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On the Electric-magnetic Goddard-Nuyts-Olive Duality*

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Abstract. We describe the mathematical model of the electric-magnetic P. Goddard-J. Nuyts-D. Olive duality in the case of compact Lie groups of symmetry. After that we give an explicit computation for the electric-magnetic P. Goddard-J. Nuyts-D. Olive duality for the pair SO(3) and Sp(1).

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1. Introduction

In 1976, M. F. Atiyah gave a hypothesis that the geometrical part of the Langlands Program in mathematics should correspond to the electric-magnetic P. Goddard-J. Nuyts-D. Olive (GNO) duality in physics. Mathematicians and physicists obtained various results in this direction, in particular, for compact Lie groups of symmetry G, P. Goddard-J. Nuyts-D. Olive observed the dual group $\mathrm{G}^{\vee}$, which is isomorphic to the Langlands dual group ${}^{L}\mathrm{G}$. Following Atiyah's ideas many researches in mathematics and in physics were done, namely, in con-

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structing model of quatum field theory, it was imposed the *'t Hooft-Polyakov quantization condition*:

$$\exp\left(4\pi i \sum_{i=1}^{r} \beta_i T_i\right) = 1,$$

where $(\beta_1, \ldots, \beta_r)$ provides a weight of the magnetic group of symmetry, $T_1, \ldots, T_r$ are the coordinates of an element of the Lie algebra of a maximal torus of the corresponding electric group of symmetry.

For a compact Lie group of symmetry G its GNO dual group $\mathrm{G}^\vee$ indeed is the same as the Langlands duality group ${}^L\mathrm{G}$. However, also in the case of the pair $\mathrm{SO}(3) - \mathrm{Sp}(1)$ it was never been exactly and precisely computed. We suppose, therefore to give an explicit computation for the electric-magnetic duality for the pair $\mathrm{SO}(3) - \mathrm{Sp}(1)$. Our main results are Theorem 3.1 and Theorem 4.1 in which we assert that *the electric group* $\mathrm{SO}(3)$ *and the magnetic groups* $\mathrm{Sp}(1)$ *in the P. Goddard-J. Nuyts-D. Olive duality are obtained from one-to-another by pairing the corresponding weight lattices. They are isomorphic to the corresponding groups in the Langlands dual pair.* In order to prove this theorem we follow the steps: From $\mathrm{SO}(3)$ we compute the root lattice and magnetic weight lattice $\Lambda(\mathrm{SO}(3))$; next compute the dual root lattice $\Phi^\vee$; and finally compute the cooresponding Lie algebras and Lie groups; The result is $\mathfrak{su}(2)$ the compact form of complexification of Lie algebra $\mathrm{Lie}(\mathrm{SO}(3)^\vee)$ and $\mathrm{SU}(2)$ which is the compact universal covering group of $\mathrm{SO}(3)^\vee$.

The paper is organized as follows. In Sec. 2 we expose a mathematical model of electric-magnetic duality. In Sec. 3 we give an explicit computation for the electric-magnetic duality for the pair $\mathrm{SO}(3) - \mathrm{Sp}(1)$. In the last section we return to the physical illustration of the electric-magnetic duality.

2. Mathematical Model of the Electric-magnetic Duality

Let G be an electric group of symmetry, which is assumed to be a compact connected Lie group. Denote by $\widetilde{\mathrm{G}}$ the universal covering group of G, and $\mathfrak{g} = \mathrm{Lie}\,\mathrm{G}$ its Lie algebra. We introduce the following notations:

- $\mathbf{T}$ is a maximal torus of G, $\mathfrak{t} = \mathrm{Lie}\,\mathbf{T}$ its Lie algebra. Since G is a compact and connected Lie group, Lie algebra $\mathfrak{t}_\mathbb{C}$ is the corresponding Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$;
- Φ is the root system of $\mathfrak{g}_\mathbb{C}$, corresponding to the Cartan subalgebra $\mathfrak{t}_\mathbb{C}$.
- $\Phi^\vee = \left\{N^{-1}\frac{\alpha}{\alpha^2}; \alpha \in \Phi\right\}$ is the dual root system of Φ, where N is a normalized parameter.
- V denotes a minimal faithful representation of G.
- $\Lambda(\mathrm{G})$ denotes the weight lattice of G in V.

Let $\mathfrak{g}_\mathbb{C}^\vee$ be the Lie algebra with the root system $\Phi^\vee$. $\widetilde{\mathrm{G}^\vee}$ is a connected, simply connected Lie group and has the complex Lie algebra $\mathfrak{g}_\mathbb{C}^\vee$. We now choose a

Cartan basis

$$T_1, T_2, \dots, T_r, E_{-\alpha_1}, E_{\alpha_1}, E_{-\alpha_2}, E_{\alpha_2}, \dots (r = \dim \mathfrak{t}),$$

whose $T_1, T_2, \dots, T_r$ is an orthonormal basis with respect to the Killing form, restricted to $\mathfrak{t}_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}$ and therefore in this basis the matrix of the Killing form is the unit one. Hence, for every $\alpha, \beta \in \mathfrak{t}^*_{\mathbb{C}}$, we can define the scalar product of two linear forms by

$$\langle \alpha, \beta \rangle = \sum_{i=1}^{r} \alpha(T_i)\beta(T_i).$$

For short, we denote $\langle \alpha, \beta \rangle$ by $\alpha\beta$. We identify a linear form ω on $\mathfrak{t}$ with a r-tuple $(\omega_1, \omega_2, \dots, \omega_r)$, where $\omega_i = \omega(T_i), i = 1, \dots, r$.

To construct the magnetic dual group $\mathrm{G}^\vee$, we need first of all the lemmas what follow and were remarked in [4].

Lemma 2.1. *Let a canonical homomorphism from the universal covering group $\widetilde{G}$ to the Lie group* G

$$k : \widetilde{\mathrm{G}} \to \mathrm{G}\,.$$

Then

$$k(\mathrm{G}) = \ker k = \{\exp\left(4\pi i \omega N T\right),\ \omega \in \Lambda(\mathrm{G})^*\}\,,$$

where

$$\Lambda(\mathrm{G})^* = \{\beta \in \mathfrak{t}^* : 2\omega N \beta \in \mathbb{Z}, \forall \omega \in \Lambda(\mathrm{G})\}\,.$$

Lemma 2.2. *Let $Z(\widetilde{\mathrm{G}})$ be the centre of Lie group $\widetilde{\mathrm{G}}$, $\Lambda(\widetilde{\mathrm{G}^\vee})$ be the weight lattice and $\Lambda_r(\widetilde{\mathrm{G}^\vee})$ be the root lattice of Lie group $\widetilde{\mathrm{G}^\vee}$. Then there is an isomorphism*

$$Z(\widetilde{\mathrm{G}}) \cong \Lambda(\widetilde{\mathrm{G}^\vee})/\Lambda_r(\widetilde{\mathrm{G}^\vee}).$$

We consider a subgroup of $Z(\widetilde{\mathrm{G}^\vee})$ defined by

$$k(\mathrm{G}^\vee) = \{\exp\left(4\pi i \omega N T^\vee\right),\ \omega \in \Lambda(\mathrm{G})\}\,, \tag{1}$$

which is a discrete normal subgroup of $Z(\widetilde{\mathrm{G}^\vee})$. We define $\mathrm{G}^\vee = \widetilde{\mathrm{G}^\vee}/k(\mathrm{G}^\vee)$ and call it the *magnetic group of symmetry.* Since $k(\mathrm{G}^\vee)$ is the kernel of the canonical homomorphism $k : \widetilde{\mathrm{G}^\vee} \to \mathrm{G}^\vee$, one deduces that

$$k(\mathrm{G}^\vee) = \{\exp\left(4\pi i \omega N T^\vee\right),\ \omega \in \Lambda(\mathrm{G}^\vee)^*\}\,. \tag{2}$$

Comparing (1) with (2) we see that

$$\Lambda(\mathrm{G}) = \Lambda(\mathrm{G}^\vee)^*.$$

On the other hand, $(\Lambda(\mathrm{G})^*)^* = \Lambda(\mathrm{G})$ [7], it follows that $\Lambda(\mathrm{G})^* = \Lambda(\mathrm{G}^\vee)$. Hence $k(\mathrm{G}) = \{\exp\left(4\pi i \omega N T\right),\ \omega \in \Lambda(\mathrm{G}^\vee)\}$. Therefore there exists a surjection from $\Lambda(\mathrm{G}^\vee)$ to $k(\mathrm{G})$ and its kernel is

$$\{\omega \in \Lambda(\mathrm{G}^\vee) : \exp(4\pi i\omega NT) = 1\} = \{\omega \in \Lambda(\mathrm{G}^\vee) : 2\beta N\omega \in \mathbb{Z}, \forall\beta \in \Lambda(\mathrm{G})\}.$$

Since $\omega \in \Lambda(\mathrm{G}^\vee)$, we write $\omega = \sum_{\alpha_i^\vee \in \Delta(\mathrm{G}^\vee)} m_i\alpha_i^\vee$, where $\Delta(\mathrm{G}^\vee)$ is a set of simple roots $\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_r^\vee$. The fundamental weights λ_i are defined, see e.g. [3]

$$2\lambda_i \cdot \alpha_j^\vee/(\alpha_j^\vee)^2 = \delta_{ij}.$$

Choosing β in turn to be $\lambda_1, \lambda_2, \ldots, \lambda_r$, we obtain

$$2\lambda_i N\omega = \sum_{\alpha_j^\vee \in \Delta(\mathrm{G}^\vee)} 2m_j\lambda_i N\alpha_j^\vee = m_i.$$

Hence, $m_i \in \mathbb{Z}$, it deduces that $\omega \in \Lambda_r(\mathrm{G}^\vee)$. Therefore $k(\mathrm{G}) \cong \Lambda(\mathrm{G}^\vee)/\Lambda_r(\mathrm{G}^\vee)$. Similarly, we have

$$k(\mathrm{G}^\vee) \cong \Lambda(\mathrm{G})/\Lambda_r(\mathrm{G}),$$
$$k(\mathrm{G}^{\vee^\vee}) \cong \Lambda(\mathrm{G}^\vee)/\Lambda_r(\mathrm{G}^\vee) \cong k(\mathrm{G}).$$

Hence $\mathrm{G}^{\vee^\vee} = \widetilde{\mathrm{G}^{\vee^\vee}}/k(\mathrm{G}^{\vee^\vee}) \cong \widetilde{\mathrm{G}}/k(\mathrm{G}) = \mathrm{G}$. The groups G and $\mathrm{G}^\vee$ are called electric-magnetic dual or GNO dual.

However, in practice it is often more convenient to use the following result.

Lemma 2.3. *Let $Z(\mathrm{G})$ be the centre of Lie group* G*, $k(\mathrm{G}^\vee)$ be the kernel of the canonical homomorphism $k : \widetilde{\mathrm{G}^\vee} \to \mathrm{G}^\vee$, then there is a natural isomorphism $Z(\mathrm{G}) \cong k(\mathrm{G}^\vee)$.*

Indeed this statement is equivalent to the previous lemmas.

3. Electric-magnetic Dual Pair SO(3) − Sp(1)

We consider the so called *electric group of symmetry* SO(3). It is a connected, compact Lie group and has Lie algebra $\mathfrak{so}(3)$.

Theorem 3.1. *The electric group* SO(3) *and the magnetic group* Sp(1) *in the GNO dual are obtained one-from-another by paring the corresponding weight lattices.*

Proof. In order to prove this theorem we follow the steps:

- From SO(3) we compute the root lattice and magnetic weight lattice $\Lambda(\mathrm{SO}(3))$.
- Compute the dual root lattice $\Phi^\vee$.
- Compute the cooresponding Lie algebras and Lie groups.

Finally, the result is $\mathfrak{su}(2)$ the compact form of complexification of Lie algebra $\mathrm{Lie}(\mathrm{SO}(3)^\vee)$ and Sp(1) is the real form of $\mathrm{SO}(3)^\vee$. ■

Remark 3.2. Lie group SO(3) and Lie group Sp(1) are isomorphic to the corresponding groups in the Langlands dual pair.

4. The Electric-magnetic Duality of Weights

In constructing a model of new quantum physics, a quantum condition 't Hooft - Polyakov was required:

$$\exp\left(4\pi i \sum_{i=1}^{r} \beta_i T_i\right) = 1, \tag{3}$$

where $(\beta_1, \beta_2, \ldots, \beta_r)$ is a weight of the magnetic symmetry group, $(T_1, T_2, \ldots, T_r)$ are the coordinates of an element in Lie algebra of a maximal torus of the electric group of symmetry.

We construct an mathematical model corresponding to that model as follows. Let G be a connected, compact Lie group and the corresponding Lie algebra $\mathfrak{g} = \mathrm{Lie}\,\mathrm{G}$. Denote a maximal torus of G by $\mathbf{T}$ and its complexified Lie algebra by $\mathfrak{t}_{\mathbb{C}} = \mathrm{Lie}\,\mathbf{T} \otimes \mathbb{C}$. We choose a Cartan basis $T_1, T_2, \ldots, T_r, E_{-\alpha_1}, E_{\alpha_1}, E_{-\alpha_2}, E_{\alpha_2}, \ldots$ such that the Killing form restricted on $\mathfrak{t}_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}$ has the unit matrix as its matrix in this basic. Then, the quantum condition (3) can be rewritten as follows:

$$\exp\left(4\pi i \sum_{i=1}^{r} \beta_i T_i\right) = e, \tag{4}$$

where $(\beta_1, \beta_2, \ldots, \beta_r)$ provides an element $\beta \in \mathfrak{t}_{\mathbb{C}}^*$, $\beta_i = \beta(T_i)$ which is called a *magnetic weight*. We denote by $\beta(\mathrm{G})$ the magnetic weights $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$. From (4), we deduce that $\beta = N\omega, \omega \in \Lambda(\mathrm{G})^*$. Furthermore, $\Lambda(\mathrm{G})^* = \Lambda(\mathrm{G}^\vee)$, then $\beta(\mathrm{G}) = N\Lambda(\mathrm{G}^\vee)$. It means that the set of magnetic weights of Lie group G is exactly the weight lattices of Lie group $\mathrm{G}^\vee$ modulo a factor which is equal to the normalized parameter N.

Let us return to the case of the pair $\mathrm{SO}(3) - \mathrm{Sp}(1)$.

Theorem 4.1. *The set of magnetic weights of Lie group* SO(3) *provides the weight lattices of Lie group* Sp(1).

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