r]$. Since

$$n - m_i \in [n_2, n_2 + m_1], \quad \text{for } n \in [n_2 + m_1, n_2 + m_1 + m_r]$$

so

$$\frac{x_{n-m_1}}{\lambda_n^{n-m_1}} \geqslant \frac{x_{n-m_2}}{\lambda_n^{n-m_2}} \geqslant \ldots \geqslant \frac{x_{n-m_r}}{\lambda_n^{n-m_r}}.$$

Therefore

$$\begin{aligned}
&\frac{x_{n+1}}{\lambda_n^{n+1}} - \frac{x_n}{\lambda_n^n} \\
&= \frac{1}{\lambda_n^{n+1}} \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) \\
&\leqslant \frac{1}{\lambda_n^{n+1}} [\alpha_1(n) \lambda_n^{n-m_1} + \alpha_2(n) \lambda_n^{n-m_2}] F\left(\frac{x_{n-m_2}}{\lambda_n^{n-m_2}}\right) + \sum_{i=3}^{r} \alpha_i(n) F(x_{n-m_i}) \\
&\leqslant \cdots\cdots\cdots\cdots\cdots\cdots \\
&\leqslant \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} F\Big(\frac{x_{n-m_r}}{\lambda_n^{n-m_r}}\Big) \leqslant 0,
\end{aligned}$$

for $n \in [n_2 + m_1, n_2 + m_1 + m_r]$. Repeating the above procedure we can show that $\left\{\frac{x_n}{\lambda_n^n}\right\}_n$ is nonincreasing for $n \in [n_2 + m_1 + \ell m_r, n_2 + m_1 + (\ell + 1) m_r]$, for $\ell \in \mathbb{N}_0$. That is, $\left\{\frac{x_n}{\lambda_n^n}\right\}_n$ is nonincreasing for $n \geqslant n_2$. On the other hand, from inequality

$$\frac{1}{\lambda_n^{n+1}} \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) \leqslant \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} F\Big(\frac{x_{n-m_r}}{\lambda_n^{n-m_r}}\Big)$$

we have the inequality (11). ∎

Theorem 2.7. *Assume that the assumptions of Proposition 2.6 hold. Further, assume that* $\sum_{j=N}^{\infty}\sum_{i=1}^{r}\alpha_i(j)\lambda_j^{j-m_i}=\infty$. *Then every nonoscillatory solution of* (1) *tends to* 0 *as* $n\to\infty$.

Proof. Let $\{x_n\}_n$ be an eventually positive solution of (1). By Proposition 2.6, $\left\{\frac{x_n}{\lambda_n^n}\right\}_n$ is eventually nonincreasing and hence there exists $\lim_{n\to\infty}\frac{x_n}{\lambda_n^n}=\beta$. We have $\beta\geqslant 0$. If $\beta>0$, by summing (1) from N to n we have

$$\begin{aligned}0&=\frac{x_{n+1}}{\lambda_n^{n+1}}-\frac{x_N}{\lambda_n^N}-\frac{1}{\lambda_n^{n+1}}\sum_{j=N}^{n}\sum_{i=1}^{r}\alpha_i(j)F(x_{j-m_i}),\\&\geqslant\frac{x_{n+1}}{\lambda_n^{n+1}}-\frac{x_N}{\lambda_n^N}+\frac{1}{\lambda_n^{n+1}}\sum_{j=N}^{n}\frac{x_{j-m_r}}{\lambda_j^{j-m_r}}\sum_{i=1}^{r}\alpha_i(j)\lambda_j^{j-m_i}\\&\geqslant\frac{x_{n+1}}{\lambda_n^{n+1}}-\frac{x_N}{\lambda_n^N}+\frac{1}{\lambda_n^{n+1}}\frac{x_{n-m_r}}{\lambda_n^{n-m_r}}\sum_{j=N}^{n}\sum_{i=1}^{r}\alpha_i(j)\lambda_j^{j-m_i}.\end{aligned}$$

Letting $n\to\infty$ we get a contradiction. Therefore $\beta=0$. The proof is complete. ■

Proposition 2.8. *In addition to the assumptions of Proposition 2.6, suppose that* $\lambda_n\geqslant 1,\forall n$ *and there exists a positive number* γ *such that*

$$\sum_{j=n-m_r}^{n}\sum_{i=1}^{r}\alpha_i(j)\lambda_j^{j-m_i}\geqslant\gamma>0\ \textit{for all large } n. \tag{12}$$

Let $\{x_n\}_n$ *be an eventually positive solution of* (1). *Then* $\left\{\frac{x_{n-m_r}}{x_n}\right\}_n$ *is eventually bounded above.*

Proof. From (12), for any large integer $\overline{N}$ there exists an integer n such that $\overline{N}\in[n-m_r,n]$ and

$$\sum_{j=n-m_r}^{\overline{N}}\sum_{i=1}^{r}\alpha_i(j)\lambda_j^{j-m_i}\geqslant\frac{\gamma}{2},\quad\sum_{j=\overline{N}}^{n}\sum_{i=1}^{r}\alpha_i(j)\lambda_j^{j-m_i}\geqslant\frac{\gamma}{2}.$$

Summing (1) from $n-m_r$ to $\overline{N}$ we have

$$\begin{aligned}\frac{x_{\overline{N}+1}}{\lambda_n^{\overline{N}+1}}-\frac{x_{n-m_r}}{\lambda_n^{n-m_r}}&=\frac{1}{\lambda_n^{n+1}}\sum_{j=n-m_r}^{\overline{N}}\sum_{i=1}^{r}\alpha_i(j)F(x_{j-m_i})\\\Rightarrow\frac{x_{n-m_r}}{\lambda_n^{n-m_r}}&=\frac{x_{\overline{N}+1}}{\lambda_n^{\overline{N}+1}}-\frac{1}{\lambda_n^{n+1}}\sum_{j=n-m_r}^{\overline{N}}\sum_{i=1}^{r}\alpha_i(j)F(x_{j-m_i})\end{aligned}$$

$$\geqslant \frac{x_{\overline{N}+1}}{\lambda_n^{\overline{N}+1}} + \frac{1}{\lambda_n^{n+1}} \sum_{j=n-m_r}^{\overline{N}} \frac{x_{j-m_r}}{\lambda_j^{j-m_r}} \sum_{i=1}^{r} \alpha_i(j)\lambda_j^{j-m_i}$$

$$\geqslant \frac{x_{\overline{N}+1}}{\lambda_n^{\overline{N}+1}} + \frac{x_{\overline{N}-m_r}}{\lambda_n^{\overline{N}-m_r}} \sum_{j=n-m_r}^{\overline{N}} \sum_{i=1}^{r} \alpha_i(j)\lambda_j^{j-m_i}.$$

Hence

$$\frac{x_{n-m_r}}{\lambda_n^{n-m_r}} \geqslant \frac{x_{\overline{N}+1}}{\lambda_n^{\overline{N}+1}} + \frac{x_{\overline{N}-m_r}}{\lambda_n^{\overline{N}-m_r}} \frac{\gamma}{2}. \tag{13}$$

Similarly, summing (1) from $\overline{N}$ to n we have

$$\frac{x_{n+1}}{\lambda_n^{n+1}} - \frac{x_{\overline{N}}}{\lambda_n^{\overline{N}}} = \frac{1}{\lambda_n^{n+1}} \sum_{j=\overline{N}}^{n} \sum_{i=1}^{r} \alpha_i(j)F(x_{j-m_i}).$$

Hence

$$\frac{x_{\overline{N}}}{\lambda_n^{\overline{N}}} \geqslant \frac{x_{n+1}}{\lambda_n^{n+1}} + \frac{x_{n-m_r}}{\lambda_n^{n-m_r}} \frac{\gamma}{2}. \tag{14}$$

Combining (13) and (14) we have $\frac{x_{\overline{N}-m_r}}{x_{\overline{N}}} \leqslant (2/\gamma)^2$. Since $\overline{N}$ is arbitrary the proof is complete. ■

Theorem 2.9. *In addition to the hypotheses of Propositon 2.8 suppose that* $\lambda_n \geqslant 1, \forall n$ *and*

$$\liminf_{n\to\infty} \frac{1}{m_r} \sum_{j=n-m_r}^{n-1} \sum_{i=1}^{r} \frac{\alpha_i(j)}{\lambda_j^{m_i+1}} > \frac{m_r^{m_r}}{(m_r+1)^{m_r+1}}.$$

Then, (1) *is oscillatory.*

Proof. Assume the contrary and let $\{x_n\}_n$ be an eventually positive solution of (1), then by Proposition 2.6,

$$\begin{aligned}
\frac{x_{n+1}}{\lambda_n^{n+1}} - \frac{x_n}{\lambda_n^n} &= \frac{1}{\lambda_n^{n+1}} \sum_{i=1}^{r} \alpha_i(n)F(x_{n-m_i}), \\
&\leqslant \frac{1}{\lambda_n^{n+1}} \sum_{i=1}^{r} \alpha_i(n)\lambda_n^{n-m_i} F\left(\frac{x_{n-m_r}}{\lambda_n^{n-m_r}}\right) \\
&= \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} F\left(\frac{x_{n-m_r}}{\lambda_n^{n-m_r}}\right) \\
&\leqslant -\sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}} \frac{x_{n-m_r}}{\lambda_n^{n-m_r}} \\
&\leqslant -\frac{x_n}{\lambda_n^n} \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}}.
\end{aligned} \tag{15}$$

Hence, eventually

$$1 - \frac{x_{n+1}}{\lambda_n x_n} \geqslant \sum_{i=1}^{r} \frac{\alpha_i(n)}{\lambda_n^{m_i+1}}$$

and so

$$\begin{aligned}\frac{1}{m_r}\sum_{j=n-m_r}^{n-1}\sum_{i=1}^{r}\frac{\alpha_i(j)}{\lambda_j^{m_i+1}} &\leqslant \frac{1}{m_r}\sum_{j=n-m_r}^{n-1}\left(1-\frac{x_{j+1}}{x_j\lambda_j}\right)\\ &= 1-\frac{1}{m_r}\sum_{j=n-m_r}^{n-1}\frac{x_{j+1}}{x_j\lambda_j}\\ &\leqslant 1-\left(\prod_{j=n-m_r}^{n-1}\frac{x_{j+1}}{x_j\lambda_j}\right)^{\frac{1}{m_r}}\\ &= 1-\left(\frac{x_n}{x_{n-m_r}}\frac{1}{\prod\limits_{j=n-m_r}^{n-1}\lambda_j}\right)^{\frac{1}{m_r}}.\end{aligned}$$

Putting $\alpha = \frac{m_r^{m_r}}{(m_r+1)^{m_r+1}}$, from the inequality

$$\liminf_{n\to\infty}\frac{1}{m_r}\sum_{j=n-m_r}^{n-1}\sum_{i=1}^{r}\frac{\alpha_i(j)}{\lambda_j^{m_i+1}} > \frac{m_r^{m_r}}{(m_r+1)^{m_r+1}}$$

we can choose a constant β such that for n sufficiently large

$$\alpha < \beta \leqslant \frac{1}{m_r}\sum_{j=n-m_r}^{n-1}\sum_{i=1}^{r}\frac{\alpha_i(j)}{\lambda_j^{m_i+1}}.$$

Therefore, for all large n, $\left(\frac{x_n}{x_{n-m_r}}\right)^{\frac{1}{m_r}}\left(\frac{1}{\prod\limits_{j=n-m_r}^{n-1}\lambda_j}\right)^{\frac{1}{m_r}} \leqslant 1-\beta$, which in particular implies that $0<\beta<1$. Since

$$\max_{0\leqslant\lambda\leqslant 1}[(1-\lambda)\lambda^{\frac{1}{m_r}}] = \alpha^{\frac{1}{m_r}},$$

we have

$$1-\lambda \leqslant \alpha^{\frac{1}{m_r}}/\beta^{\frac{1}{m_r}}$$

and

$$\frac{1}{\prod\limits_{j=n-m_r}^{n-1}\lambda_j}\frac{\beta}{\alpha}x_n \leqslant x_{n-m_r},$$

or

$$\frac{1}{\lambda_n^{n-m_r}\prod\limits_{j=n-m_r}^{n-1}\lambda_j}\frac{\beta}{\alpha}x_n \leqslant \frac{x_{n-m_r}}{\lambda_n^{n-m_r}}. \tag{16}$$

Substituting (16) into (15) we obtain

$$\frac{x_{n+1}}{\lambda_n^{n+1}} - \frac{x_n}{\lambda_n^n} \leqslant -\frac{x_n}{\lambda_n^n}\frac{1}{\lambda_n^{-m_r}\prod\limits_{j=n-m_r}^{n-1}\lambda_j}\frac{\beta}{\alpha}\sum_{i=1}^{r}\frac{\alpha_i(n)}{\lambda_n^{m_i+1}}.$$

Hence

$$1-\frac{x_{n+1}}{x_n\lambda_n} \geqslant \frac{1}{\lambda_n^{-m_r}}\frac{1}{\prod\limits_{j=n-m_r}^{n-1}\lambda_j}\frac{\beta}{\alpha}\sum_{i=1}^{r}\frac{\alpha_i(n)}{\lambda_n^{m_i+1}},$$

and so

$$\frac{1}{\lambda_n^{-m_r}}\frac{1}{\prod\limits_{j=n-m_r}^{n-1}\lambda_j}\frac{\beta}{\alpha}\frac{1}{m_r}\sum_{j=n-m_r}^{n-1}\sum_{i=1}^{r}\frac{\alpha_i(j)}{\lambda_j^{m_i+1}} \leqslant 1-\left(\frac{x_n}{x_{n-m_r}}\frac{1}{\prod\limits_{j=n-m_r}^{n-1}\lambda_j}\right)^{\frac{1}{m_r}}.$$

Thus

$$\left(\frac{x_n}{x_{n-m_r}}\right)^{\frac{1}{m_r}}\left(\frac{1}{\prod\limits_{j=n-m_r}^{n-1}\lambda_j}\right)^{\frac{1}{m_r}} \leqslant 1-\frac{1}{\lambda_n^{-m_r}}\frac{1}{\prod_{j=n-m_r}^{n-1}\lambda_j}\frac{\beta^2}{\alpha}$$

and eventually

$$\frac{1}{\lambda_n^{-m_r}}\frac{1}{\prod\limits_{j=n-m_r}^{n-1}\lambda_j}\left(\frac{\beta}{\alpha}\right)^2 x_n \leqslant \frac{x_{n-m_r}}{\lambda_n^{n-m_r}}.$$

By induction, for every $m=1,2,\dots$, there exists an integer n_k such that

$$\frac{1}{\lambda_n^{-m_r}}\frac{1}{\prod\limits_{j=n-m_r}^{n-1}\lambda_j}\left(\frac{\beta}{\alpha}\right)^k x_n \leqslant \frac{x_{n-m_r}}{\lambda_n^{n-m_r}}, \quad n \geqslant n_k,$$

or

$$\frac{\lambda_n^n}{\prod\limits_{j=n-m_r}^{n-1}\lambda_j}\left(\frac{\beta}{\alpha}\right)^k \leqslant \frac{x_{n-m_r}}{x_n}, \quad n \geqslant n_k,$$

which implies that $\left\{\frac{x_{n-m_r}}{x_n}\right\}_n$ is eventually unbounded. But this, in view of Proposition 2.8, is impossible. ■

3. Conclusion

In this paper, we obtain some new results for oscillation of the difference equation (1) in two following cases

- $\{\lambda_n\}_n$ is an arbitrary sequence of positive real numbers and $\alpha_i(n) \geqslant 0, \quad \forall n \in \mathbb{N}_0$ (Theorems 2.1, 2.2, 2.3);
- $\lambda_n \geqslant 1, \quad \forall n \in \mathbb{N}_0$ and the positivity of $\alpha_i(n)$ is not required (Theorems 2.7, 2.9).

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