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Approximation on a Bidisk

Nguyen Quang Dieu[1] **and Kieu Phuong Chi**[2]

[1]*Department of Mathematics, Hanoi University of Education, Hanoi, Vietnam*

[2]*Department of Mathematics, Vinh University, Vinh, Nghe An, Vietnam*

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Abstract. In this paper we are concerned with uniform approximation on a closed bidisk in $\mathbb{C}^2$. A sufficient condition was given on functions f, g defined near the origin in $\mathbb{C}^2$ so that the algebra generated by $z^{n_1}, f^{n_2}, w^{n_3}$ and g^{n_4} $(n_1, n_2, n_3, n_4 \in \mathbb{N})$ is dense in the space of continuous functions on Δ for all bidisks Δ close enough to the origin in $\mathbb{C}^2$.

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1. Introduction and Statement of Results

We recall that for a given compact K in $\mathbb{C}^n$, by $\hat{K}$ we denote the polynomial convex hull of K i.e.,

$$\hat{K} = \{z \in \mathbb{C}^n : |p(z)| \le \|p\|_K \text{ for every polynomial } p \text{ in } \mathbb{C}^n\}.$$

We say that K is polynomially convex if $\hat{K} = K$. A compact K is called locally polynomially convex at $a \in K$ if there exists a closed ball $B(a)$ centered at a, such that $B(a) \cap K$ is polynomially convex. A compact $K \subset \mathbb{C}$ is polynomially convex if its complement is connected. In $\mathbb{C}^n$ $(n > 1)$ there is no general condition that a compact subset is polynomially convex. For a compact set $K \subset \mathbb{C}^n$, let $C(K)$ denote the algebra of all continuous complex valued functions on K, with the norm

$$\|g\|_K = \max\{|g(z)| : z \in K\}, \quad \text{for every } g \in C(K),$$

and let $P(K)$ denote the closure of the set of polynomials in $C(K)$. It is well-known that $K, \hat{K}$ respectively can be identified with the space of maximal ideals of $C(K), P(K)$ (see [1, 5]). If $f_1, f_2, \dots, f_k$ are continuous functions on K then $[f_1, f_2, \dots, f_k; K]$ denotes the closure in $C(K)$ of the polynomials in $f_1, f_2, \dots, f_k$. The interesting problem is finding the conditions on $f_1, \dots, f_k$ such that every continuous function on K can be approximated uniformly by polynomials in $f_1, f_2, \dots, f_k$, i.e $[f_1, f_2, \dots, f_k; K] = C(K)$. Many results are known if one considers two functions on a disk in $\mathbb{C}$. Let D be a closed disk in the complex plane centered at the origin, let f be a $\mathcal{C}^1$ function defined in a neighborhood of the origin of $\mathbb{C}$ with $\frac{\partial f}{\partial \bar{z}}(0) \neq 0$, $\frac{\partial f}{\partial z}(0) = 0$ and $f(0) = 0$. Wermer [16] showed that $[z, f; D] = C(D)$ for D small enough. In [13] De Paepe proved that $[z^m, f^n; D] = C(D)$ if m, n are coprime positive integers and z^m, f^n separated points near the origin. Using polynomial convexity theory, it can be shown that $[z^2, f^2; D] \neq C(D)$ for some choices of f (see [9, 10, 12],...), while for other choices of g we have $[z^2, g^2; D] = C(D)$ (see [3, 4, 9],...).

Let $\Delta = \{(z, w) \in \mathbb{C}^2 : |z| \leq r, |w| \leq r, r > 0\}$ be a closed bidisk in $\mathbb{C}^2$ centered at the origin. Let f, g be continuous functions on Δ. In this work, we concern with finding conditions on f, g to ensure that $[z^{n_1}, g^{n_2}, w^{n_3}, f^{n_4}; \Delta] = C(\Delta)$ with $n_1, n_2, n_3, n_4 \in \mathbb{N}$. As in a previous work, we rely heavily on the theory of polynomial convexity. The line of proof is the same as the algebras on disks. However, we note that the most difficult step in the above approach is to show the polynomial convexity of a union of totally real graphs. Unlike the one dimension case, in high dimension, we have difficulties while applying Kallin's lemma (mentioned later), since the pairwise intersections of these graphs may be of real dimension two.

We come to state the results of this work. The following proposition provides a simple result which is similar as algebra on disks (see [9]).

Proposition 1.1. *Let n_1, n_2, n_3 and n_4 be positive intergers such that* $\gcd(n_i, n_j) = 1$ *for all $i \neq j$. Let f, g be $\mathcal{C}^1$ functions defined in a neighborhood U of the origin of $\mathbb{C}^2$ satisfying the conditions:*

(i) *$f(z, w) = \overline{z} + \phi_1(z, w)$ and $g(z, w) = \overline{w} + \phi_2(z, w)$ for every $(z, w) \in U$.*

(ii) *$\phi_1(z, w) = o(zw)$ and $\phi_2(z, w) = o(zw)$ then (z, w) tends to $(0, 0)$.*

(iii) *The functions $z^{n_1}, f^{n_2}, w^{n_3}$ and g^{n_4} separate points near the origin.*

Then

$$[z^{n_1}, f^{n_2}, w^{n_3}, g^{n_4}; \Delta] = C(\Delta)$$

if Δ is a sufficiently small bidisk centered at the origin of $\mathbb{C}^2$.

Remark 1.2. 1) There are a lot of functions f, g verifying the assumption of Proposition 1.1. Indeed, it is easy to check that the functions

$$f(z, w) = \overline{z} + z^2 w^2, \quad g(z, w) = \overline{w} + z^2 w^2$$

satisfy the conditions of Proposition 1.1.

2) If the pairs of n_1, n_2, n_3 and n_4 are not coprime and the functions f, g as above then we are interested in proving

$$[z^k, g^k, w^k, f^k; \Delta] = C(\Delta),$$

where k is the greatest common divisor of n_1, n_2, n_3 and n_4. Note that by the work of O'Farrell and Garcia (see [8]) we get that $X = \{z^2, (\overline{z} + \overline{z}^2)^2, w^2, (\overline{w} + \overline{w}^2)^2 : (z, w) \in \Delta\}$ is not polynomially convex for all small bidisks Δ centered at the origin. So

$$[z^2, (\overline{z} + \overline{z}^2)^2, w^2, (\overline{w} + \overline{w}^2)^2; \Delta] \neq C(\Delta).$$

The following theorem gives a class of f, g such that

$$[z^2, g^2, w^2, f^2; \Delta] = C(\Delta).$$

Theorem 1.3. *Suppose that*

$$f(z, w) = z^2 + \varphi_1(z, w),$$
$$g(z, w) = w^2 + \varphi_2(z, w),$$

where φ_1, φ_2 are smooth functions in a neighborhood of the origin of $\mathbb{C}^2$, $\varphi_1(z, w) = o(|zw|^2)$ and $\varphi_2(z, w) = o(|zw|^2)$. Then for all small bidisks Δ the functions $z^2, (\overline{z} + f)^2, w^2$ and $(\overline{w} + g)^2$ separate points near the origin. Moreover

$$[z^2, (\overline{z} + f)^2, w^2, (\overline{w} + g)^2; \Delta] = C(\Delta).$$

2. Preliminaries

We will frequently invoke the following useful result (see [10]).

Theorem 2.1. [Kallin's lemma] *Suppose that:*

(i) X_1, X_2 *are polynomially convex subsets of* $\mathbb{C}^n$*;*

(ii) Y_1, Y_2 *are polynomially convex subsets of* $\mathbb{C}$ *such that* 0 *is a common boundary point for* Y_1 *and* Y_2*, and* $Y_1 \cap Y_2 = \{0\}$;

(iii) $p : \mathbb{C}^n \to \mathbb{C}$ *is a polynomial map such that* $p(X_1) \subset Y_1$, $p(X_2) \subset Y_2$*;*

(iv) $p^{-1}(0) \cap (X_1 \cup X_2)$ *is polynomially convex.*

Then $X_1 \cup X_2$ *is polynomially convex.*

It should be noticed that Kallin's lemma is applicable mostly in the case where $X_1 \cap X_2$ is "small". One can improve the conclusion of the Theorem 2.1 by strengthening the conditions. We have the following variant due to Stout (see [10]).

Theorem 2.2. [Stout] *Suppose that:*

(i) X_1, X_2 *are compact subsets of* $\mathbb{C}^n$ *with* $P(X_1) = C(X_1)$ *and* $P(X_2) = C(X_2)$*;*

(ii) Y_1, Y_2 *are polynomially convex subsets of* $\mathbb{C}$ *such that* 0 *is a common boundary point for* Y_1 *and* Y_2*, and* $Y_1 \cap Y_2 = \{0\}$;

(iii) $p : \mathbb{C}^n \to \mathbb{C}$ *is a polynomial map such that* $p(X_1) \subset Y_1,\ p(X_2) \subset Y_2$*;*

(iv) $p^{-1}(0) \cap (X_1 \cup X_2) = X_1 \cap X_2$.

Then $P(X_1 \cup X_2) = C(X_1 \cup X_2)$.

Recall that a manifold M is *totally real* if for each point $a \in M$ the (real) tangent space T_aM contains no complex line. An obvious example of totally real manifolds is the real Euclidean space $\mathbb{R}^n$. Let D be a region in $\mathbb{C}^n$. Let M be a $2n$-dimensional surface of class $\mathcal{C}^1$ defined by

$$M = \{(z_1, \dots, z_n, f_1, \dots, f_n) : (z_1, \dots, z_n) \in D\},$$

where $f_i \in \mathcal{C}^1(D)$ for every $i = 1, \dots, n$. It is easy to see that M is totally real if and only if

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial \overline{z}_k}\right)_{i,k} = n.$$

A well-known result of Wermer and Hörmander (see [1]) states that every smooth totally real manifold M of $\mathbb{C}^n$ is locally polynomially convex. Moreover every continuous function on a polynomially convex subset of a smooth totally real manifold M in $\mathbb{C}^n$ can be approximated uniformly by polynomials. The next result is a far reaching generalization of the above statement.

Theorem 2.3. [O'Farrell, Preskenis and Walsh] *Let* X *be a compact polynomially convex set in* $\mathbb{C}^n$ *and* E *be a closed subset of* X*. Assume that* $X \backslash E$ *is totally real* (*that is, locally contained in a totally real manifold*)*. Then*

$$\mathcal{P}(X) = \{f \in \mathcal{C}(X) : f|_E \in \mathcal{P}(E)\}.$$

We need the follwing result of Weinstock (see [15, p. 60]).

Theorem 2.4. *Let* $f = (f_1, \dots, f_n)$ *be* n*-tube of functions of class* $\mathcal{C}^1$ *on a neighborhood of* 0 *in* $\mathbb{C}^n$*. If*

$$\det\left(\frac{\partial f_i}{\partial \overline{z}_k}(0)\right)_{i,k} \neq 0$$

then there exists a closed neighborhood of 0 *in* $\mathbb{C}^n$ *on which every continuous function is a uniform limit of polynomial* $z_1, \dots, z_n, f_1, \dots, f_n$*.*

The following lemma is a generalization of the result of [10].

Lemma 2.5. *Let* X *be a compact subset of* $\mathbb{C}^m$*, and a polynomial mapping* $\pi : \mathbb{C}^m \to \mathbb{C}^m$ *defined by*

$$\pi(z_1,\dots,z_m)=\left(z_1^{k_1},\dots,z_m^{k_m}\right).$$

Let $\pi^{-1}(X)=X_{1,\dots,1}\cup\dots\cup X_{i_1,\dots,i_m}\cup\dots\cup X_{k_1,\dots,k_m}$*, with* $X_{1,\dots,1}$ *compact, and*

$$X_{i_1,\dots,i_m}=\left\{\left(\rho_{k_1}^{k_1-1}z_1,\dots,\rho_{k_m}^{k_m-1}z_m\right):(z_1,\dots,z_m)\in X_{1,\dots,1}\right\}$$

for $1\le i_1\le k_1,\dots,1\le i_m\le k_m$*, where* $\rho_{k_i}=\exp\left(\frac{2\pi i}{k_j}\right)$ *with* $j=1,\dots,m$*. If*

$$P\left(\pi^{-1}(X)\right)=C\left(\pi^{-1}(X)\right)$$

then $P(X)=C(X)$*.*

Proof. Let $Q(z_1,\dots,z_m)=\sum a_{j_1,\dots,j_m}z_1^{j_1}\dots z_m^{j_m}$ be a polynomial in m variables. For each $(i_1,\dots,i_m)$ with $1\le i_1\le k_1,\dots,1\le i_m\le k_m$, we put

$$Q_{i_1,\dots,i_m}(z_1,\dots,z_m):=Q\big(\rho_{k_1}^{i_1-1}z_1,\dots,\rho_{k_m}^{i_m-1}z_m\big),\ (z_1,\dots,z_m)\in X_{1,\dots,1}.$$

We claim that

$$\frac{1}{k_1\dots k_m}\sum Q_{i_1,\dots,i_m}(z_1,\dots,z_m)=\sum a_{p_1k_1,\dots,p_mk_m}z_1^{k_1p_1}\dots z_m^{k_mp_m}.$$

Indeed, since every polynomial Q can be written as a finite sum of monomials of the forms $az_1^{s_1}\dots z_m^{s_m}$ we only need to check for $Q(z_1,\dots,z_n)=az_1^{s_1}\dots z_m^{s_m}$. We have

$$\begin{aligned}&\frac{1}{k_1\dots k_m}\sum Q_{i_1,\dots,i_m}(z_1,\dots,z_m)\\&=\frac{a}{k_1\dots k_m}z_1^{s_1}\dots z_m^{s_m}\sum\rho_{k_1}^{(i_1-1)s_1}\dots\rho_{k_m}^{(i_m-1)s_m}\\&=\frac{a}{k_1\dots k_m}z_1^{s_1}\dots z_m^{s_m}\prod_{j=1}^{m}\left(\sum_{1\le i_j\le k_j}\left(\rho_{k_j}^{s_j}\right)^{i_j-1}\right).\end{aligned}$$

If there exists $1\le j\le m$ such that $s_j\ne p_jk_j$ then

$$\sum_{1\le i_j\le k_j}\left(\rho_{k_j}^{s_j}\right)^{i_j-1}=\frac{(\rho_{k_j}^{s_j})^{k_j}-1}{\rho_{k_j}^{s_j}-1}=0,$$

where $\rho_{k_j}^{k_j}=(\exp\frac{2\pi i}{k_j})^{k_j}=1$. It follows that

$$\frac{1}{k_1\dots k_m}\sum Q_{i_1,\dots,i_m}(z_1,\dots,z_m)=0.$$

In the case $s_j=p_jk_j$ for all $j=1,\dots,m$ we have

$$\sum_{1\le i_j\le k_j} (\rho_{k_j}^{s_j})^{i_j-1} = \sum_{1\le i_j\le k_j} (\rho_{k_j}^{k_j})^{p_j(i_j-1)} = k_j.$$

It implies that

$$\frac{1}{k_1...k_m}\sum Q_{i_1,\dots,i_m}(z_1,\dots,z_m) = \frac{a}{k_1...k_m} z_1^{s_1}\dots z_m^{s_m}\prod_{j=1}^m k_j$$
$$= a z_1^{p_1k_1}\dots z_n^{k_mp_m}.$$

The claim is proved. Now, suppose that $P\Big(\pi^{-1}(X)\Big) = C\Big(\pi^{-1}(X)\Big)$. Let $f\in C(X)$. Then $f\circ\pi\in C\Big(\pi^{-1}(X)\Big)$, so there is a polynomial Q in m variables with $f\circ\pi\approx Q$ in $\pi^{-1}(X)$. In particular, this is true for $X_{i_1,\dots,i_m}$, so

$$f(z_1^{k_1},\dots,z_m^{k_m}) \approx Q(\rho_{k_1}^{i_1-1}z_1,\dots,\rho_{k_m}^{i_m-1}z_m) = Q_{i_1,\dots,i_m}(z_1,\dots,z_m)$$

on $X_{1,\dots,1}$. It follows that

$$f(z_1^{k_1},\dots,z_m^{k_m}) \approx \frac{1}{k_1\dots k_m}\sum_{1\le i_1\le k_1,\dots,1\le i_m\le k_m} Q_{i_1,\dots,i_m}(z_1,\dots,z_m) \text{ on } X_{1,\dots,1}. \tag{1}$$

If Q has the form

$$Q(z_1,\dots,z_m) = \sum a_{r_1,\dots,r_m} z_1^{r_1}\dots z_m^{r_m}$$

then the right hand side of (1) equals

$$\sum a_{p_1k_1,\dots,p_mk_m} z_1^{k_1p_1}\dots z_m^{k_mp_m},$$

so equals $P(z_1^{k_1},\dots,z_m^{k_m})$, where P is a polynomial in m variables. We conclude that

$$f(z_1^{k_1},\dots,z_m^{k_m}) \approx P(z_1^{k_1},\dots,z_m^{k_m}) \text{ on } X_{1,\dots,1}.$$

That is $f\approx P$ on X. So $P(X) = C(X)$. The lemma is proved. ∎

3. The Proofs of the Results

Proof of Proposition 1.1. Consider the polynomial mapping

$$\varPhi : (z_1,z_2,z_3,z_4)\in\mathbb{C}^4 \mapsto \varPhi(z_1,z_2,z_3,z_4) = (z_1^{n_1}, z_2^{n_2}, z_4^{n_3}, z_3^{n_4})\in\mathbb{C}^4.$$

For small bidisks $\varDelta$ around the origin of $\mathbb{C}^2$ we set

$$X = \Big\{\Big(z^{n_1}, \big(\overline{z}+\phi_1(z,w)\big)^{n_2}, w^{n_3}, \big(\overline{w}+\phi_2(z,w)\big)^{n_4}\Big) : (z,w)\in\varDelta\Big\}.$$

Then we have

$$\Phi^{-1}(X) = \bigcup X_{j_1 j_2 j_3 j_4},\ 1 \le j_1 \le n_1,\ 1 \le j_2 \le n_2\ 1 \le j_3 \le n_3,\ 1 \le j_4 \le n_4,$$

where

$$X_{j_1 j_2 j_3 j_4} = \Big\{ \Big(\rho_1^{j_1} z, \rho_2^{j_2} \big(\overline{z} + \phi_2(z,w) \big), \rho_3^{j_3} w, \rho_4^{j_4} \big(\overline{w} + \phi_1(z,w) \big) \Big) : (z,w) \in \Delta \Big\},$$

and $\rho_k = \exp\left(\frac{2\pi i}{n_k}\right), \quad k = 1,2,3,4$. From Theorem 2.4 we infer that

$$P(X_{j_1 j_2 j_3 j_4}) = C(X_{j_1 j_2 j_3 j_4})$$

if Δ is sufficiently small. We will show that $P\big(\Phi^{-1}(X)\big) = C\big(\Phi^{-1}(X)\big)$. To do this consider the polynomial in four variables

$$p(z_1, z_2, z_3, z_4) = z_1 z_2 z_3 z_4.$$

We obtain

$$\begin{aligned} p(X_{j_1 j_2 j_3 j_4}) &= \Big\{ \rho_1^{j_1} \rho_2^{j_2} \rho_3^{j_3} \rho_4^{j_4} \big(|z|^2 + z\phi_1(z,w) \big) \big(|w|^2 + w\phi_2(z,w) \big) : (z,w) \in \Delta \Big\} \\ &= \Big\{ \rho_1^{j_1} \rho_2^{j_2} \rho_3^{j_3} \rho_4^{j_4} |zw|^2 + o(z^2 w^2) : (z,w) \in \Delta \Big\}. \end{aligned}$$

We see that the set $p(X_{j_1 j_2 j_3 j_4})$ is contained in a sector at 0, situated near the half line through 0 with the argument

$$2\pi \left(\frac{j_1}{n_1} + \frac{j_2}{n_2} + \frac{j_3}{n_3} + \frac{j_4}{n_4} \right).$$

By the pairwise coprimeness of n_1, n_2, n_3 and n_4 we get that

$$\frac{j_1}{n_1} + \frac{j_2}{n_2} + \frac{j_3}{n_3} + \frac{j_4}{n_4} \ne \frac{i_1}{n_1} + \frac{i_2}{n_2} + \frac{i_3}{n_3} + \frac{i_4}{n_4}$$

if $j_s \ne i_s$ for some $s = 1,2,3,4$. It follows that

$$p(X_{j_1 j_2 j_3 j_4}) \cap p(X_{i_1 i_2 i_3 i_4}) = \{0\}$$

with $j_s \ne i_s$ for some $1 \le s \le 4$. Moreover, this fact implies that

$$\big(X_{j_1 j_2 j_3 j_4} \cap X_{i_1 i_2 i_3 i_4} \big) = p^{-1}(0) \cap \big(X_{j_1 j_2 j_3 j_4} \cup X_{i_1 i_2 i_3 i_4} \big)$$

with $j_s \ne i_s$ for some $1 \le s \le 4$. Applying Theorem 2.2 repeatedly we get that

$$P\big(\pi^{-1}(X)\big) = C\big(\pi^{-1}(X)\big).$$

By Lemma 2.5 we infer that $P(X) = C(X)$. In view of condition (iii) the map Φ is a diffeomorphism from Δ onto X. It follows that $C(X) = C(\Delta)$, or equivalently

$$[z^{n_1}, f^{n_2}, w^{n_3}, g^{n_4}; \Delta] = C(\Delta).$$

∎

Proof of Theorem 1.3. First we check that z^2, f^2, w^2 and g^2 separate points near $(0,0)$. If $(a_1, b_1) \neq (a_2, b_2)$ then $a_1 \neq a_2$ or $b_1 \neq b_2$. Suppose that $a_1 \neq a_2$. Clearly the points a_1 and a_2 with $a_2 \neq -a_1$ are separated by z^2. Now assume that f^2 takes the same value at a_1 and $-a_1$ for some $a_1 \neq 0$. This means that

$$\left(\overline{a}_1 + a_1^2 + o(a_1^2)\right)^2 = \left(-\overline{a}_1 + a_1^2 + o(a_1^2)\right)^2.$$

It implies that $2\overline{a}_1 = o(a_1^2)$ or $2a_1^2 = o(a_1^2)$. We arrive at a contradiction if we choose a bidisk Δ sufficiently small. Hence (a_1, b_1) and (a_2, b_2) are separated by z^2 and f^2. Similarly, if $b_1 \neq b_2$ then (a_1, b_1) and (a_2, b_2) are separated by w^2 and g^2.

Next consider the polynomial mapping

$$\pi : \mathbb{C}^4 \to \mathbb{C}^4$$

defined by $\pi(z_1, z_2, z_3, z_4) = (z_1^2, z_2^2, z_3^2, z_4^2)$. For a closed bidisk Δ centered at the origin of $\mathbb{C}^2$, we set

$$S(\Delta) = \Big\{\left(z^2, (\overline{z} + f)^2, w^2, (\overline{w} + g)^2\right) : (z, w) \in \Delta\Big\}.$$

It is easy to compute that $\pi^{-1}\left(S(\Delta)\right) = \bigcup_{i=1}^{16} X_i$, where X_i are totally real graphs around the origin of $\mathbb{C}^4$. By the results of Wermer mentioned in Preliminaries we infer that each X_i is polynomially convex for Δ small enough. More precisely,

$$\begin{aligned}
X_1 &= \{\left(z, \overline{z} + z^2 + \varphi_1(z,w), w, \overline{w} + w^2 + \varphi_2(z,w)\right) : (z,w) \in \Delta\}.\\
X_2 &= \{\left(z, -\overline{z} - z^2 - \varphi_1(z,w), w, \overline{w} + w^2 + \varphi_2(z,w)\right) : (z,w) \in \Delta\}.\\
X_3 &= \{\left(z, \overline{z} + z^2 + \varphi_1(z,w), -w, \overline{w} + w^2 + \varphi_2(z,w)\right) : (z,w) \in \Delta\}\\
&= \{\left(z, \overline{z} + z^2 + \varphi_1(z,-w), w, -\overline{w} + w^2 + \varphi_2(z,-w)\right) : (z,w) \in \Delta\}.\\
X_4 &= \{\left(z, \overline{z} + z^2 + \varphi_1(z,w), w, -\overline{w} - w^2 - \varphi_2(z,w)\right) : (z,w) \in \Delta\}.\\
X_5 &= \{\left(z, -\overline{z} - z^2 - \varphi_1(z,w), -w, \overline{w} + w^2 + \varphi_2(z,w)\right) : (z,w) \in \Delta\}\\
&= \{\left(z, -\overline{z} - z^2 - \varphi_1(z,-w), w, -\overline{w} + w^2 + \varphi_2(z,-w)\right) : (z,w) \in \Delta\}.\\
X_6 &= \{\left(z, \overline{z} + z^2 + \varphi_1(z,w), -w, -\overline{w} - w^2 - \varphi_2(z,w)\right) : (z,w) \in \Delta\}\\
&= \{\left(z, \overline{z} + z^2 + \varphi_1(z,-w), w, \overline{w} - w^2 - \varphi_2(z,-w)\right) : (z,w) \in \Delta\}.\\
X_7 &= \{\left(z, -\overline{z} - z^2 - \varphi_1(z,w), w, -\overline{w} - w^2 - \varphi_2(z,w)\right) : (z,w) \in \Delta\}.\\
X_8 &= \{\left(z, -\overline{z} - z^2 - \varphi_1(z,w), -w, -\overline{w} - w^2 - \varphi_2(z,w)\right) : (z,w) \in \Delta\}\\
&= \{\left(z, -\overline{z} - z^2 - \varphi_1(z,-w), w, \overline{w} - w^2 - \varphi_2(z,-w)\right) : (z,w) \in \Delta\}.\\
X_9 &= \{\left(-z, \overline{z} + z^2 + \varphi_1(z,w), w, \overline{w} + w^2 + \varphi_2(z,w)\right) : (z,w) \in \Delta\}
\end{aligned}$$

$$= \{(z, -\overline{z} + z^2 + \varphi_1(-z, w), w, \overline{w} + w^2 + \varphi_2(-z, w)) : (z, w) \in \Delta\}.$$

$$X_{10} = \{(-z, -\overline{z} - z^2 - \varphi_1(z, w), w, \overline{w} + w^2 + \varphi_2(z, w)) : (z, w) \in \Delta\}$$
$$= \{(z, \overline{z} - z^2 - \varphi_1(-z, w), w, \overline{w} + w^2 + \varphi_2(-z, w)) : (z, w) \in \Delta\}.$$

$$X_{11} = \{(-z, \overline{z} + z^2 + \varphi_1(z, w), -w, \overline{w} + w^2 + \varphi_2(z, w)) : (z, w) \in \Delta\}$$
$$= \{(z, -\overline{z} + z^2 + \varphi_1(-z, -w), w, -\overline{w} + w^2 + \varphi_2(-z, -w)) : (z, w) \in \Delta\}.$$

$$X_{12} = \{(-z, \overline{z} + z^2 + \varphi_1(z, w), w, -\overline{w} - w^2 - \varphi_2(z, w)) : (z, w) \in \Delta\}$$
$$= \{(z, -\overline{z} + z^2 + \varphi_1(-z, w), w, -\overline{w} - w^2 - \varphi_2(-z, w)) : (z, w) \in \Delta\}.$$

$$X_{13} = \{(-z, \overline{z} - z^2 - \varphi_1(z, w), -w, \overline{w} + w^2 + \varphi_2(z, w)) : (z, w) \in \Delta\}$$
$$= \{(z, \overline{z} - z^2 - \varphi_1(-z, -w), w, -\overline{w} + w^2 + \varphi_2(-z, -w)) : (z, w) \in \Delta\}.$$

$$X_{14} = \{(-z, -\overline{z} - z^2 - \varphi_1(z, w), w, -\overline{w} - w^2 - \varphi_2(z, w)) : (z, w) \in \Delta\}$$
$$= \{(z, \overline{z} - z^2 - \varphi_1(-z, w), w, -\overline{w} - w^2 - \varphi_2(z, w)) : (-z, w) \in \Delta\}.$$

$$X_{15} = \{(-z, \overline{z} + z^2 + \varphi_1(z, w), -w, -\overline{w} - w^2 - \varphi_2(z, w)) : (z, w) \in \Delta\}$$
$$= \{(z, -\overline{z} + z^2 + \varphi_1(-z, -w), w, \overline{w} - w^2 - \varphi_2(-z, -w)) : (z, w) \in \Delta\}.$$

$$X_{16} = \{(-z, -\overline{z} - z^2 - \varphi_1(z, w), -w, -\overline{w} - w^2 - \varphi_2(z, w)) : (z, w) \in \Delta\}$$
$$= \{(z, \overline{z} - z^2 - \varphi_1(-z, -w), w, \overline{w} - w^2 - \varphi_2(-z, -w)) : (z, w) \in \Delta\}.$$

We will show that $\pi^{-1}(S(\Delta))$ is polynomially convex if Δ is sufficiently small.
Claim 1. For Δ small enough, $M_1 = Y_1 \cup Y_2$ is polynomially convex where $Y_1 = X_1 \cup X_3 \cup X_4 \cup X_6$ and $Y_2 = X_{10} \cup X_{13} \cup X_{14} \cup X_{16}$.

First we show that $X_1 \cup X_6$ is polynomially convex. Let p_1 be a polynomial in four variables defined by

$$p_1(z_1, z_2, z_3, z_4) = i(z_3^3 - z_4^3).$$

We obtain

$$p_1(X_1) = \Big\{i\Big((w^3 - \big(\overline{w} + w^2 + \varphi_2(z, w)\big)^3\Big) : (z, w) \in \Delta\Big\}$$
$$= \Big\{i(w^3 - \overline{w}^3) - 3i|w|^4 + o(|w|^4) : (z, w) \in \Delta\Big\}$$

and

$$p_1(X_6) = \Big\{i\Big((w^3 - \big(\overline{w} - w^2 - \varphi_2(z, -w)\big)^3\Big) : (z, w) \in \Delta\Big\}$$
$$= \Big\{i(w^3 - \overline{w}^3) + 3i|w|^4 + o(|w|^4) : (z, w) \in \Delta\Big\}.$$

It follows that $p_1(X_1)$ is contained in the lower half plane and $p_1(X_6)$ is contained in the upper half plane for Δ small enough. Hence $p_1(X_1)$ only meets $p_1(X_6)$ at 0 if Δ is sufficiently small. On the other hand, it is easy to see that

$$p_1^{-1}(0) \cap (X_1 \cup X_6) = \{(z, \overline{z} + z^2, 0, 0) : z \in D\},$$

where D is a disk centered at the origin in $\mathbb{C}$. So $p_1^{-1}(0) \cap (X_1 \cup X_6)$ is a totally real graph. It implies that $p_1^{-1}(0) \cap (X_1 \cup X_6)$ is polynomially convex for Δ small enough. Applying Kallin's lemma we can deduce that $Z_1 := X_1 \cup X_6$ is polynomially convex if Δ is small enough. In the similar way with polynomial $p_2(z_1, z_2, z_3, z_4) = i(z_3^3 + z_4^3)$ we get that $Z_2 := X_3 \cup X_4$ is polynomially convex.

Now consider the polynomial $q(z_1, z_2, z_3, z_4) = z_3 z_4$. We have

$$q(Z_1) = \big\{|w|^2 + o(|w|^2) : (z, w) \in \Delta\big\}$$

and

$$q(Z_2) = \big\{ -|w|^2 + o(|w|^2) : (z, w) \in \Delta\big\}.$$

Thus $q(Z_1) \cap q(Z_2) = \{0\}$ if Δ is sufficiently small. Moreover

$$q^{-1}(0) \cap (Z_1 \cup Z_2) = \{(z, \overline{z} + z^2, 0, 0) : z \in D\},$$

where D is a small disk centered at the origin in $\mathbb{C}$. It implies that $q^{-1}(0) \cap (Z_1 \cup Z_2)$ is polynomially convex. From Kallin's lemma we get that $Y_1 = Z_1 \cup Z_2$ is polynomially convex.

Similarly we can show that $Y_2 = X_{10} \cup X_{13} \cup X_{14} \cup X_{16}$ is polynomially convex. Now consider the polynomial $p(z_1, z_2, z_3, z_4) = i(z_1^3 - z_2^3)$. We have

$$\begin{aligned} p(Y_1) &= \Big\{ i\Big(z^3 - \big(\overline{z} + z^2 + \varphi_1(z, w)\big)^3\Big) : (z, w) \in \Delta\Big\} \\ &\quad \cup \Big\{ i\Big(z^3 - \big(\overline{z} + z^2 + \varphi_1(z, -w)\big)^3\Big) : (z, w) \in \Delta\Big\} \\ &= \Big\{ i(z^3 - \overline{z}^3) - 3i|z|^4 + o(|z|^4) : (z, w) \in \Delta\Big\}, \end{aligned}$$

and

$$\begin{aligned} p(Y_2) &= \Big\{ i\Big(z^3 - \big(\overline{z} - z^2 + \varphi_1(z, w)\big)^3\Big) : (z, w) \in \Delta\Big\} \\ &\quad \cup \Big\{ i\Big(z^3 - \big(\overline{z} - z^2 + \varphi_1(z, -w)\big)^3\Big) : (z, w) \in \Delta\Big\} \\ &= \Big\{ i(z^3 - \overline{z}^3) + 3i|z|^4 + o(|z|^4) : (z, w) \in \Delta\Big\}. \end{aligned}$$

It follows that $p(Y_1)$ is contained in the lower half plane and $p(Y_2)$ is contained in the upper half plane for Δ small enough. Hence $p(Y_1)$ only meets $p(Y_2)$ at 0 for Δ small enough. On the other hand, it is easy to check that

$$\begin{aligned} p^{-1}(0) \cap (Y_1 \cup Y_2) = &\{(0, 0, w, \overline{w} + w^2) : w \in D\} \cup \{(0, 0, w, \overline{w} - w^2) : w \in D\} \\ &\cup \{(0, 0, w, -\overline{w} + w^2) : w \in D\} \\ &\cup \{(0, 0, w, -\overline{w} - w^2) : w \in D\}, \end{aligned}$$

where D is a small disk centered at the origin of the complex plane. Invoking the Kallin's lemma respectively with polynomials

$$p_3(z_1, z_2, z_3, z_4) = i(z_3^3 - z_4^3),\ p_4(z_1, z_2, z_3, z_4) = i(z_3^3 + z_4^3),$$

we get that

$$N_1 = \{(0,0,w,\overline{w}+w^2) : w \in D\} \cup \{(0,0,w,\overline{w}-w^2) : w \in D\}$$

and

$$N_2 = \{(0,0,-w,-\overline{w}+w^2) : w \in D\} \cup \{(0,0,w,-\overline{w}-w^2) : w \in D\}$$

are polynomially convex. Applying Kallin's lemma with the polynomial $p_5(z_1, z_2, z_3, z_4) = z_3 z_4$ we can conclude that $p^{-1}(0) \cap (Y_1 \cup Y_2) = N_1 \cup N_2$ is polynomially convex. Using Kallin's lemma we conclude that $M_1 = Y_1 \cup Y_2$ is polynomially convex if Δ is small enough.

<u>Claim 2.</u> Put $Y_3 = X_2 \cup X_5 \cup X_7 \cup X_8$ and $Y_4 = X_9 \cup X_{11} \cup X_{12} \cup X_{15}$. Then $M_2 = Y_3 \cup Y_4$ is polynomially convex if Δ is sufficiently small.

Indeed, in the way as in the proof of Claim 1 we can deduce that $X_2 \cup X_8$ and $X_5 \cup X_7$ are polynomially convex (using Kallin's lemma respectively with polynomials $q_1(z_1, z_2, z_3, z_4) = i(z_3^3 - z_4^3)$, $q_2(z_1, z_2, z_3, z_4) = i(z_3^3 + z_4^3)$). Applying Kallin's lemma with the polynomial $q_3(z_1, z_2, z_3, z_4) = z_3 z_4$ we infer that $Y_3 = (X_2 \cup X_8) \cup (X_5 \cup X_7)$ is polynomially convex. Similarly Y_4 is polynomially convex. Next, applying Kallin's lemma with the polynomial $q(z_1, z_2, z_3, z_4) = i(z_1^3 - z_2^3)$ we can get that $M_2 = Y_3 \cup Y_4$ is polynomially convex if Δ is small enough.

Last consider the polynomial $F(z_1, z_2, z_3, z_4) = z_1 z_2$. We have

$$F(M_1) = \{|z|^2 + o(|z|^2) : (z, w) \in \Delta\}$$

and

$$F(M_2) = \{-|z|^2 + o(|z|^2) : (z, w) \in \Delta\}.$$

It implies that $F(M_1) \cap F(M_2) = \{0\}$ if Δ is small enough. On the other hand, it is easy to see that

$$\begin{aligned} F^{-1}(0) \cap (M_1 \cup M_2) = &\{(0,0,w,\overline{w}+w^2) : w \in D\} \cup \{(0,0,w,\overline{w}-w^2) : w \in D\} \\ &\cup \{(0,0,w,-\overline{w}+w^2) : w \in D\} \\ &\cup \{(0,0,w,-\overline{w}-w^2) : w \in D\} \\ = &N_1 \cup N_2, \end{aligned}$$

where N_1 and N_2 are defined as in the proof of Claim 1. Hence $F^{-1}(0) \cap (M_1 \cup M_2)$ is polynomially convex. Using Kallin's lemma we can conclude that $\pi^{-1}\big(S(\Delta)\big) = M_1 \cup M_2$ is polynomially convex for Δ small enough. Clearly $\pi^{-1}\big(S(\Delta)\big)$ is totally real outside the origin. Applying Theorem 2.3 (with $E = \{0\}, 0 \in \mathbb{C}^4$) we get that $P\big(\pi^{-1}\big(S(\Delta)\big) = C\big(\pi^{-1}\big(S(\Delta)\big)$. By Lemma 2.5 we have $P\big(S(\Delta)\big) = C\big(S(\Delta)\big)$ or equivalently

$$[z^2, (\overline{z}+f)^2, w^2, (\overline{w}+g)^2; \Delta] = C(\Delta).$$

The theorem is proved. ■

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